Ideal Photon Detectors

Astronomy 6525

Lecture 4

Outline

- Radiation Transport: Terminology
- Detectors Attributes
- Electrical Bandwidth
- Ideal Photon detection
  - Poisson Noise
  - Signal-to-Noise Ratio
  - Notes on signal extraction
  - Bose-Einstein corrections
- Supplemental material
  - Addition detector performance measures
  - Optimal aperture photometry extraction
  - Bose-Einstein statistics - details
Radiation Transport: Terminology

- $I_\nu = \text{specific intensity}$
  - energy from a given direction
  - number density of photons per unit frequency interval ($\#$/cm$^3$/Hz)
  
  $I_\nu = h\nu c \frac{dn_\nu}{d\Omega}$

- $F_\nu = \text{Flux density}$
  
  $F_\nu = \int d\Omega I_\nu \cos \theta$
  
  $\rightarrow \int d(\cos \theta) 2\pi I_\nu \cos \theta$ (az. sym.)
  
  $\rightarrow \pi I_\nu$ ($I_\nu = \text{const.}$)

  
  $d\Omega = 2\pi \sin \theta \, d\theta$

  $dA_{\text{proj}} = dA \cos \theta$

- $f_\nu = \text{Flux from a star}$
  
  $f_\nu = \int d\Omega I_\nu \quad (\cos \theta = 1)$

  $d\Omega = \frac{dA_{\text{proj}}}{d^2} = \frac{2\pi r dr}{d^2} = -\frac{2\pi R^2 \cos \phi \, d(\cos \phi)}{d^2}$

  $\Rightarrow f_\nu = 2\pi \frac{R^2}{d^2} \int_0^1 I_\nu(0, \mu) \mu d\mu \quad (\mu = \cos \phi)$

  $\Rightarrow f_\nu = \frac{R^2}{d^2} F_\nu$  
  
  $\Rightarrow f_\nu = \frac{R^2}{d^2} \pi B_\nu$  
  
  ($I_\nu = B_\nu$)

- $f_\nu = \text{monochromatic flux seen by an observer}$
- $f_\nu = \text{observed flux density}$
- $f_\nu = \text{flux seen by an observer}$
### Attributes of Detectors

- **Responsivity,** $R_o$ [amps/Watt]
  - ratio of output current to input power (due to photons)
  - gives no information about the noise properties of a detector, hence does not indicate sensitivity
  - measure at constant (DC) power
  - for one electron per photon: $R_o = e/h\nu = 0.81 \lambda (\mu m)$

- **Spectral Response,** $R_o(\lambda)$
  - wavelength response of $R_o$

- **Frequency Response,** $R(\lambda, f)$
  - response to a modulated signal, e.g. chopped radiation

### Attributes (cont’d)

- **Quantum Efficiency,** $\eta$ (QE)
  - Average number of electrons generated per photon
  - Fraction of photons absorbed for bolometers

- **Detective Quantum Efficiency,** (DQE)
  - Square of output S/N relative to input S/N ratio

- **Dark Current,** $i_d$
  - Detector output with no photons falling on the device

- **Read Noise,** $R_N$ (RN)
  - Detector noise w/ no input photons
  - Usually independent of integration time
Electrical Bandwidth

We wish to characterize the electrical bandwidth of the system.

If a detection system responds uniformly to modulation frequency between \( f_1 \) and \( f_2 \) and no response outside, then the bandwidth is:

\[ \Delta f = f_2 - f_1 \]

Otherwise, if the response function, \( R(f) \) varies continuously then the equivalent bandwidth is:

\[ \Delta f = \int_{0}^{\infty} \left| \frac{R(f)}{R_{\text{max}}} \right|^2 df \]

where \( R_{\text{max}} = \max(R(f)) \)

Frequency Response: RC Circuit

One can characterize the response function of a detector by specifying the dependence of its output on the frequency of a sinusoidally varying input photon power.

The frequency response can be limited by many factors in the detector system:

- However, most factors can be modeled as RC circuits where the R and C are in parallel.
- RC circuits have an exponential time rise response, so that charge deposited on the capacitor bleeds off through the resistor with an exponential time constant, \( \tau_{\text{RC}} = RC \)
Frequency Response

- If we input a voltage pulse to the system: \( v_{in}(t) = v_0 \delta(t) \), then the output voltage, as observed with an oscilloscope will have the form:

\[
v_{out}(t) = \begin{cases} 
  0, & t < 0 \\
  \frac{v_0}{\tau_{RC}} e^{-t/\tau_{RC}}, & t \geq 0
\end{cases}
\]

- The same event can be viewed in terms of the effect of the circuit on the input frequencies instead of in the time dependence of the voltage.
- To do so, we take a Fourier transform of the input and output voltages to change to the frequency domain:

\[
V_{in}(f) \equiv FT\{v_{in}(t)\} = FT\{v_0 \delta(t)\}
\]

Frequency Response

- The \( \delta(t) \) function has contributions from all frequencies at equal strength, so that:

\[
V_{in}(f) = v_0 \int_{-\infty}^{\infty} \delta(t) e^{-2\pi f t} dt = v_0
\]

