Physics 318: Problem Set 12
Due Wednesday, April 30, 2008

1. Poisson Brackets:
   a. Show that the Poisson bracket \( \{ F, G \} \) of two functions \( F \) and \( G \) on phase space can be written as
   \[
   \{ F, G \} = \frac{\partial F}{\partial \eta_i} J_{ij} \frac{\partial G}{\partial \eta_j}.
   \]
   Here the vector \( \eta \) and the \( 2f \times 2f \) matrix \( J \) are the quantities defined in lecture, given by
   \[
   \eta_i = q_i, \eta_{f+i} = p_i, \quad J_{ij} = 0, \quad J_{i,f+j} = \delta_{ij}, \quad J_{f+i,f+j} = 0 \quad \text{for} \quad 1 \leq i, j \leq f.
   \]
   b. Show that a mapping from the phase space coordinates \( \eta_i \) to new phase space coordinates \( \zeta_i \) is canonical if and only if it preserves Poisson brackets, i.e. if
   \[
   \frac{\partial F}{\partial \eta_i} J_{ij} \frac{\partial G}{\partial \eta_j} = \frac{\partial F}{\partial \zeta_i} J_{ij} \frac{\partial G}{\partial \zeta_j}
   \]
   for any functions \( F \) and \( G \) on phase space.

2. The wave equation:
   a. Show that the wave equation
   \[
   \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0
   \]
   is solved by \( u(x,t) = f(x-ct) + g(x+ct) \) where \( f \) and \( g \) are arbitrary functions of a single variable.
   b. For the case of the string discussed in lectures, determine the functions \( f \) and \( g \) such that the boundary conditions \( u(0,t) = u(l,t) = 0 \) and the initial conditions \( u(x,0) = F(x) \) and \( \dot{u}(x,0) = G(x) \) are satisfied.

3. Action principle for electromagnetic fields: Consider the action functional
   \[
   S[\Phi(x,t), A(x,t)] = \int dt \int d^3x \left[ \frac{1}{2} \varepsilon_0 (\nabla \Phi + \dot{A})^2 - \frac{1}{2\mu_0} (\nabla \times A)^2 - \rho \Phi + j \cdot A \right].
   \]
   This is a functional of the scalar potential \( \Phi \) and the vector potential \( A \), and depends also on the charge density \( \rho(x,t) \) and current density \( j(x,t) \). Here \( \varepsilon_0 \) is the permittivity of empty space and \( \mu_0 \) is the permeability of empty space.
   a. By varying the action with respect to the scalar potential \( \Phi \), derive the equation
   \[
   \nabla^2 \Phi + \nabla \cdot \dot{A} = -\rho/\varepsilon_0.
   \]
   b. By varying the action with respect to the scalar potential \( A \), derive the equation
   \[
   \mu_0 \varepsilon_0 (\nabla \Phi + \dot{A}) + \nabla (\nabla \cdot A) - \nabla^2 A - \mu_0 \dot{j} = 0.
   \]
   c. Using the expressions
   \[
   E = -\nabla \Phi - \dot{A}, \quad B = \nabla \times A
   \]
   for the electric and magnetic fields in terms of the potentials, deduce from a. and b. the four Maxwell equations
   \[
   \nabla \cdot E = \rho/\varepsilon_0, \quad \nabla \cdot B = 0,
   \]
   \[
   \nabla \times E = -\dot{j}, \quad \nabla \times B = \mu_0 \dot{A}.
   \]
\[ \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dot{\mathbf{E}} + \mu_0 \mathbf{j}. \]

d. Now suppose that the electric and magnetic fields are coupled to \( N \) point particles of masses \( m_n \), charges \( q_n \), and positions \( \mathbf{x}_n(t) \) for \( 1 \leq n \leq N \). The charge density and current density are then

\[
\rho(\mathbf{x}, t) = \sum_{n=1}^{N} q_n \delta^3[\mathbf{x} - \mathbf{x}_n(t)], \quad \mathbf{j}(\mathbf{x}, t) = \sum_{n=1}^{N} q_n \dot{\mathbf{x}}_n(t) \delta^3[\mathbf{x} - \mathbf{x}_n(t)].
\]

The total action for the system consisting of the electric and magnetic fields and the point particles can be obtained by adding to the above action the kinetic energy

\[
\sum_{n=1}^{N} \int dt \frac{1}{2} m_n \dot{\mathbf{x}}_n^2
\]

for the particles. By varying this total action with respect to the positions of the particles, derive the equation of motion

\[
m_n \ddot{\mathbf{x}}_n(t) = q_n \mathbf{E}[\mathbf{x}_n(t), t] + q_n \dot{\mathbf{x}}_n(t) \times \mathbf{B}[\mathbf{x}_n(t), t]
\]

for \( 1 \leq n \leq N \).

4. Consider a simple one-dimensional model of a crystal consisting of an infinite chain of identical point masses of mass \( m \) connected by identical springs of length \( a \) and force constant \( f \). In equilibrium, the distance between two successive is the lattice spacing \( a \). Let the displacement of the \( n \)th point mass from its equilibrium position be \( x_n(t) \).

\[
\begin{array}{c}
\begin{array}{cccc}
\text{m} & \text{m} & \text{m} & \text{m} \\
\text{f} & \text{f} & \text{f} \\
X_{n-1} & X_n
\end{array}
\end{array}
\]

a. Derive the Lagrangian and equations of motion for the chain.

b. Show that for an arbitrary real constant \( k \), the Bloch wave

\[
x_n(t) = \text{Re} \left[ Q_k(t) \exp(ikna) \right]
\]

defines a normal mode, i.e. that the equations of motion can be satisfied by an ansatz of the form \( Q_k(t) = A_k \exp(i\omega_k t) \). Argue that without loss of generality the constant \( k \) can be restricted to the range \(-\pi/a \leq k \leq \pi/a\). Make a sketch of the eigenfrequencies \( \omega_k(k) \) as a function of \( k \). This relation is called a dispersion relation.

c. Consider now a finite chain of \( N \) masses subject to the periodic boundary condition \( x_n(t) = x_{n+N}(t) \). Show that this boundary condition leads to a discrete set of values \( k_m, m = 1, 2, 3, \ldots \), and determine these values. How many physically different values of \( k_m \) exist, and what are the corresponding eigenfrequencies \( \omega_m = \omega(k_m) \)? Write down the general solution for the motion of the chain and describe its properties.