Physics 7683 : Problem Set 1
Due Thursday, Sept 17, 2009

1. Bogolubov Transformations for Special Cases: Consider the time dependent harmonic oscillator
\[ H = \frac{\hat{p}^2}{2} + \omega(t)^2 \frac{\hat{q}^2}{2}, \]
where \( \omega(t) \) is a smooth function taking the values \( \omega(t) = \omega_{\text{in}} \) at early times and \( \omega(t) = \omega_{\text{out}} \) at late times. The Bogolubov coefficients are defined by taking the solution
\[ q(t) = \exp[-i\omega_{\text{in}}t]/\sqrt{2\omega_{\text{in}}} \]
which is purely positive frequency at early times, and writing it at late times as a linear combination of positive and negative frequency solutions:
\[ \sqrt{2\omega_{\text{out}}}q(t) = \alpha^* \exp[-i\omega_{\text{out}}t] - \beta \exp[i\omega_{\text{out}}t]. \]
In this problem we will compute the coefficients in some special cases.

a. Consider the adiabatic regime \( \dot{\omega}/\omega^2 \ll 1 \). By performing a WKB type analysis show that the leading order solution that is purely positive frequency at early times is
\[ q(t) = \frac{1}{\sqrt{2\omega(t)}} \exp \left[ -i \int dt' \omega(t') \right]. \]
Deduce that in this approximation the transformation is trivial, \( \beta = 0 \). Does this result continue to hold when one computes the subleading WKB (post adiabatic) corrections? [Hint: Replace \( \omega \) by \( \omega/\epsilon \) in the differential equation, use an ansatz of the form \( q(t) = [A(t) + \epsilon B(t) + \ldots] \exp[i\phi(t)/\epsilon] \), and expand the differential equation in powers of \( \epsilon \).]

b. Suppose that the frequency can be written as
\[ \omega(t) = \omega_{\text{in}} + \Delta \omega(t), \]
where the frequency perturbation is small, \( \Delta \omega \ll \omega_{\text{in}} \) and also \( T \Delta \omega \ll 1 \) where \( T \) is the duration of the period of time evolution. Derive an expression for \( |\beta| \) in terms of the Fourier transform of \( \Delta \omega \). Use your result to argue that if \( \omega(t) \) is smooth, then \( |\beta| \) goes to zero faster than any power of \( 1/(\omega_{\text{in}} \tau) \) in the limit \( \tau \rightarrow \infty \), where \( \tau \) is the timescale over which \( \omega(t) \) changes. Also argue that if any finite-order derivative of \( \omega(t) \) has a discontinuity, then \( |\beta| \) will scale as a power law \( \propto 1/(\omega_{\text{in}} \tau)^n \) for some finite integer \( n \).

c. Suppose that the frequency \( \omega(t) \) changes instantaneously from \( \omega_{\text{in}} \) to \( \omega_{\text{out}} \) at \( t = 0 \). Show that for this case the Bogolubov coefficients are given by
\[ \alpha = \frac{1}{2} \left( \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} + \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \right), \quad \beta = \frac{1}{2} \left( \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} - \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} \right). \]

2. Normal Ordered Form of Squeezing Operator: Consider the Hilbert space of a harmonic oscillator whose annihilation operator is \( \hat{a} \). In this problem you will derive the normal ordered form of the squeezing operator \( \hat{S} \) which is defined by the property
\[ \hat{S}^\dagger \hat{a} \hat{S} = \alpha \hat{a} + \beta^* \hat{a}^\dagger, \]
where $\alpha$ and $\beta$ are complex numbers. In other words, you will derive the function $\bar{f}^{(n)}(\mu, \mu^*)$ of a complex variable $\mu$ and its complex conjugate $\mu^*$ for which

$$\hat{S} = : \bar{f}^{(n)}(\hat{a}, \hat{a}^\dagger) :$$

a. Using the fact that coherent states $|\mu\rangle$ are eigenstates of the annihilation operator, argue that

$$\bar{f}^{(n)}(\mu, \mu^*) = (\mu | \hat{S} | \mu).$$

b. Show that for any function $g$, we have

$$[g(\hat{a}), \hat{a}^\dagger] = g'(\hat{a}).$$

c. Write hermitian conjugate of the defining relation (1) in the form $\hat{a}^\dagger \hat{S} = \hat{S} (\alpha^* \hat{a}^\dagger + \beta \hat{a})$, multiply on the left by $(\mu |$ and on the right by $| \mu)$ Show using parts a. and b. that this gives the differential equation

$$\mu^* \bar{f}^{(n)}(\mu, \mu^*) = \beta \mu \bar{f}^{(n)}(\mu, \mu^*) + \alpha^* (\mu^* + \frac{\partial}{\partial \mu}) \bar{f}^{(n)}.$$

Similarly from $\hat{a} \hat{S} = \hat{S} (\alpha \hat{a} + \beta^* \hat{a}^\dagger)$ derive the differential equation

$$\left(\mu + \frac{\partial}{\partial \mu} \right) \bar{f}^{(n)}(\mu, \mu^*) = \alpha \mu \bar{f}^{(n)}(\mu, \mu^*) + \beta^* \left( \mu^* + \frac{\partial}{\partial \mu} \right) \bar{f}^{(n)}.$$

d. Solve this pair of differential equations using an ansatz of the form

$$\bar{f}^{(n)} = \mathcal{N} \exp \left[ A \mu^2 + B \mu \mu^* + C (\mu^*)^2 \right],$$

where $\mathcal{N}$, $A$, $B$ and $C$ are constants. Deduce the value of the normalization constant $\mathcal{N}$ from $1 = \langle 0 | \hat{S}^\dagger \hat{S} | 0 \rangle$ as in lecture. Thereby deduce that

$$\hat{S} = |\alpha|^{-1/2} : \exp \left[ -\frac{\beta}{2\alpha} \hat{a}^2 + \frac{\beta^*}{2\alpha} \hat{a}^\dagger 2 + \left( \frac{1}{\alpha^2} - 1 \right) \hat{a}^\dagger \hat{a} \right] :.$$

3. **Time-dependent, Driven Harmonic Oscillator:** In this problem we will generalize the analysis given in lecture to an oscillator which is driven in addition to having a time-varying frequency. The Hamiltonian of the system is

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega(t)^2 \hat{q}^2 - J(t) \hat{q}.$$

Assume that the source $J(t)$ vanishes and that the frequency $\omega(t)$ is constant at early times and at late times, with values $\omega_{in}$ and $\omega_{out}$. For any state $|\psi\rangle_{in}$ defined on the (Heisenberg picture) in basis, define a corresponding state $|\psi\rangle_{out}$ defined on the out basis by replacing in by out everywhere in the definition of the state. Then we have

$$|\psi\rangle_{in} = \hat{S}^\dagger |\psi\rangle_{out},$$

where $\hat{S}$ is a generalization of the operator derived in lecture. Derive an expression for $\hat{S}$ in terms of squeeze operators, rotation operators and displacement operators.