- Since the input, \( V_{in}(f) = \text{constant} \), any deviations from a flat spectrum in the output must be due to the action of the circuit, i.e. the output spectrum yields the frequency response of the circuit directly:

\[
V_{out}(f) = \int_{-\infty}^{\infty} v_{out}(t) e^{-2\pi f t} dt = \frac{v_0}{\tau_{RC}} \int_{0}^{\infty} e^{-t/\tau_{RC}} e^{-2\pi f t} dt
\]
Frequency Response

- So that
  \[ V_{out}(f) = V_o \int_0^\infty e^{-(1+2\pi f)\tau_0} \cdot df = \frac{V_0}{1 + 2\pi f \tau_{RC}} \]

- The imaginary part is just a phase shift.
- We are only concerned here with the amplitude of the frequency response which is given by:
  \[ |V_{out}(f)| = \left( V_{out} \cdot V_{out}^* \right)^{1/2} = \frac{V_0}{\left(1 + (2\pi f \tau_{RC})^2\right)^{1/2}} \]

- The frequency response can be characterized by a cutoff frequency:
  \[ f_c = \frac{1}{2\pi \tau_{RC}} \]

- At which the amplitude drops to 1/sqrt(2) of its value at \( f = 0 \)
  \[ |V_{out}(f_c)| = \frac{1}{\sqrt{2}} |V_{out}(0)| \]
E-BW: Exponential Decay

- For such a system, with exponential decay time, $\tau$, the response function, $R(f)$, is therefore:

$$R(f) = \frac{R_0}{1 + 2\pi f \tau}$$

Or using the definition of equivalent electrical bandwidth introduced earlier:

$$\Delta f = \int_0^\infty \frac{df}{1 + (2\pi f \tau)^2} \Rightarrow \Delta f = \frac{1}{4\tau}$$

E-BW: Integrator over time T

- For a system that integrates over a time, $T$, the response is:

$$v_{out}(t) = \begin{cases} 
1 & \text{for } -T/2 < t < T/2 \\
0 & \text{otherwise}
\end{cases}$$

- The frequency response is:

$$R(f) = \int_{-T/2}^{T/2} \frac{R_0}{T} e^{-2\pi f t} dt = R_0 \frac{\sin \pi f T}{\pi f T}$$

- The electrical bandwidth is then:

$$\Delta f = \int_0^\infty \left[ \frac{\sin \pi f T}{\pi f T} \right]^2 df \Rightarrow \Delta f = \frac{1}{2T}$$
Ideal Photon Detector

- No output current in the absence of incident power
- No noise except that due to the randomness of emission times of photoelectrons
- Let $P$ be the power falling onto the detector with quantum efficiency, $\eta$, in a small bandwidth, $\Delta\nu$.
- Assume emission events are probabilistic in the sense that they are uncorrelated and occur at an average rate $r$, given by:

$$r = \frac{\eta P}{h\nu}$$

Ideal Photon Detector: Shot Noise

- The average number of photoevents occurring in a time $T$ is:

$$\bar{N} = rT$$
- The actual number of events will fluctuate around $\bar{N}$ for any one particular interval of length $T$.
- The probability, $P(N)$, that in any one such interval exactly $N$ photoevents occur is given by the Poisson probability distribution:

$$P(N) = \frac{\bar{N}^N}{N!} e^{-\bar{N}}$$
Applicability of Poisson Distribution

- Divide the time interval $T$ into $n$ segments. The average number of photons per segment is $\overline{N}/n$, where $\overline{N}$ is the average number of photon events for the time interval $T$.
- For sufficiently large $n$, $\overline{N}/n << 1$, so that $\overline{N}n$ can be interpreted as the probability that one photoevent occurs in a given segment.
- The probability that exactly $N$ events occur in the total interval $T$ is given by the binomial distribution

$$P_n(N) = \frac{n!}{N!(n-N)!} \left( \frac{\overline{N}}{n} \right)^N \left[ 1 - \frac{\overline{N}}{n} \right]^{n-N}$$

Poisson Distribution

Combination of $n$ things taken $N$ at a time – the total number of ways that $N$ indistinguishable events can occur in $n$ segments

$$P_n(N) = \frac{n!}{N!(n-N)!} \left( \frac{\overline{N}}{n} \right)^N \left[ 1 - \frac{\overline{N}}{n} \right]^{n-N}$$

- Rewriting gives

$$P_n(N) = \frac{1(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n+1}{n})}{N!} \frac{\overline{N}^N}{\overline{N}} \left[ 1 - \frac{\overline{N}}{n} \right]^{n-N}$$

- Taking the limit as $n \to \infty$, we arrive at the Poisson distribution

$$P(N) = \lim_{n \to \infty} P_n(N)$$

$$= \frac{\overline{N}^N}{N!} \lim_{n \to \infty} \left[ 1 - \frac{\overline{N}}{n} \right]^{n-N} \Rightarrow P(N) = \frac{\overline{N}^N}{N!} e^{-\overline{N}}$$
Poisson Probability Distribution

- Figure 8.4 in Boyd – the Poisson probability distribution
- Note how as the average number of photons gets large, the width of the distribution (noise) approaches $\sqrt{N}$

