1. For the first part of the problem we seek a relation between $T$ and $z$. A good strategy is to work toward eliminating $\rho$ between the hydrostatic and state equations

$$\frac{dp}{dz} = -\rho g, \quad p = \rho RT. \tag{1}$$

If successful this will lead to a relation between $p$ and $T$. This will involve $z$ as an independent variable. Then Part 2 of the strategy would be to use the no-heat-added condition

$$c_v \, dT + pd \left( \frac{1}{\rho} \right) = 0, \quad \text{[Problem Set(3)]}$$

to find another relation between $T$ and $p$. At that point it should be possible to solve for either $T(z)$ or $p(z)$ by eliminating one or the other.

Part 1 of the strategy gives, as in lecture,

$$\frac{1}{p} \frac{dp}{dz} = -\frac{g}{RT}. \tag{2}$$

Part 2 starts with the heat equation given in the problem set,

$$c_v dT + pd \left( \frac{1}{\rho} \right) \approx 0. \tag{3}$$

It is clear from the fact that $1/\rho$ is volume per unit mass that this is a specific energy statement about the energy per unit mass. The units of $c_v$ must therefore be energy (temperature, mass)$^{-1}$. The units of $R$ in Eq. (1) of the problem set must be the same. Thus $R = R^{(\text{universal})}/\bar{m}$.

Next, eliminate pressure and density from the heat equation (3) by use of (1) and (2). This gives

$$\frac{dT}{dz} = -\frac{g}{c_p}, \quad \text{[Problem Set(4)]}$$

after use of the ideal gas relation $c_p = c_v + R$.

The heat capacity at constant volume of an ideal gas is $k_B/2$ for each fully excited degree of freedom. For a cool diatomic gas, (for example, the atmospheres of Earth, Mars and Jupiter) there are three translational degrees of freedom and two rotational degrees of freedom. Thus the specific heat (per mass) at constant volume is $c_v = (5/2)(k_B/\bar{m}) = (5/2)R$. Finally, use of $c_p = c_v + R$ gives $c_p/R = 7/2$.

For the numerical section (d), I got $dT/dz = -9.76 \, \text{K km}^{-1}$, based on an assumed $\bar{\mu} = 29 \, \text{gm mole}^{-1}$. This implies that an elevation difference of 0.1 km would produce adiabatic cooling of about 1 K (or 9/5 F).
2. We need to solve for \( p(z) \). Start with Eq. (2) from the previous problem. Rearrange to get all the \( p \)'s on the left hand side.

\[
\frac{dp}{p} = \frac{g}{RT(z)} \, dz.
\]

We are given \( T(z) = T_0 - g z / c_p \). Eliminate \( T \) and rearrange.

\[
\frac{dp}{p} = -\frac{c_p}{R} \frac{dz}{D - z}, \quad \text{where} \quad D = \frac{c_p T_0}{g} = \frac{c_p}{R} H_0. \tag{4}
\]

Here \( H_0 \) is the scale height at the base temperature \( T_0 \). We defined \( D \) for convenience. It turns out to be the finite depth of this strange atmosphere. We discover this when we integrate (4) and get

\[
p = p_0 \left( 1 - \frac{z}{D} \right)^{c_p/R},
\]

where the surface pressure \( p_0 \) is arbitrary (a constant of integration). Using the equation of state, the density is found to be

\[
\rho = \rho_0 \left( 1 - \frac{z}{D} \right)^{(c_p - R)/R}.
\]

Plots are attached. It is a bit mysterious. We were all taught as children that an exponential is more powerful than any power law. But here the exponential isothermal atmosphere does not drop off as fast as the algebraic power law atmosphere.

