1. The thermal wind in water.

The Gulf Stream leaves the coast near Cape Hatteras (about 36° N) and meanders into the open ocean. Current meters show that the flow speed is about 2 m s⁻¹ and that the stream extends through the top 1000 m of the ocean. Assume a linear profile with speed equal to zero at 1000 m depth. The stream is typically 50 km wide. Derive the thermal wind equation for a simple unidirectional current in water, using x as the direction along the current and y as the direction across the current. Use the equation of state

\[ \rho = \rho_0 [1 - \alpha (T - T_0)] \]

where \( \rho_0 = 1000 \text{ kg m}^{-3} \), and \( \alpha = 2 \times 10^{-4} \text{ K}^{-1} \). Use your thermal wind equation to estimate the temperature difference horizontally across the current. Observed contrasts are about 3 K.

2. The Venus atmosphere.

The atmosphere of Venus at a height of about 8 pressure scale heights (pressure about 30 millibars, surface pressure about 90 bars) is observed to rotate with a period of about 4 days. The solid planet has a radius of about 6000 km and rotates with a 244 day period in the same direction. Assume that the atmosphere rotates as a rigid spherical shell at each pressure level, and that the rotation rate varies linearly with \( \hat{z} = \ln(p_0/p) \). The rotation of the atmosphere is the same as that of the solid surface at \( \hat{z} = 0 \). Note that these statements completely specify \( u(\phi, \hat{z}) \), where \( \phi \) is latitude and \( u \) is zonal velocity.

We wish to evaluate the latitudinal temperature variation. The atmosphere is 95% CO₂, 5% N₂. Find the cyclotrophic balance equation for purely zonal motion on a sphere. Centrifugal as well as Coriolis accelerations balance against the pressure gradient. You will want to use the “log p” form of the latitudinal force balance equation. Start with the vector form of \( F = ma \) for a fluid, assume that the velocity is purely zonal and axisymmetric, and derive the appropriate latitudinal force balance equation.

Calculate the temperature variation with latitude in the Venus atmosphere. In particular, find the pole to equator temperature difference at 8 scale heights elevation. This exercise is useful because we have data on wind velocities beneath the clouds but little on horizontal temperature contrasts.

3. Hurricane

Use a local cylindrical coordinate system centered on the hurricane’s axis of symmetry, with \( r, \lambda, z \) and \( u, v, w \) denoting radius, azimuth, height and the corresponding velocities. Write down the two balance equations for the swirling system. You can probably do this by inspecting the Venus balance equations and putting certain sin’s or cosine’s equal to unity. Then eliminate pressure and find the thermal wind equation for the system.

A typical mature hurricane has a cyclonic swirl velocity just above the boundary layer that increases inward toward the eye wall. Assume that the increase is in proportion to \( 1/r \) (see
footnote *). Presumably the increase comes about because of inward drift that tends to conserve angular momentum. The velocity decreases with height, and at the tropopause, \( \hat{z} = \hat{z}_t \), the swirl is negligibly small (or even slightly anticyclonic). Assume

\[
v(r, \hat{z}_t) = 0, \quad v(r, \hat{0}) = \frac{f_0 r_e^2}{r},
\]

where \( f_0 \) is the local Coriolis parameter and \( r_e \) is the radius to the outer edge of the whole system, usually about 500 km.

The tropopause height at tropical latitudes is typically \( \hat{z}_t = 1.5 \). A hurricane has a warm core, with vertically averaged temperature near the core typically 5 K above that at the edge of the system. Presumably the elevated temperature comes from latent heat release due to ascent and precipitation near the core. There is usually a well defined core, or eye, with radius \( r_c \), where the swirl reaches a maximum speed.

a) Assume simple azimuthal motion with cylindrical symmetry, so that \( v(r, \hat{z}) \) is the dominant velocity component. Derive the thermal wind equation that relates \( \partial T / \partial r \) to \( \partial v \partial \hat{z} \).

\textit{Hint:} You can get the momentum equation from our discussion of gradient balance. Just be careful about signs.

b) Integrate vertically from \( \hat{z} = 0 \) to \( \hat{z} = \hat{z}_t \). Define a vertically averaged temperature

\[
\langle T \rangle = \frac{1}{\hat{z}_t} \int_0^{\hat{z}_t} T \, d\hat{z}.
\]

Express the vertical integral of your thermal wind equation in terms of \( \langle T \rangle(r) \).

c) Integrate horizontally from \( r = r_c \) to \( r = r_e \). Assume \( r_c / r_e \ll 1 \) and keep only the leading term involving this parameter. Then find an expression for the maximum flow speed at the eye wall, and show that it is independent of \( r_e \) and \( f_0 \), but only depends on thermodynamic quantities. Make a numerical estimate of the maximum velocity, assuming \( \langle T \rangle(r_c) - \langle T \rangle(r_e) = 5 \) K.

d) Find an expression for the core radius, showing that the core becomes smaller if the temperature contrast increases. Make a numerical estimate of the core size for a storm at 15 degrees latitude with properties as mentioned above.

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* Anthes, R. A., 1982, *Tropical Cyclones*, American Meteorological Society, Boston, 208 pp., gives an exponent of \( x \sim 0.6 \) or 0.7 for \( 1/r^x \) based on matches with real hurricane examples.