Properties of Poisson Dist’n

- Normalization: $\sum_{N=0}^{\infty} P(N) = e^{-N} \sum_{N=0}^{\infty} \frac{N^N}{N!} = e^{-N} Ne^N = 1$
- One can show:
  - Expectation value: $\langle N \rangle = \sum_{N=0}^{\infty} N P(N) = N$
  - Variance: $(\Delta N)^2 \equiv \langle (N - \langle N \rangle)^2 \rangle = N$
  - RMS noise: $\Delta N_{rms} = \sqrt{(\Delta N)^2} = N^{1/2}$
  - Signal to noise ratio: $\frac{S}{N} = \frac{\sqrt{N}}{\Delta N_{rms}} = N^{1/2}$
    $\propto T^{1/2}$, since $\bar{N} = rT$
Noise for a Photodetector

- The photoevents give rise to a photocurrent (and likewise noise) in a photon detector.
- Suppose a photon detector is characterized by an averaging time $T$, the average current is:
  $$\bar{I} \equiv \frac{e\bar{N}}{T}$$
- The noise in the photocurrent is:
  $$i_N^2 \equiv (\Delta i)^2 \equiv (i - \bar{I})^2 = \frac{e^2}{T^2} \left( \frac{N - \bar{N}}{N} \right)^2 = \frac{e^2 \bar{N}}{T^2}$$

So that
$$i_N^2 = \frac{e\bar{I}}{T} = 2e\bar{I} \Delta f$$  (using $\Delta f = 1/2T$)

White noise - constant noise power per unit frequency interval.

Signal-Limited Detection

- Suppose $P_s$ is the signal power falling onto an ideal detector of quantum efficiency, $\eta$, the signal current is:
  $$i_s = \frac{\eta e P_s}{h\nu} \quad \text{w/ noise} \quad i_N = \sqrt{2e i_s \Delta f} = \sqrt{\frac{2\eta e^2 P_s \Delta f}{h\nu}}$$

The signal-to-noise ratio is then
$$\frac{S}{N} = \frac{i_s}{i_N} = \sqrt{\frac{\eta P_s}{2 h\nu \Delta f}} \quad \Rightarrow \quad \text{NEP} \equiv P_s \bigg|_{S/N = 1} = \frac{2 h\nu \Delta f}{\eta}$$

$$\Rightarrow \text{NEP} = \frac{h\nu}{\eta T}$$

Minimum detectable power (or NEP) will produce on average one photo-detection per measurement time $T$. 
Background Limited Instrument Performance (BLIP)

- In addition to $P_S$, there may be an unwanted background power, $P_B$. The signal and noise currents are now:

$$i_s = \frac{\eta e P_s}{h\nu} \quad i_n = \sqrt{\frac{2\eta e^2 (P_S + P_B)\Delta f}{h\nu}}$$

The signal-to-noise ratio is then

$$\frac{S}{N} = \sqrt{\frac{\eta P_s^2}{2h\nu\Delta f (P_S + P_B)}}$$

BLIP occurs for $P_B >> P_S$, such as looking through the atmosphere in the infrared.

$$\Rightarrow NEP = \sqrt{\frac{2h\nu P_B\Delta f}{\eta}}$$

Insight into Signal-to-Noise Ratio

- Photoelectron events are collected for an integration time given by $t = 1/(2\Delta f)$, so the total number collected for (monochromatic radiation) the source and background are:

$$N_S = \frac{\eta P_s}{h\nu} t \quad N_B = \frac{\eta P_b}{h\nu} t$$

- Each component has shot noise which add in quadrature

$$\Delta N_{rms} = \sqrt{(\Delta N_S)^2 + (\Delta N_B)^2} = \sqrt{N_S + N_B} = \sqrt{\frac{\eta (P_S + P_B) t}{h\nu}}$$

$$\Rightarrow \frac{S}{N} = \frac{\eta P_s^2 t}{h\nu (P_S + P_B)} \quad \Rightarrow \quad P_B >> P_S \Rightarrow P_S = \frac{S}{N} \sqrt{\frac{h\nu P_B}{\eta t}}$$
Point Source Sensitivity

- So for background limited instrument performance we have

\[ P_S = \frac{S}{N} \sqrt{\frac{h \nu P_B}{\eta t}} \]

BLIP

Now for a **point source** and a small bandwidth \((\Delta \lambda)\)

\[ P_S = f_\lambda \Delta \lambda A_T \tau_w \tau_i \]

- \( f_\lambda = \text{ergs/cm}^2/\text{s}/\mu\text{m} \)
- \( A_T = \text{area of telescope} \)
- \( \tau_w, \tau_i = \text{transmissions} \)

\[ \Rightarrow f_\lambda = \frac{S}{N \Delta \lambda A_T \tau_w \tau_i} \sqrt{\frac{h \nu P_B}{\eta t}} \]

 Flux density to achieve a given S/N ratio in time \(t\) for BLIP.

Note: \( f_\nu \Delta \nu = f_\lambda \Delta \lambda \), so we can equally use \( f_\nu \) rather than \( f_\lambda \)

Including Other Noise Sources

- In addition to the intrinsic shot noise associate with photons from the source and background, there can be other sources of noise, e.g.
  - Shot noise from dark current
  - “Read noise” associated with the electronic circuitry associated with the detector

- In many cases, these noises are uncorrelated so we can added them in quadrature to get the total noise
  - There may be “excess” noise above that expected from pure shot noise for some processes.
  - We will use the factor \( \beta (> 1) \) to characterize this excess.

