1. You need to assume that $z$ denotes the geometric height, as we have been using it in class. The kernel $K$ is then

$$K = \frac{dT}{dz}. \quad (1)$$

The transmissivity is given by

$$T = \exp(-\tau). \quad (2)$$

If the optical depth is due to absorption by some atmospheric gas that is well mixed and has an absorption cross-section $\sigma$ independent of pressure $p$, then

$$\tau = \int_{z}^{\infty} \sigma n \, dz,$$

where $n$ is the gas number density. But from hydrostatics, the pressure is related to the vertical integral of the mass density $mn$:

$$p = \int_{z}^{\infty} mng \, dz.$$

Combining with the previous equation gives

$$\tau = \frac{\sigma p}{mg}.$$

Here $\sigma$ and $g$ are assumed constant. Thus $\tau$ is simply proportional to the pressure, and we can write

$$\tau = \tau_0 \frac{p}{p_0},$$

where subscript 0 denotes the base of the atmosphere. Expressing $K$ in terms of $p$ by using (1) and (2) gives

$$K = \frac{1}{H} \frac{\tau_0 p_{/p_0}}{p_0} \exp \left( -\tau_0 \frac{p}{p_0} \right).$$

If $\tau_0$ is greater than unity, this function decays toward the top of the atmosphere because of the factor of $p$, and it decays toward them bottom because of the exponential. Therefore it peaks somewhere, and that will be the height of maximum information at that particular wavelength. See Figure on next page.
Weighting functions for four different choices of optical depth.
2. This is about your old friend the optical depth, 

\[ \tau = \sigma n s, \]

where \( s \) is the distance. When \( \tau = 1 \), the radiance is reduced by a factor of \( e^{-1} \). This is probably the limit on “good imaging”. Solving for \( s \) and plugging in numbers gives \( s = 6.4 \) km.

3. Evaluate the column mass, \( m \), for each atmosphere (kg m\(^{-2}\)), and then ask how deep \( (D) \) a layer of material with the density of water, \( \rho_w = 1000 \) kg m\(^{-3}\) would have the same column mass. The easiest way to evaluate the column mass is to use hydrostatics: \( p_0 = mg \). This gives \( D = p_0/(\rho_w g) \). For Venus, Earth, Mars and Titan a table in our lecture notes gives \( g = 8.9, 9.8, 3.7 \) and \( 1.4 \) m s\(^{-2}\). Surface pressures are 90., 1., 0.006 and 1.6 bars. The precipitable meters come out to be 1000., 10., 0.16 and 114.

4. To get to Equation (3) is straightforward under the assumption that \( L \) is constant. In fact, \( L \) is a function of \( T \), although not a strong one, and therefore (3) is really the leading term in an approximation valid for \( T \) near \( T_0 \). The story is similar for the scale height expression. To be rigorous, expand the exponent in (3) in a Taylor series and keep only the leading term in powers of \( z \).

With the scale height formulation, the height integral of the exponential is easy and the column abundance (mass area\(^{-1}\)) is given by

\[ m = \int_0^{\infty} \rho \, dz = \rho v_0 H_v = \frac{p v_0}{R_v T_0} H_v. \]

The water vapor scale height comes out to be about 3 km. The mean atmospheric scale height on Mars is about 10 km. The column abundance for saturated water vapor should come out to be about 0.05 kg m\(^{-2}\) if you use a temperature of about 215 K. This is about 50 precipitable microns.

5. This parallels the Mars case. I used 288 K as a mean surface temperature, and I got 2.1 km for the \( H_2O \) scale height, 20 mb for the surface saturation vapor pressure, and 3.3 cm for the precipitable water. The mean atmospheric scale height is about 8 km, so water vapor is strongly trapped near the surface.

6. The heat loss by radiation is balanced by latent heat released during ice formation:

\[ \sigma T^4 \Delta t = Lm, \]

where \( m \) is mass per unit area. The depth is then \( m/\rho \). I used half a Mars orbital period for \( \Delta t \) (667 days / 2) and got 2.6 meters for the ice depth. Upon comparing this with the column abundance obtained in Problem 3, you will see that major shifts in atmospheric mass are implied.