3. In an isothermal atmosphere, pressure is given by

\[
p = p_0 e^{-z/H},
\]

where \( H = RT/g \). Use the equation of state \( p = \rho RT \), eliminate the pressure, and solve for the density:

\[
\rho = \frac{p_0}{RT} e^{z/H}.
\]

Assume that the surface pressure \( p_0 \) is fixed. Form the ratio of densities for atmospheres with two different temperatures \( T_1 \) and \( T_2 \), and one finds

\[
\frac{\rho_2}{\rho_1} = \frac{T_1}{T_2} \exp \left[ \left( 1 - \frac{T_1}{T_2} \right) \frac{z}{H_1} \right].
\]

It is clear that a large multiplying factor arises when \( z/H_1 \gg 1 \). For \( T_1 = 150 \) K, \( T_2 = 165 \) K, \( z = 120 \) km, I found \( \rho_2/\rho_1 \sim 3.7. \)
4. The plot shows wavenumber units. We know from classroom discussion that wavenumber is frequency divided by a constant (the speed of light). Therefore start with the Planck function in frequency units,

\[ B_\nu = \frac{2h\nu^3/c^2}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}, \]

where

\[ c = 3. \times 10^8 \text{ m s}^{-1}, \]
\[ h = 6.626 \times 10^{-34} \text{ J s}, \]
\[ k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}. \]

Now write \( B_\bar{\nu} \):

\[ B_\bar{\nu} = \left| \frac{d\nu}{d\bar{\nu}} \right| B_\nu \rightarrow cB_\nu. \]

Solve for \( T \) in terms of \( \bar{\nu} \), call the result \( T_b \), and replace \( B_\bar{\nu} \) with \( I_\bar{\nu} \), and one finds

\[ T_b = \frac{A_2 \bar{\nu}}{\ln \left( 1 + A_1 \bar{\nu}^3 \right)}. \]

For convenience we have defined the two constants

\[ A_1 = 2hc^2 = 1.193 \times 10^{-16} \text{ W m}^{-2} (\text{m}^{-1})^{-1} \text{ m}^3, \]
\[ A_2 = \frac{hc}{k_B} = 0.0144 \text{ K m}. \]

Changing to the mixed units of the Cassini spectrum puts a factor of 100 in front of both \( A_1 \) and \( A_2 \), due to the cm \( \rightarrow \) m rescaling. Substituting numbers for the spectral points indicated gives \( T_b \)'s of about 135 and 155 K.

5. Since there is no vertical flux of any constituent,

\[ D \left( \frac{\partial n_i}{\partial z} + \frac{n_i}{H_i} \right) + K \left( \frac{\partial n_i}{\partial z} + \frac{n_i}{H} \right) = 0, \]

where \( n_i \) is the minor constituent number density, \( H_i \) is its scale height, and \( H \) is the mean scale height, dominated by nitrogen and assumed constant in height. \( D \) and \( K \) are the molecular and turbulent diffusivities respectively. We are told to assume that \( K \) is constant and that \( D = K \exp(z/H) \). This implies that the origin of our height coordinate is chosen to be where \( D = K \). Simplifying and rearranging gives
\[
\frac{dn_i}{n_i} = -\frac{(H/H_i) \exp(z/H) + 1 \, dz}{\exp(z/H) + 1 \, H}.
\]

This gives

\[
n_i = C \frac{1 + \exp(-z/H)}{[1 + \exp(z/H)]^{H/H_i}},
\]

where \( C \) is a constant. Since the mean scale height \( H \) is constant, \( n \) (no subscript) is proportional to \( \exp(-z/H) \). Thus the concentration of a minor molecule, given by the ratio \( n_i/n \), is

\[
n_i = n_i(-\infty) \frac{1 + \exp(z/H)}{n(-\infty) [1 + \exp(z/H)]^{H/H_i}} = n_i(-\infty) \left[1 + \exp(z/H)\right]^{1-H/H_i}.
\]

A plot for helium is attached.

6. The “solar constants” must be reduced by the square of the distances. This gives

\[
S_M = 593 \, \text{W m}^{-2} \quad \text{and} \quad S_J = 50.7 \, \text{W m}^{-2}.
\]

For Mars, use the global heat balance discussed in lecture.

\[
\sigma T^4_{eM} = \frac{(1 - A)}{4} S_M.
\]

This gives \( T_{eM} = 216 \, \text{K} \).

For Jupiter, averaging the absorbed insolation over the total planetary area brings in a divisor of 4 and gives 8.13 W m\(^{-2}\). The average emitted flux is \( \sigma T^4_{eJ} = 13.6 \, \text{W m}^{-2} \). The difference is a flux from the planetary interior of 5.45 W m\(^{-2}\). Its origin is primarily the heat of formation. Smaller contributions are from gravitational contraction and, possibly, gravitational separation of materials of different densities.