- For the present purpose we (ideally!) include read noise and dark current in a simple way in the signal-to-noise ratio estimate.
Signal-to-Noise Ratio

- The signal-to-noise ratio is on a source with $S$ collected photoelectrons:
  \[
  \frac{S}{N} = \frac{S}{\sqrt{N_S^2 + N_B^2 + N_d^2 + R_N^2}}
  \]

- Since the (uncorrelated) noises add in quadrature
- Thus, solving for $S$ and plugging in gives
  \[
  \sqrt{\frac{2}{12\cdot22\cdot2}} = \frac{S}{\sqrt{N_S^2 + N_B^2 + N_d^2 + R_N^2}}
  \]

- If one of the noise terms dominates the others then the system is called: background, dark current or read noise limited depending on which dominates.
- The best condition is signal-noise-limited

Point Source Sensitivity (cont’d)

- **BLIP** (Background Limited Infrared Performance) is defined as the background dominating all other noise sources.
- Assuming the background emission is extended and “thermal”
  \[
  P_B = \varepsilon \lambda B_{\lambda}(T) \Delta \lambda A_T \Omega \tau_i
  \]
- If there are more sources of background, add them to get $P_B$. Putting the above in for $P_B$ gives
  \[
  F_{\lambda} = \frac{S}{N} \frac{1}{\tau_i} \sqrt{\frac{\hbar v \varepsilon \lambda B_{\lambda} \Omega}{(\eta/\beta) \tau_i \Delta \lambda A_T}}
  \]
- BLIP is the best you can do (unless your source is bright enough to reach the signal-noise-limited regime)
Point Source Extraction

How do we “extract” the photo-electrons for a point source? (determines background in S/N determination)

**Options**

- Add-up signal in pixels that contain the source.
- Fit point spread function (PSF) to source (best way – more in a later)
- Optimize extraction radius to get best S/N ratio (if not using PSF).
- Don’t have to get “all” the flux - but must account for this in calibration and S/N calculations.

Point Sources: At the telescope

- Consider a detector operating at a telescope.
- Use the measured parameters to estimate sensitivity (or integration time). We have:

\[
\text{Signal} = St \\
\text{Noise} = \left( S_t + p S_b t + p R_N^2 \right)^{1/2}
\]

- \( S = \text{e-/sec from source} \)
- \( S_b = \text{e-/sec/pixel from sky} \)
- \( p = \text{number of pixels} \) (seeing dependent)
- \( R_N = \text{read noise (e-/pixel)} \)

\[
\frac{S}{N} = \frac{St}{\left( S_t + p S_b t + p R_N^2 \right)^{1/2}}
\]

sum over \( p \) pixels
Matched Filtering/Optimal Extraction

- For point source extraction,
  - A weighted extraction which matches the point spread function provides the optimal signal-to-noise ratio
  - Need to determine center and amplitude in fit
  - Can ignore bad pixels (hard for aperture photometry)

- But …
  - At high S/N (> 50-100?) small errors in PSF dominate S/N ratio => likely better off using aperture photometry

- Also …
  - At low S/N (< 5-10?) exact PSF shape doesn’t matter, sometimes a Gaussian or other simple function will do

Optical Imaging Palomar

- Suppose we have a imaging CCD at Palomar with the following characteristics (the “old” COSMIC instrument).
  - CCD: 2048x2048
  - Scale: 0.2856 arcsec/pixel
  - 20th mag. star: 637 e-/sec at R-band ($S_{R20}$)
  - sky: 18 e-/sec/pix at R-band ($B_R$)

- Ignoring the read noise (okay for $t > 25$ sec), the time to obtain a give S/N at R-band on a source of magnitude $m$ is

$$t_R = \left[ \frac{S}{N} \right]^2 \frac{S_{R20} 10^{-(m-20)/2.5} + pB_R}{S_{R20}^2 10^{-(m-20)/1.25}}$$

e.g. $t_R = 430$ seconds to get $S/N = 5$ for $m = 25$ with an extraction aperture of 10 pixels containing 50% of the flux.
### Scaling Laws for Point Sources

For BLIP

\[ f_\lambda = \frac{S}{N \tau_e} \sqrt{\frac{h \nu \epsilon B_\lambda}{\eta / \beta} \frac{\Omega}{\tau_i \Delta \lambda A_r}} \]

\[ A_r = \frac{\pi}{4} D^2 \quad \Omega = \frac{\pi}{4} \theta_b^2 \]

- For diffraction limited performance:

\[ A_r \Omega = 0.92 \lambda^2 \implies f_\lambda \propto \frac{1}{D^2 \sqrt{t}} \]

\[ t \propto \left( \frac{S}{N} \right)^2 \frac{1}{f_r D^2} \]

- For a fixed beam size (e.g. seeing limited):

\[ \Omega = \frac{\pi}{4} \theta_b^2 \implies f_\lambda \propto \frac{\theta_b}{D \sqrt{t}} \]

\[ t \propto \left( \frac{S}{N} \right)^2 \frac{\theta_b^2}{f_r D^2} \]

### Extended Source Sensitivity

- For an extended source on the sky. Let

\[ f_\lambda = I_\lambda \Omega \quad I_\lambda = \text{specific intensity} \quad \text{(ergs/cm}^2\text{/sec/sr/\mu m}) \]

- Then

\[ I_\lambda = \frac{S}{N} \frac{1}{\tau_e} \sqrt{\frac{h \nu \epsilon B_\lambda}{(\eta / \beta) \tau_i \Delta \lambda A_r \Omega}} \]

