Interferometry and Synthesis Imaging

Arrays of Antennas
To consider:
1) how $\Gamma_{\nu\nu}$ is related to the radiation field
2) consider how phase fluctuations caused by various
   incoherent media affect $\Gamma_{\nu\nu}$
3) how phase fluctuations caused by incoherent local oscillators
   and
   clocks affect $\Gamma_{\nu\nu}$.

Wien - Kohnlein Theorem

\[ V(t) \leftrightarrow FT \rightarrow \tilde{V}(\omega) \]
\[ R(\tau) \leftrightarrow FT. \rightarrow S(\omega) \]

\[ R(\tau) = \text{autocorrelation function} \]
\[ = \langle V(t) V^*(t+\tau) \rangle \quad \text{if stationary statistics} \]
\[ \approx T^{-1} \int_0^T dt \, V(t) V^*(t+\tau) \]
Arrays

- Beam forming vs. interferometry.
- Array factor contribution to the PSF.
- Visibility functions.
- Van Cittert-Zernike theorem.
- Synthesis imaging and calibration.

Main Points for Beam Forming

- The voltages (e.g. the baseband signals) from $N$ antennas can be summed to provide an effective area $= N \times$ (single-antenna area).
- The PSF of the array can be shaped and directed through appropriate complex weights for each signal in the sum.
- With digital beam forming, multiple beams can be formed (multiple source sampling).
- All operations must occur at rates $\sim 1 / \text{bandwidth}$ $\rightarrow$ enormous processing requirements.
### Main Points for Interferometry

- The cross correlation of voltages between antennas (the visibility $\Gamma(u,v)$) in an array is related to the mutual coherence of the radiation field.
- Cross correlations between polarizations and antennas yield Stokes visibilities.
- Local oscillator stability + ionosphere + atmosphere limit the coherence time of the cross correlations.

### Main Points for Interferometry

- **Advantages of aperture synthesis over single dish observations:**
  - Spatial filtering.
  - RFI rejection (RFI doesn’t correlate over long baselines).
  - Flexibility in image construction.
- **Disadvantages:**
  - Limited collecting area (so far).
  - No ‘zero-spacing’ flux density.
  - Computation intensive (data sets ~ Terabytes with VLA).
Main Points for Interferometry

- “Phase center”.
- Effects due to finite bandwidths and integration times (distortion of visibility function + reduction of field of view).
- Calibration methods:
  - Amplitude, phase, and polarization calibration.
- Image formation:
  - Van Cittert-Zernike theorem:
    \[ \text{Image} = \text{Fourier transform(visibility function)}. \]
  - Sidelobes and dynamic range:
    \[ \text{CLEAN} = \text{deconvolution of PSF from raw image}. \]
  - Self calibration (hybrid mapping).

Arrays of Antennas

Pair of antennas with incident plane wave:
Phase difference due to path length difference (PLD).

\[ \Delta \phi = 2\pi \frac{PLD}{\lambda} = k PLD = kb \sin \theta. \]
Arrays of Antennas

\[ E = E_1 + E_2 \]
\[ = E_0(e^{i\phi_1} + e^{i\phi_2}) \]
\[ = E_0(e^{i\phi_1} + e^{i(\phi_1 + \Delta\phi)}) \]
\[ = E_0 e^{i\phi_1}(1 + e^{i\Delta\phi}). \]

\[ I = |E|^2 = |E_0|^2 2(1 + \cos \Delta\phi) = 2I_0(1 + \cos \Delta\phi) \]
\[ \Rightarrow I = 2I_0(1 + \cos \Delta\phi) \begin{cases} 4I_0 & \text{if } \Delta\phi = 0, \\ 2I_0 & \text{if } \Delta\phi = \pi/2, \\ 0 & \text{if } \Delta\phi = \pi. \end{cases} \]

The Fringe Pattern

\[ I(\theta) = 2I_0(1 + \cos(kb \sin \theta)). \]

Fringe spacing \( \Delta\theta \approx \frac{\lambda}{b} \) for \( \Delta\theta \ll 1. \)
The Fringe Pattern

\[ I(\theta) = 2I_0(1 + \cos(kb \sin \theta)). \]

Fringe spacing \( \Delta \theta \approx \frac{\lambda}{b} \) for \( \Delta \theta \ll 1 \).
Also need to include antenna power pattern \( P_n(\theta) \):

\[ I(\theta) \rightarrow I(\theta)P_n(\theta). \]

