Rossby Waves

\[ \frac{\partial p}{\partial t} = -\frac{H_0}{\rho_0} \frac{\partial}{\partial z}, \quad H_0, \rho_0 \text{ constant} \]

thin layer, \( N^2 \text{ constant} \)

\[ \frac{\partial}{\partial t} \left( \frac{\alpha^2}{N^2} \frac{\partial \Phi}{\partial z^2} + \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z} \right) = 0 \]

\[ \frac{\partial}{\partial t} \left( \frac{\alpha^2}{N^2} \frac{\partial \Phi}{\partial z^2} + \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z} \right) = \frac{\partial}{\partial t} \left( \frac{\alpha^2}{N^2} \frac{\partial \Phi}{\partial z^2} \right) + \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z} \]

\[ u = -\frac{1}{f_0} \frac{\partial \Phi}{\partial y} \]

\[ v = \frac{1}{f_0} \frac{\partial \Phi}{\partial x} \]

Linearize

Consider small amplitude perturbations about \( \Phi = 0 \), \( \frac{\partial \Phi}{\partial t} = 0 \), \( \frac{\partial \Phi}{\partial x} = 0 \).

\[ \frac{\partial}{\partial t} \Phi + \frac{\partial}{\partial z} (f_0 \beta y + \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z}) = 0 \]

\[ = f_0 \beta y + \left( \frac{\alpha^2}{N^2} \frac{\partial^2 \Phi}{\partial z^2} \right) \Phi' \]

Drop squared terms \( \Phi' \)

\[ \frac{\partial}{\partial t} \Phi + \frac{\partial}{\partial z} (f_0 \beta y + \frac{f_0^2}{N^2} \frac{\partial \Phi}{\partial z}) \Phi' = 0 \]

\[ \frac{1}{f_0} \frac{\partial \Phi}{\partial y} + \left( \frac{\alpha^2}{N^2} \frac{\partial^2 \Phi}{\partial z^2} \right) \Phi' = 0 \]

\[ \frac{\omega}{k} = -\frac{\beta}{k^2 + \frac{f_0^2}{N^2}} \]

Mode

\[ e^{ikx + \omega t} \]

\[ i f_0 \beta + \left( k^2 + k^2 + \frac{f_0^2}{N^2} \right)i \omega = 0 \]

Wave. No instabilities!
\( c \) instead of \( w \)

\[
\omega = kc, \quad \frac{\omega}{k} = C \quad \times \text{phase velocity}
\]

\( \rightarrow e^{i k(x-c t)+i y+imz} \)

\[
C = -\frac{\beta}{k^2+\ell^2+\frac{\omega^2}{N^2}}m^2
\]

Mean wind \( \bar{u} \) in \( x \) direction

1) \( \frac{\partial \Phi}{\partial y} = -f \bar{u} \quad \Rightarrow \quad \Phi = -f \bar{u} y + \text{const} \)

\( \Rightarrow \frac{\partial^2 \Phi}{\partial y^2} + f \beta y + \frac{f^2}{N^2} \frac{\partial^2 \Phi}{\partial z^2} = f \beta y \quad \text{as before} \)

2) \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{v} \frac{\partial}{\partial y} \)

\( \Rightarrow \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \frac{u'}{\delta x} + \frac{v'}{\delta y} \uparrow \text{new term} \)

\( \therefore \text{Only change} \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \)

\( -ikc \rightarrow -ikc + ik \bar{u} \)

\( \therefore (c \rightarrow c - \bar{u}) \)

New dispersion relation

\[
C = \bar{u} - \frac{\beta}{k^2+\ell^2+\frac{\omega^2}{N^2}}m^2
\]

Propagation \((-x)\) direction relative to mean flow.
Phase + group velocities

\[ \omega = k \hat{u} - \frac{\beta k}{k^2 + k^2 + \frac{f_0^2}{N^2} \omega^2} = \omega_1 + \omega_2 \]