**Diffraction limited beam**

\[ A_r \Omega = 0.92 \lambda^2 \]

\[ I_\lambda \propto \frac{1}{\sqrt{t}} \]

\[ t \propto \left( \frac{S}{N} \right)^2 \frac{1}{I_\lambda} \]

\[ \implies \text{Independent of D!} \]

**Scaling Laws**

**Fixed beam size**

\[ \Omega = \frac{\pi}{4} \theta_b^2 \]

\[ I_\lambda \propto \frac{1}{\theta_b D \sqrt{t}} \]

\[ t \propto \left( \frac{S}{N} \right)^2 \frac{1}{\theta_b^2 I_\lambda D^2} \]
Spectral Lines

- For best sensitivity
  - Narrow spectral bandpass to remove extraneous flux without reducing line flux

- The integrated flux will be roughly: \( f = f_\lambda \Delta \lambda \) so that the sensitivity is now (BLIP case):

\[
f = \frac{S}{N} \frac{1}{\tau_n} \sqrt{\frac{h \nu \epsilon_\lambda B_\lambda}{\eta \tau_i A_i}}
\]

narrower spectral BW

\[ \Rightarrow \text{better sensitivity, unless line is resolved} \]

But Photons are Bosons

- Photons follow Bose-Einstein Probability Distribution:

\[
p(n) = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \quad \bar{n} \text{ is average occupation number} \quad \bar{n} = \frac{1}{e^{h \nu / kT} - 1} \quad \text{when thermally generated}
\]

- For which:

\[
\Delta n_{rms} = \sqrt{(\Delta n)^2} = \sqrt{\bar{n}(\bar{n} + 1)}
\]

for \( h \nu >> kT, \ i.e. \ n << 1 \) \( \Delta n_{rms} = \sqrt{\bar{n}} \)

for \( h \nu << kT, \ i.e. \ n >> 1 \) \( \Delta n_{rms} = \bar{n} \) Photon “bunching” causes increased dispersion
Boson Effects: Signal-to-Noise Ratio

- Including Bose-Einstein statistics

\[(\Delta N)^2_{\text{rms}} \equiv N_B^2 = \frac{P_B \eta \epsilon \tau \nu t}{h \nu} (1 + \eta \epsilon \tau \nu \bar{n})\]

- The number of signal electrons generated will be

\[S = \frac{\eta \tau \nu \bar{P}_B}{h \nu} t \quad \tau_w = 1 - \epsilon\]

- The signal to noise is then given by

\[\frac{S}{N} \equiv \frac{S}{N_B} \quad \epsilon = \text{emissivity of warm background (atmosphere and telescope combined)}\]

See supplemental material for more details

Supplemental Material

- Reference:

Appendices

- Measures of Detector Performance
- Extraction size for aperture photometry
- Bose-Einstein statistics and S/N
Measures of Detector Performance

- **NEP - Noise Equivalent Power (W/Hz^{1/2})**
  - The input power that produces an output signal-to-noise ratio of unity in a specified electrical bandwidth, $\Delta f$.
  - Often $\Delta f = 1$ Hz then the NEP is the minimum signal detectable in a 1/2 second measurement.

- **NEF – Noise Equivalent Flux (W/m^2/Hz^{1/2})**
  - $NEF = NEP / (A_{tel} \eta_{atm} \eta_{tel})$

- **D* - Specific Detectivity (cm-Hz^{1/2}/W)**
  - $D^* = \sqrt{A \Delta f / NEP}$

- **NEFD - Noise Equivalent Flux Density**
  - Input flux that produces a signal-to-noise ratio of unity in 1/2 second measurement
  - e.g. NEFD = 10 mJy (1 Jy = $10^{-26}$ W m$^{-2}$ Hz$^{-1}$)

Choosing the extraction region

- The size of the extraction region for counting the source photons is important.
- Choosing it too large or too small will degrade the signal-to-noise ratio
- If the PSF is known, determine the location and amplitude of best fit solution.
  - PSF could be determined from other (bright sources) in the field or from separate reference measurement (if it is stable)
- Essentially the choice is the number of pixels to sum, with possible weighting for each according to the PSF.
- Let’s look at the Gaussian and Airy functions
What is $\Omega$, the extraction beam?

- This choice of an extraction beam is science dependent. For point sources, in the seeing limited case we might expect a Gaussian PSF while for diffraction limited observations the Airy diffraction pattern will hold.
- We then integrate the flux (either by summing pixels or fitting a PSF).

![Plot of Gaussian and Airy diffraction pattern, and the encircled energy (weighted area integral) vs. $x$, where $x$ is in units of FWHM for the Gaussian and $\lambda/D$ for the Airy pattern.]

The HWHM of the Airy pattern is $0.51\lambda/D$ (and of course $0.5$ for the Gaussian).

The Airy pattern is for $D_{\text{obscur}}/D_{\text{tel}} = 0.12$ or 1.44% obscured area.

Optimum Signal-to-Noise Ratio

- For BLIP observations the signal-to-noise ratio will vary as the encircled energy divided by $x$ (since the background power will scale as $x^2$ and the noise scales as the square root of the power - However at longer wavelengths this is no longer true because Bose-Einstein statistics become important).

![Plot of relative signal-to-noise ratio for a Gaussian and Airy diffraction pattern vs. $x$, where $x$ is in units of FWHM for the Gaussian and $\lambda/D$ for the Airy pattern.]

