Remote Sensing:
Inverse Problems in Radiative Transfer

- Measurements of electromagnetic radiation are performed external to the body of interest.
- Parameters of interest are functions of the measurements.
- For planetary atmospheres, retrieval of parameters (for example a vertical profile) usually requires inversion of a radiative transfer equation.
Voyager IRIS Thermal Emission Spectra
Radiative Transfer Equation:

\[ \mu \frac{dI_v}{d\tau_v} = I_v - J_v \]

Optical depth:

\[ \tau_v = \int z \sum_i \sigma_i n_i d\tau \]

Formal solution:

\[ I_v(\mu) = \int_0^\infty J_v(\tau) e^{-\tau/\mu} \frac{d\tau}{\mu} \]

\( J_v = \text{source function} \)
\( \sigma_i = \text{cross section} \)
\( n_i = \text{gas number density} \)
Finite band-pass spectral instrument:

Atmospheric transmittance:

\[ Tr \equiv \int_{\Delta \nu} d\nu \phi(\nu) e^{-\tau/\mu} \]

Consider thermal emission only.

Assume local thermodynamic equilibrium (LTE).

\[ J_\nu = B_\nu(T) \]

\[ B_\nu(T) = \text{Planck Function} \]

\[ T = \text{Temperature} \]
\[ I_v(\mu) = \int_0^\infty B_v[T(z)] \frac{\partial Tr_v(\mu)}{\partial z} \, dz \]

Write in finite difference form

\[ I_i \equiv \sum_{j=1}^n B(T_j) \left[ \frac{\partial Tr}{\partial z} \right]_{ij} w_j \quad i = 1, m \]

Basic information retrieval problem:

Given measurements of radiance \( I_i \) as a function of frequency (wavelength) and/or geometry, infer information on atmospheric parameters such as temperature, composition, aerosols, etc.
Observational Geometries

“Limb”

“Nadir”

Limb: geometric + spectral
Nadir: spectral only
Forward modeling

Measurement

(A) ATMOS. STRUCTURE AND COMPOSITION → RADIATIVE TRANSFER PROCESS (LOW PASS FILTER) → MEASUREMENT (NOISE, SYSTEMATIC ERROR) → $I_{\nu}(\mu)$

(B) MODEL 1
   MODEL 2
   ...
   MODEL n → RADIATIVE TRANSFER EQUATION → $\{i_1^\nu, i_2^\nu, \ldots, i_n^\nu\} → I_{\nu} →$ SELECT "BEST FIT" MODEL

Inversion

(C) ESTIMATE OF ATMOS. PROFILES → INVERSION ALGORITHM → RADIATIVE TRANSFER INTEGRAL EQUATION → $I_{\nu}(\mu)$

CONSTRAINTS, FILTERING, ETC.
Information Content of Spectral Measurements

Let \( \mathbf{x} \) be a vector of atmospheric parameters

Linearize about a reference value \( \mathbf{x}^o \):

\[
I_i - I_i^o = \sum_j \frac{\partial I_i}{\partial x_j} (x_j - x_j^o)
\]

\[
g_i = I_i - I_i^o
\]

\[
K_{ij} = \frac{\partial I_i}{\partial x_j} 
\quad i = 1, m 
\quad j = 1, n
\]

\[
f_j = (x_j - x_j^o)
\]

\[
g = Kf
\]
JUPITER WEIGHTING FUNCTIONS

\[
\begin{align*}
\text{PRESSURE (mb)} & \quad \text{d}B & \quad \text{d}T \\
& \quad \text{d}\theta & \quad \text{d}n \text{p}
\end{align*}
\]

\[
\begin{align*}
1304.3 \text{ cm}^{-1} & \\
1301.1 & \\
1297.3 & \\
602 & \\
340 & \\
310 & \\
280 & \\
\end{align*}
\]
If
\[ g = K \cdot f_1 \]
and there exists an \( f_2 \) such that
\[ K \cdot f_2 = 0 \]
then \( f = f_1 + f_2 \) is a solution
\[ \Rightarrow \text{Non-uniqueness} \]