The direction is peculiar. First term \( \frac{d\omega_1}{dk} = \hat{u} = \text{mean flow drift} \)

Second term

Reflection from western boundary

Incident

Reflected

Frequency \( \omega \sim \frac{\Omega}{k} \sim \frac{\Omega L}{a} \sim \left( \frac{1}{a} \right) \Omega \) \text{ low frequency}

Speed \( \frac{\omega}{k} \sim \frac{\Omega L^2}{a} \sim \left( \frac{L}{a} \right)^2 \Omega a \) \text{ compare to } U

\[ \frac{\omega}{U} \sim \frac{\Omega L}{U a} \sim \frac{L/a}{R_0} \sim 1 \]
Stationary wave

Important application

\[ \begin{align*}
  \vec{u} & \rightarrow \\
  \text{surface with continents etc} \\
  k, \beta, \bar{u}, \rho, N, f_0 \text{ imposed by configuration, mean flow} \\
  \text{Stationary topography } \Rightarrow c = 0. \\
  m \text{ selected by dispersion relation} \\
  \frac{f_0^2}{N^2} m^2 = \frac{\beta}{\bar{u}} - (k^2 + \ell^2) \\
\end{align*} \]

Plot for various parameters on next page

\[ \left( \frac{f_0}{HN} \right)^2 \left( H m \right)^2 = \frac{\beta}{\bar{u}} - \frac{1}{\ell^2} \left( \frac{\bar{k}^2 + \bar{\ell}^2}{\ell^2} \right) \]

\[ \left( \frac{10^2}{10^4} \right) = 10^{-12} \text{ m}^{-2} \]
Eady problem in baroclinic instability

1. Scale the QG equations

\[
\begin{align*}
    x &= L \hat{x} \\
    y &= L \hat{y} \\
    p &= \frac{\rho_0}{\rho_0} \hat{p} \\
    t &= \frac{L}{U} t \\
    u &= U \hat{u} \\
    v &= V \hat{v} \\
    \Phi &= f_0 U L \hat{\Phi} \\
    T &= \frac{f_0 U L}{R} T
\end{align*}
\]

- Neglect variation of \( p \) as coefficient, for example:

\[
\frac{\partial \Phi}{\partial p} = -RT \implies \rho_0 \frac{\partial \Phi}{\partial \hat{p}} = -RT
\]

\[
\text{scaling} \quad \frac{\partial \hat{\Phi}}{\partial \hat{p}} = -\hat{T}
\]

- Switch to \( \hat{z} = 1 - \hat{\rho} \), use \( \hat{\omega} \) instead of \( \omega \), \( \hat{\omega} = \frac{D\hat{\Phi}}{D\hat{\rho}} \)

- Choose \( L = \frac{HN}{f_0} \), assume \( HN \) constant.

- Finally, drop hats!

\[
\begin{align*}
    u &= -\frac{\partial \Phi}{\partial y} \\
    v &= \frac{\partial \Phi}{\partial x} \\
    T &= \frac{\partial \hat{\Phi}}{\partial \hat{z}} \quad \left\{ \begin{array}{l}
        \frac{\partial u}{\partial \hat{z}} = -\frac{2T}{\hat{z}} \\
        \frac{\partial u}{\partial \hat{y}} = 0
    \end{array} \right. \\
    \frac{\partial \hat{\omega}}{\partial \hat{t}} &= 0, \quad \frac{\partial \hat{\omega}}{\partial \hat{y}} = 0
\end{align*}
\]

\[
\frac{D}{Dt} \left( \nabla^2 \Phi + \rho y + \frac{\partial^2 \Phi}{\partial z^2} \right) = 0
\]
2. Choose basic state + linearize

Instability of thermal wind is question

\[ T = T + T' \]
\[ u = \bar{u} + u' \]
\[ \frac{\partial \bar{u}}{\partial z} = -\frac{2 \bar{T}}{\partial y} \]