The maximum S/N occurs at $x = 0.673$ and $0.660$ for the Gaussian and Airy pattern respectively. Choosing $x = 0.5$ for the extraction radius causes a 6% decrease in S/N.

The Airy pattern is for $D_{\text{obscur}}/D_{\text{tel}} = 0.12$ or 1.44% obscured area.
Choosing $\Omega$

- Our first guess for an extraction diameter for a diffraction limited beam might have been $1.22\lambda/D$ which would give

$$A_i\Omega = \frac{\pi}{4} D_i^2 \frac{\pi}{4} \left[\frac{1.22 \lambda}{D_i}\right]^2 = 0.92 \lambda^2$$

- From the results in optimizing S/N a better choice of the extraction diameter is $1.294 \times \text{FWHM} = 1.32 \lambda/D_i$. Now we have

$$A_i\Omega = \frac{\pi}{4} D_i^2 \frac{\pi}{4} \left[\frac{1.32 \lambda}{D_i}\right]^2 = 1.08 \lambda^2$$

- The table below shows how the fractional flux and relative S/N change with extraction aperture diameter for the Airy diffraction pattern.

<table>
<thead>
<tr>
<th>$\theta (\lambda/D)$</th>
<th>Flux Frac.</th>
<th>S/N</th>
<th>$\theta (\lambda/D)$</th>
<th>Flux Frac.</th>
<th>S/N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.45</td>
<td>0.90</td>
<td>1.50</td>
<td>0.71</td>
<td>0.94</td>
</tr>
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<td>1.25</td>
<td>0.59</td>
<td>0.92</td>
<td>1.75</td>
<td>0.78</td>
<td>0.89</td>
</tr>
<tr>
<td>1.345</td>
<td>0.64</td>
<td>0.96</td>
<td>2.00</td>
<td>0.81</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Choosing $\Omega$

- The table below summarizes the results for Gaussian and diffraction limited PSFs with the addition of a final column for the fraction of the flux in the extracted beam.

- Nicely, the ratio of the optimum extraction radius to HWHM is about the same for both so adopting an average value will result in only a few percent error.

<table>
<thead>
<tr>
<th>PSF</th>
<th>HWHM</th>
<th>Opt. S/N radius</th>
<th>Opt S/N over HWHM</th>
<th>Fraction of Flux in Extraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>0.500</td>
<td>0.673</td>
<td>1.346</td>
<td>0.715</td>
</tr>
<tr>
<td>Airy</td>
<td>0.510</td>
<td>0.660</td>
<td>1.294</td>
<td>0.632</td>
</tr>
</tbody>
</table>

The units for the HWHM and optimum extraction radius are FWHM for the Gaussian and $\lambda/D$ for the Air diffraction pattern.

- A combined expression for the extraction diameter (when diffraction and Gaussian terms are both present) might look like

$$d_{ext} = \sqrt{(1.294 \text{FWHM}_A)^2 + (1.346 \text{FWHM}_G)^2}$$

$$= 1.32 \sqrt{(\lambda/D)^2 + (1.02 \text{FWHM}_G)^2}$$
The Complete Story: Photons are Bosons

- The probability distribution function for getting \( n \) photons per mode, that is, the probability that \( n \) photons are excited in a mode of angular frequency \( \omega \) is given by:

\[
p(n) = \frac{e^{-\hbar \omega / kT}}{\sum_{n=0}^{\infty} e^{-\hbar \omega / kT}} \quad \nu = \frac{\omega}{2\pi}
\]

- \( n \) is called the mode occupation number, and the energy of each mode is (quantized as):

\[
E = n\hbar \omega \quad n = 0, 1, 2, \ldots \quad \omega = kc / N
\]

- Summing the series gives:

\[
p(n) = \left(1 - e^{-\hbar \omega / kT}\right) e^{-\hbar \omega / kT}
\]

Bose-Einstein Probability Distribution

- The average occupation number is:

\[
\bar{n} = \sum_{n=0}^{\infty} np(n) \quad \Rightarrow \quad \bar{n} = \frac{1}{e^{\hbar \nu / kT} - 1}
\]

- We could then write:

\[
p(n) = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \quad \text{Bose-Einstein probability distribution}
\]

- Now

\[
(\Delta n)^2 = \bar{n}^2 - (\bar{n})^2 \quad \text{where} \quad \bar{n}^2 = \sum_{n=0}^{\infty} n^2 p(n)
\]
Noise in Bose-Einstein Distribution

- So we have
  \[ \overline{n^2} = \sum_{n=0}^{\infty} n^2 p(n) = (1-e^{-x}) \frac{d^2}{dx^2} \sum_{n=0}^{\infty} e^{-nx} = (1-e^{-x}) \frac{d^2}{dx^2} \frac{1}{1-e^{-x}} = e^x + 1 \]
  \[ \Rightarrow \overline{n^2} = 2\overline{n}^2 + \overline{n} \]
- The rms dispersion becomes:
  \[ \Delta n_{\text{rms}} = \sqrt{\overline{n}}\sqrt{\overline{n} + 1} \]
- For \( h\nu >> kT \), i.e. \( n << 1 \)
  \[ \Delta n_{\text{rms}} = \sqrt{n} \]
- For \( h\nu << kT \), i.e. \( n >> 1 \)
  \[ \Delta n_{\text{rms}} = n \]
- Photon "bunching" causes increased dispersion