Since \( g \) contains measurement error,
actual condition on \( f_2 \) is
\[ K \cdot f_2 \leq \text{noise level} \]
$\lim_{K \to \infty} \int_0^{z_t} K(v, z) \sin kz \, dz = 0$
Define signal variance

\[ s^2 = \sum_i g_i^2 = g^T g \]

\[ = \frac{f^T_k K^T_k f_k}{2} \]

Eigenvalues

\[ K^T_k \Psi_k = \lambda_k \Psi_k \rightarrow \Psi_k^T \Psi_k' = s_k k' \]

Expand \( f \)

\[ f = \sum_k \hat{f}_k \psi_k \]

Then

\[ s^2 = \sum_k \lambda_k \hat{f}_k^2 \]
Singular Value Decomposition

\[
\begin{align*}
\text{PRESSURE (mb)} & & \text{PRESSURE (mb)} & & \text{PRESSURE (mb)} & & \text{PRESSURE (mb)} \\
k = 1 & & k = 2 & & k = 3 & & k = 4 \\
\lambda = 48.6 & & \lambda = 13.1 & & \lambda = 3.97 & & \lambda = 1.40 \\
\end{align*}
\]

RELATIVE AMPLITUDE

\[
\begin{align*}
\text{PRESSURE (mb)} & & \text{PRESSURE (mb)} & & \text{PRESSURE (mb)} & & \text{PRESSURE (mb)} \\
k = 5 & & k = 6 & & k = 7 & & k = 8 \\
\lambda = 0.37 & & \lambda = 0.040 & & \lambda = 0.0054 & & \lambda = 0.00049 \\
\end{align*}
\]

RELATIVE AMPLITUDE
Noise Limitation

\[ \lambda_{p_k} \hat{f}_k \]

\[ \text{Noise Variance} \]

\[ 10^6 \]

\[ 10^5 \]

\[ 10^4 \]

\[ 10^3 \]

\[ 10^2 \]

\[ 10^1 \]

\[ 10^0 \]

\[ 10^{-1} \]

\[ 10^{-2} \]

\[ 10^{-3} \]

\[ k \]

1 2 3 4 5 6 7 8
Methods of solving "ill-posed" inverse RT problem

Constrained inversion

\[ Q = \left( \hat{q} - K \hat{f} \right)^T E^{-1} \left( \hat{q} - K \hat{f} \right) + \gamma \hat{f}^T S^{-1} \hat{f} \]

\[ \chi^2 \]

\( \gamma \) is relative weight of fit to data vs. constraint

\( E \) = measurement error covariance

\( S \) = depends on specific constraint chosen

Minimizing \( Q \):

\[ \hat{f} = \left( K^T E^{-1} K + \gamma S^{-1} \right)^{-1} K^T E^{-1} \hat{q} \]
Important matrix identity! [Can you prove it?]

\[ C = (K^T E^{-1} K + \gamma E) \]  
\[ \times (K^T E^{-1})^T = \Sigma K^T (K \Sigma K^T + \gamma E) \]

Propagation of measurement error:

\[ \hat{E}_f = C E \hat{C} \]

Relation of \( \hat{f} \) to \( f \):

\[ \hat{f} = \Sigma K^T (K \Sigma K^T + \gamma E)^{-1} K f \]

A "averaging kernel"
Choice of constraints

- Minimize $\hat{\mathbf{f}}^T \mathbf{f}$ $\rightarrow \mathbf{S} = \mathbf{1}$ ("zero order regularization")
- Minimize curvature (2nd differences)
- Use a priori statistics of $\mathbf{f}$ $\rightarrow \mathbf{S} = \text{covariance}
- Let $S_{ij} = \exp\left\{-(z_i - z_j)^2 / 2\sigma^2\right\}$

Choice of $\mathbf{S}$

Usually determined empirically

Frequent choice is value for which

$\chi^2 \approx m$
Why does it work?