Simple is goal \( \Rightarrow \) \[ \bar{T} = -y \], \[ \bar{u} = z \]

\[ \bar{\Phi} = -yz \]
\[ \bar{v} = \bar{w} = 0 \]

Finally, neglect \( \beta \). Then \[ \frac{\partial}{\partial x} \bar{\Phi} + \beta y + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = 0 \]

\[ \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left( \bar{T}' + \frac{\partial^2 \Phi}{\partial z^2} \right) = 0 \]

\[ \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) T' + \bar{v}' \frac{\partial \bar{T}}{\partial y} + \bar{w}' = 0 \]

\[ \bar{T}' = \frac{\partial \bar{\Phi}'}{\partial z} \]
\[ \bar{u}' = \frac{\partial \bar{\Phi}'}{\partial x} \]

B.C. \( \bar{w}' = 0 \) at \( z = 0, z = 1 \)

Formulation complete.
3. Linearized equations in terms of $\Phi'$

$$\frac{\partial^2 \Phi'}{\partial t^2} + (\vec{u} + \vec{u}') \frac{\partial \Phi'}{\partial x} + (\vec{v} + \vec{v}') \frac{\partial \Phi'}{\partial y}$$

$\vec{u} = 2$, $\vec{v} = 0$

$$\nabla^2 \Phi' = 0 \quad (\phi' = -\Phi')$$

Using these, and recalling that $\nu' = \frac{\partial \Phi'}{\partial x}$,

$$\left( \frac{\partial^2}{\partial t^2} + 2 \frac{\partial}{\partial x} \right) \nabla^2 \Phi' = 0$$

$$\left( \frac{\partial^2}{\partial t^2} + 2 \frac{\partial}{\partial x} \right) \Phi' \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] = 0$$

at $z = 0, 1$

4. Solution.

Let $\Phi' = \Psi(z)e^{i(kx + ly - \omega t)}$

$$(\omega - k^2) (\frac{\partial^2}{\partial z^2} - k^2 \ell^2) \Psi = 0$$

$$\omega \frac{\partial \Psi}{\partial z} + k \Psi = 0 \quad \text{at} \quad z = 1$$

$$\omega \frac{\partial^2 \Psi}{\partial z^2} + k \Psi = 0 \quad \text{at} \quad z = 0$$

Let $\mu^2 = k^2 + \ell^2$. Then DE is solved by

$$\Psi = C \sinh \mu z + D \cosh \mu z$$

Substitute, find dispersion relation for $\omega$ from B.C.

$$\omega = \frac{k}{2} \left[ 1 \pm \left( 1 + \frac{4}{\mu^2} - \frac{4}{\mu \ell \mu} \right)^{1/2} \right]$$

Also, the ratio of $C$ to $D$ is easily expressed from $z = 0$ B.C.

$$C \omega \mu + D \ell = 0 \Rightarrow \Psi = C \left[ \sinh \mu z - \frac{\omega}{k} \cosh \mu z \right]$$
5. Growth rate and phase speed

a. For $\mu >> 1$, $(1 + \frac{4}{\mu} - \frac{4}{\mu^2})^2 \rightarrow (1)^2$ \(\omega\) is real

b. For $\mu << 1$, \[ \text{tangly} = \mu = \frac{\mu^3}{3} + \frac{2}{15} \mu^5 + \ldots \]

\[ \frac{4}{\mu} = \frac{1}{\mu^2} \left( 1 - \frac{\mu^2}{1} - \frac{\mu^4}{5} + \ldots \right) \approx \frac{4}{\mu} \left[ 1 + \left( \frac{\mu^2}{3} - \frac{1}{5} \mu^4 + \frac{1}{3} \ldots \right) \right] \]

\[ 1 - \frac{1}{3\times 5x} \]

\[ \frac{4}{\mu} \left( 1 + \frac{\mu^2}{3} - \frac{\mu^4}{45} \ldots \right) \]

\[ \frac{4}{\mu} \left( 1 + \frac{1}{3} - \frac{\mu^2}{3} + \frac{\mu^4}{45} \ldots \right) = (\frac{4}{\mu}) \left( 1 + \frac{1}{3} - \frac{\mu^2}{3} + \frac{\mu^4}{45} \ldots \right) \]