Noise in Ideal Detector

- So we have
  \[ (\Delta n)^2_{\text{rms}} = \overline{n}(\overline{n} + 1) \]
- Now consider the power falling onto a detector
  \[ P_G = A\Omega \Delta \nu B_\nu(T) = A\Omega \Delta \nu \frac{2\hbar \nu^3}{c^2} \frac{1}{e^{\hbar \nu/kT} - 1} = h\nu \overline{n} r \]
  where
  \[ r = \frac{2A\Omega}{\Delta^2} \Delta \nu \]
- \( \Omega \) = solid angle for a single transverse field, \( \Delta \nu \cdot t \) = number of possible physically independent measurements of the field amplitude in time \( t \)
- The fluctuations in the photon number occur over the coherence time of the radiation field, \( \tau_c \sim (\Delta \nu)^{-1} \). (see Boyd & heterodyne detection)
Transmission and QE effects

- However, the sampling time is much longer than the coherence time. If, $t$, is the integration time, the we sample a total of $rt$ modes.
- If we add together $rt$ modes the noise is then
  \[(\Delta N)^2_{rms} = rt(\Delta n)^2_{rms} = rt\bar{n}(\bar{n} + 1)\]
  \[= \frac{rtP_B}{h\nu \tau} (1 + \bar{n})\]
- If the detector has quantum efficiency, $\eta$, and optical transmission, $\tau$, and the emissivity of the background is, $\varepsilon$, then the number of modes “reaching” the detector and producing photoelectrons is reduced by the product of the factors, i.e. $\bar{n} \rightarrow \eta \varepsilon \tau \bar{n}$

Boson Effects: Signal-to-Noise Ratio

- Including Bose-Einstein statistics
  \[(\Delta N)^2_{rms} \equiv N_B^2 = \frac{P_B \eta \varepsilon \tau_w t}{h\nu} (1 + \eta \varepsilon \tau \bar{n})\]
- The number of signal electrons generated will be
  \[S = \frac{\eta \varepsilon \tau_w P_S}{h\nu} \tau_w \quad \tau_w = 1 - \varepsilon\]
- The signal to noise is then given by
  \[\frac{S}{N} \equiv \frac{S}{N_B}\]

$\varepsilon =$ emissivity of warm background (atmosphere and telescope combined)
Signal detection with B-E Stats

- Thus we have
  \[ P_s = \frac{S}{N} \frac{1}{1-e^{-\frac{1}{\eta \tau_i}}} \sqrt{\frac{\varepsilon P_B h \nu}{\eta \tau_i t}} (1+\eta \tau_i \bar{n}) \]

where

\[ P_B = A_i \Omega \Delta \nu B_v(T) = 2 \frac{A_i \Omega}{\lambda^2} \Delta \nu h \nu \bar{n} \quad \& \quad \bar{n} = \frac{1}{e^{\frac{h \nu}{kT}} - 1} \]

- Thus we can write:
  \[ P_s = \frac{S}{N} \frac{1}{1-e^{-\frac{1}{\eta \tau_i}}} \sqrt{\frac{\nu}{(1+\bar{n})}} \left( \frac{\Delta \nu}{t} \right) \frac{2 \frac{A_i \Omega}{\lambda^2}} \bar{n}_p = \eta \tau \bar{n} \]

The factor of two enters since we have both polarizations. Calling \( N_p \) the number of polarization measured (1 or 2), we have for point sources

\[ P_s = A_i f_\nu \Delta \nu \frac{N_p}{2} \quad \text{We reference to the unpolarized flux.} \]

Sensitivity with B-E Stats

- Putting it all together gives
  \[ f_\nu = \frac{S}{N} \frac{1}{1-e^{-\frac{1}{\eta \tau_i}}} \frac{1}{A_i} \sqrt{\frac{\nu}{(1+\bar{n})}} \frac{2}{\Delta \nu} \frac{A_i \Omega}{\lambda^2 \sqrt{t}} \]

Where for diffraction limited performance

\[ A_i \Omega \propto \lambda^2 \]

- Note that for \( h \nu >> kT \) the Bose-Einstein correction factor is unimportant. Let

\[ x = \frac{h \nu}{kT} = \frac{14388}{\lambda(\mu m) T} \]

So the dividing line is \( \sim 50 \mu m \) for whether to worry about the extra factor (and depends upon \( \eta, \epsilon, \) and \( \tau \)).

for \( T = 290 \) K; \( x = 49.6 / \lambda(\mu m) \)
Additional Complications

- Multiple background contributions
  - The derivation on the previous page assumed a single background source with emissivity $\varepsilon$ [and transmission $(1-\varepsilon)$].
  - Emission is also usually contributed by the telescope (and other sources) which may or may not be at the same temperature as the atmosphere.
- Lost light
  - In the radio and submm parts of the spectrum, typically the surface roughness of the “dish” is large enough to scatter light from the source out of the beam (but the background will be unchanged because light from the background is also scattered into the beam).
- Point source extraction
  - The optimal extraction of a point source will not include the entire PSF – so we must account for this.
  - Pixilation intrinsically widens the source PSF. To first order this can be modeled by increasing the effective extraction size (by adding the pixel angular size in quadrature with the nominal extraction beam size).