Consider "zero order regularization":

\[ \hat{f} = (K^T K + \epsilon I)^{-1} K^T q \]

Expand in eigenvectors of \( K^T K \) as before:

\[ \hat{f} = \sum_k \hat{a}_k \psi_k \]

\[ \hat{a}_k (\psi_k (\lambda_k + \epsilon) = \psi_k^T K^T q \]

For \( \epsilon = 0 \), least squares solution:

\[ \hat{a}_k (\psi_k) = \frac{\lambda_k}{\lambda_k + \epsilon} \hat{a}_k (\psi_k = 0) \]

Filtered least squares with filter \( F_k \).
\[ F_k = \frac{\lambda_k}{\lambda_k + \sigma} \]
Mars Global Surveyor Thermal Emission Spectrometer

Limb & Nadir Weight Functions

15-μm CO₂ band

pressure (mb)

normalized weight function
Mars Global Surveyor

TES Limb+Nadir Temperatures (K), \( L_s = 270 \)

Gradient Wind (m/s), \( L_s = 270 \)

Plate 7
Titan: Temperature Inversion Kernels

Wavenumber [cm$^{-1}$]

Pressure [mbar]

[Color bar with pressure and wavenumber scales]
T4: Mid-IR Limb Temperature Map

- Request: CIRS_00561_MIRLMBMAP002_PRIME
- Target: Titan
- Observation/Footprint Time: (2005 APR 01) 2005-091T00:41:31.37

- Target RA Dec: 353.76 - 3.24
- Spacecraft - Target Distance: 94040 km
- Spacecraft Velocity: 164865 km/s
- Sub SC Lat Lon: -1.329, 211.600
- Sub Solar Lat Lon: -21.796, 84.273
- Target Phase Angle: 123.72

Created by ODD (MSS D10.3.1-LINUX) on: Wed Oct 20 16:29:43 2004
CIRS Cassini Limb Kernels

Functional Derivative; Tangent Pt. Lat. = 3N

1300 cm\(^{-1}\) CH\(_4\) band
9 Tangent Heights
8 Spectral Points
Cassini CIRS Titan Limb Spectrum

Tangent Height = 200 km
Latitude = 45° N
Warm winter polar Stratopause Why?

Cassini CIRS Titan Temperature Retrievals
Cassini CIRS Jupiter

ATMOS02A 65N - 75N FP3

Brightness Temperature (K)

Wavenumber (cm⁻¹)

600 800 1000

ATMOS02A 65N - 75N FP4

Brightness Temperature (K)

Wavenumber (cm⁻¹)

1000 1100 1200 1300 1400 1500
Cassini CIRS Jupiter Temperature Retrievals

Zonal Mean Temperatures (K)

Pressure (mbar)

Latitude

Tropopause

Zonal Wind (m/s)

Pressure (mbar)

Latitude

Stratospheric Equatorial jet
Cassini CIRS Saturn Spectrum

- $\text{H}_2\text{H}_2$
- $\text{C}_2\text{H}_2$
- $\text{C}_2\text{H}_6$
- $\text{NH}_3$

Brightness Temperature (K) vs. Wavenumber (cm$^{-1}$)

Troposphere retrieval
Cassini CIRS Saturn Spectrum FP4

\[
\text{CIRS\_000SA\_TEMPSIT001\_PRIME}
\]

\[
\begin{align*}
\text{CH}_4 \\
\text{Used for stratosphere retrieval}
\end{align*}
\]
Cassini CIRS Temperature Retrievals

**Saturn Temperature (K)**

- **Warm Polar Stratosphere**
- **Tropopause**

- **Tropospheric jet decays**
- **Stratospheric jet**

**Zonal Wind (m/s)**
References

[Excellent introduction to inverse problems.]

[Good treatment of a broad range of inverse problems.]

[Exhaustive treatment of the subject, primarily from a Bayesian viewpoint.]