Thus $\omega$ has imaginary part for $\mu << 1$.

c. Since $(\frac{4}{\mu})^2$ has decreasing imaginary value as $\mu$ increases, away from zero, we conclude that for fixed $k$ the max growth rate will occur for minium $\mu$. \[ \mu^2 = \frac{l^2 + \lambda^2}{\lambda^2} \Rightarrow l = 0 \text{ gives max growth rate.} \]

d. Note that the growing modes have $\frac{\omega}{k} = \frac{1}{2}$. They drift with the average velocity of $\bar{u}$.

e. Numerically:

\[ \omega = \frac{4k}{\mu} \left( 1 + \frac{1}{3} - \frac{\mu^2}{3} + \frac{\mu^4}{45} \ldots \right) \]

\[ \omega \rightarrow \frac{4k}{\mu} \left( 1 + \frac{1}{3} - \frac{\mu^2}{3} + \frac{\mu^4}{45} \ldots \right) \]

1.7 2.5 3

f. Dimensionally

\[ \omega_{\text{max}} = \frac{U \omega}{l} = \frac{10^3 (0.31)}{10^8} \approx 1 \frac{1}{3 \times 10^5} \approx \frac{1}{3 \text{ day}} \]
6. Structure of solutions

\[ \Phi = \Psi(z)e^{i(kx+ly-wt)} \quad \Psi = C \left[ \sinh(kz) - \frac{w}{k} \cosh(kz) \right] \]

Consider \( l = 0 \), \( k \ll 1 \) \( \rightarrow \) \[ C \left[ \sinh(kz) - w\cosh(kz) \right] \]

\[ \omega = \frac{k}{2} + \frac{k}{2} \frac{1}{\sqrt{3}} \left( 1 - \frac{2}{5}k^2 \right) \]

Expand, find

\[ \Phi' = kC \left[ z - \frac{1}{2} + \frac{i}{2\sqrt{3}} \right] e^{i(kx-\omega t)} \]

\[ T = \frac{\partial \Phi}{\partial x} = kC e^{i(kx-\omega t)} \]

\[ \nu = \frac{\partial \Phi}{\partial x} = ik \Phi \]

\[ w = \frac{1}{i} kC \epsilon (1-z) \left[ \frac{1}{3} (z-\frac{1}{2}) + \frac{i}{2\sqrt{3}} \right] \]

Example: \( \Phi = |\Phi| e^{i(kx-\omega t + \phi(z))} \), \( |\Phi|^2 = (z-\frac{1}{2})^2 + \frac{1}{4}, \tan \Phi = -\frac{1}{2\sqrt{3}} \)

At \( t = 0 \), phase \( = 0 \) is \( kx = -\Phi \)

---

a. Rain before low \( \Phi \)
b. Max vorticity at \( \Phi_{\min} \) \( (\gamma = \nabla^2 \Phi) \), convergence before \( \Phi_{\min} \)
(Convergence cancels advective tendency to straighten up \( \Phi \) lines)
c. \( T_{\max} \) before \( T_{\min} \) brings up warm air from south.
7. Energetics

Vorticity equation

\[
\frac{\partial}{\partial t}(w_x - w_y) - \frac{\partial \Phi}{\partial z} = 0
\]

\[
\frac{\partial}{\partial t} (\Phi_x + \Phi_y) = i(k_x - \omega)(-k^2 \Phi)
\]

\[
\Rightarrow \quad -i(k_x - \omega)k^2 \Phi^* = w_x^* \Phi^*
\]

\[
i(k_x - \omega)k^2 \Phi = w_z^* \Phi
\]

Add

\[
-ik^2(\omega - k_x) \Phi \Phi^* = w_z^* \Phi^* + w_z^* \Phi
\]

Recall that \( \frac{w_z^* \Phi^* + w_z^* \Phi}{4} = \text{Re}\{w_z \} \text{Re}\{\Phi\} \).