Additional Complications (continued)

- Detector Noise
  - Detectors may have read noise, generation-recombination noise, or other sources of noise but we will ignore these for now.
- Chopping / Nodding
  - In the infrared/submm a source is move rapidly (chopped) between two sky positions. The difference is taken to remove atmospheric variation, telescope offsets, etc.
  - This differencing adds the noises, thus decreasing the sensitivity by $\sqrt{2}$.
  - Additionally if the source is moved off the detector (array) for the second sky position, a factor of two in time is also lost resulting in another factor of $\sqrt{2}$ change in sensitivity.
  - These factors are not included in what follows.
Sensitivity with B-E Stats

- The light loss is characterized by the Ruze factor, $g_R$, which is given by
  \[ g_R = e^{-(4\pi s_{\text{rms}}/\lambda)^2} \]

- Let the telescope have emissivity, $\varepsilon_T$ and temperature $T_T$, we then have

  \[ \bar{n}_r = \eta T \left[ \frac{\varepsilon_A}{e^{h\nu/kT_A} - 1} + \frac{\varepsilon_T}{e^{h\nu/kT_T} - 1} \right] \]

  where the atmospheric component is now explicitly labeled and attenuation by the telescope is neglected (normally a 2nd order effect here).

- Other backgrounds (emissivity sources) can be added in a similar way.

- Finally letting $g_f$ be the fraction of the source flux in the extraction beam (typically ~ 70%) and $\tau_T$ the telescope transmission, we have

  \[ f_v = \frac{(S/N)}{(1-\varepsilon_A)\tau_T g_f g_R \eta \tau_i A_T} \sqrt{\bar{n}_r(1+\bar{n}_r)} \frac{A_T \Omega}{\lambda^2} \frac{1}{\sqrt{t}} \]

Point Source Sensitivity

- In summary, the sensitivity for instrument on a telescope including the Bose-Einstein contribution is:

  \[ f_v = \frac{(S/N)}{\tau_i \tau_T g_f g_R \eta \tau_i A_T} \sqrt{\bar{n}_r(1+\bar{n}_r)} \frac{A_T \Omega}{\lambda^2} \frac{1}{\sqrt{t}} \]

  \[ g_R = e^{-(4\pi s_{\text{rms}}/\lambda)^2} \]

  \[ \bar{n}_r = \eta T \left[ \frac{\varepsilon_A}{e^{h\nu/kT_A} - 1} + \frac{\varepsilon_T}{e^{h\nu/kT_T} - 1} \right] \]

$S/N, t$ = signal-to-ratio and integration time 
$\varepsilon_T, \varepsilon_A$ = telescope and atmospheric emissivity 
$\tau_T, \tau_i$ = telescope and atmospheric transmission ($\tau_i = 1 - \varepsilon_i$) 
$T_T, T_A$ = telescope and atmospheric temperature 
$\tau_f, \eta, N_p$ = instrument transmission, detector QE, number of polarizations (1 or 2) 
$A_T, g_f$ = Telescope area and fraction of flux in extraction (~ 0.7 typically) 
$s_{\text{rms}}$ = rms surface roughness
Chopping and Pixilation

- We can add a chopping degradation factor fairly easily since this directly affects the noise through a difference (sqrt(2)) or integration time on source (for a given wall clock time).
- As a first order assumption for “pixilation” noise assume that there are $p$ pixels across the diffraction limit. Then

$$\Omega_{pix} = \Omega \left[ 1 + \frac{1}{p^2} \right]$$

- Where $\Omega$ is the non-pixilated beam area. $p = 2.0$, $2.5$, and $3.0$ yields factors of 1.12, 1.08, and 1.05 change in sensitivity respectively.
- Note that this formulation assumes that a set of randomly “dithered” images are combined so the effective beam is widened which is done to eliminate systematic effects associated with pixel position.
- This effect is negated with “perfect” pointing so that this is no “smearing” of the beam, however, quantization effects (finite size pixels) will still enter.

Point Source Sensitivity, again

- Including pixilation and chopping loses the point source sensitivity is

$$f_r = \frac{S/N}{\sqrt{t}} \frac{C_L}{g_R g_A \tau_d \tau_e \eta \tau_i} \frac{h \nu}{2} \left[ \frac{\tau_i (1 + \tau_i)}{\tau_d} \right] \frac{A_f \Omega}{\lambda^2 \Delta \nu} \sqrt{\left[ 1 + \frac{1}{p^2} \right]}$$

$$g_R = e^{-4 \pi \Delta \omega / A^2}$$

$$\tau_i = \eta \tau_i \left[ \frac{\epsilon_f}{e^{h \nu / k T_f} - 1} + \frac{\epsilon_f}{e^{h \nu / k T} - 1} \right]$$

$S/N$, $t$ = signal-to-ratio and integration time
$\epsilon_r$, $\epsilon_d$ = telescope and atmospheric emissivity
$\tau_r$, $\tau_d$ = telescope and atmospheric transmission ($\tau_d = 1 - \epsilon_d$)
$T_f$, $T_d$ = telescope and atmospheric temperature
$\tau_i$, $\eta$, $N_p$ = instrument transmission, detector QE, number of polarizations (1 or 2)
$A_f$, $g_f$ = Telescope area and fraction of flux in extraction (~ 0.7 typically)
$s_{rms}$, $C_L$ = rms surface roughness and chopping loss (1, 1.414, or 2 typically)
$p$ = pixels across beam (= 2 for Nyquist sampling)