But first integrate in \( z \):

\[
-2k^2 \int \limits_0^l [\Phi \Phi^*] dz = \int \limits_0^l [w_x^* + w_z^* \Phi] dz - \int \limits_0^l (w_x^* + w_z^* \Phi) dz
\]

\[
= -\int \limits_0^l (w_x^* + w_z^* \Phi) dz
\]

\[
= \frac{1}{2} \ln \{w_z\} \int \limits_0^l \Phi \Phi^* dz = \int \limits_0^l \text{Re}\{w_x\} \text{Re}\{\Phi\} dz
\]

\[
= \frac{\partial}{\partial t} \langle k^2 \rangle = \langle \omega_T \rangle = \text{PE release.}
\]

Comment: Energy release is due to vortical heat flux.

Although not explained in detail, identified \( \cdots \)

8. Horizontal heat flux

\[
\int \limits_0^l \Phi \Phi^* dz = \frac{1}{4} \int \limits_0^l [(i \Phi_x \Phi_x^* + i \Phi_z \Phi_z^*)] dz
\]

\[
\int \limits_0^l \Phi \Phi^* dz = \frac{k^2 C^2}{2} \ln \{w_z\}
\]

\[
\Phi_x = \int \left[ \sin(k_x x - \omega t) \cos(k_z z) \right] e^{-i(k_x - \omega)t} dz
\]
The vertical heat flux can be written in an analogous manner using
\[ \int_0^1 \phi \phi' \, dz = \frac{e^2}{2} \left( \frac{k}{\tan k} - 1 \right). \] (from solution)

We find
\[ \int_0^1 \bar{w} \bar{T} \, dz = \frac{k^2 c^2}{4} \left( \frac{k}{\tan k} - 1 \right) \ln \{w\} \]

Also, for reference
\[ \int_0^1 \frac{\bar{w}^2}{\bar{T}} \, dz = \frac{e k^2}{2} \left( \frac{k}{\tan k} - 1 \right). \]

The angle of the heat flux vector is
\[ \frac{\int_0^1 \bar{w} \bar{T} \, dz}{\int_0^1 \bar{w}^2 \, dz} = \frac{1}{2} \left( \frac{k}{\tan k} - 1 \right) \quad \text{most rapidly growing wave, } k \approx 1.7 \]

9. What is the slope of isentropes?

Dimensionally
\[ \frac{\theta y}{\ell_p} = \frac{T_y}{\ell_p} - \frac{T_y g}{\ell_p} = \frac{U_f L T}{\ell_p} \]

\[ = \frac{U_f}{N^2 H} = \frac{U_f}{L F} \frac{L f^2}{H N} \]

Nondimensionally
\[ \text{Slope} \times \text{(slope)}_{\text{dim}} = \frac{U_f}{L F} \frac{L f^2}{H N} = R_0 \]

\[ \frac{\theta y}{\ell_p} = \frac{U_f}{L F} \frac{L f^2}{H N} \]
Thus the comparison of slopes shows:

(Remember that $w$ is really $w^0$, of order $Ro$)

Heat flux slope $= \frac{1}{2} \left( \frac{k}{\text{adv} k} - 1 \right) (Ro) \sim 0.37 \, Ro$

Isentrope slope $= Ro$

$\text{adv} \approx \text{heat flux}$

Warming regions $\rightarrow$

Cooling regions $\rightarrow$

Upward, yet parcels are moved to locations where they are warmer than the environment in spite of stable stratification.

This instability utilizes buoyancy and overcomes the Coriolis constraint against direct overturning.