Relative widths: \( \lambda/b \) vs. \( \lambda/D \)
- VLA: \( D = 25m; b \sim 1–30 \text{ km.} \)
- VLBA: \( D = 25m; b \sim 6000 \text{ km.} \)

N-element Arrays

For an N-element array,

\[ E = \sum_{j=1}^{N} E_j e^{i\phi_j} = \sum_{j=1}^{N} E_j e^{ikX_j}. \]

\[ I = \left| \sum_{j=1}^{N} E_j e^{ikX_j} \right|^2 \]

\( \Rightarrow \) N “auto” terms, N(N-1) “cross” terms, N(N-1)/2 fringe patterns.

The array response to the sky brightness distribution is,
for a given pair of antennas,
the integral of the product of
the fringe pattern and the sky brightness.

This is \( \approx \) the Fourier transform of the sky.
Phase Offsets

Add a (controllable) phase difference at each antenna, $\hat{\phi}_j$.

With a suitable $\hat{\phi}_j$ we can control the directionality of the array.

$$E = \sum_{j=1}^{N} E_j e^{i(\phi_j - \hat{\phi}_j)} = \sum_{j=1}^{N} E_j e^{i(k.X_j - \hat{\phi}_j)}.$$  
Let $E_j = E_0$ for all $j$, and $\Delta \phi_j = (k.X_j - \hat{\phi}_j)$.

Phase Offsets

Then, with $E = E_0 \sum_{j=1}^{N} e^{i\Delta \phi_j}$,

- Case 1: Perfect phase compensation, $\Delta \phi_j = 0$.
  Then $E = N E_0$ and $I = N^2 I_0$.

- Case 2: Random $\Delta \phi_j$ (why?).

  $$I = I_0 \left| \sum_j e^{i\Delta \phi_j} \right|^2$$  
  $$= I_0 \sum_j \sum_k e^{i(\Delta \phi_j - \Delta \phi_k)} \left| ^2 \right.$$
  $$= NI_0 + I_0 \sum_j \sum_k \left( k \neq j \right) e^{i(\Delta \phi_j - \Delta \phi_k)} \left| ^2 \right.$$  

  For large $N$, final $N(N - 1)$ terms sum $\approx 0$; $I = NI_0$.  

Coherent and Incoherent Sums

With $E = E_0 \sum_{j=1}^{N} e^{i \Delta \phi_j}$,
- "Incoherent" case: $I = NI_0$.
- "Coherent" case: $I = N^2 I_0$.

If the phase offset at each antenna is noise-like, incoherent sum.
If the phase offset precisely compensates for a given direction, we have a coherent sum in that direction: beam steering.

Note that each antenna has a power pattern, and we end up with a net power pattern for the array:
$P_n^{(A)} \equiv$ the Fourier transform of the antenna locations.

Beam Steering

Control the direction of the main lobe of the array by the control phases $\hat{\phi}_j$.
- Reference direction: direction of main lobe
- Array beam can be steered to any location that is within the primary beam of each antenna.
- How? Analog delay lines, or digital delays.
- If $\exists$ multiple sets of control phases, then multiple beams on the sky! Cost: Extra processing.
- If each antenna has a wide FoV (e.g., dipoles $\theta_{FWHM} \sim 1$ radian) then much of the sky can be viewed simultaneously.
- Steerable antennas: entire sky is accessible. (cf., SKA).
Alternative Usage of Arrays

Rather than summing antenna outputs in real time, process each pair of antennas separately, and do all beam forming offline in software.

⇒ Fourier transformation vs. correlation (convolution).
→ Cost: much, much more processing.
(Also, not usually real-time.)

Schematically:

\[
\text{Array}(X_j) \xrightarrow{\mathcal{F} \ vs. \ X_j} P_n^{(A)}(\theta) \xrightarrow{\text{Cross correlate pairs}} \Gamma_{ij} \xrightarrow{\mathcal{F} \ vs. \ b} \mathcal{F} \xrightarrow{\text{Convolve with sky}} I(\theta)
\]
Visibilities and Sky Images

UV Coverage

Synthesized ("Dirty") Beam

Convolved

Sky brightness distribution

Output dirty map

Cross Correlation

The "visibility" $\Gamma$ is function of $(b_{ij}) = X_j - X_i$.

$$\Gamma_{ij} \equiv \langle E_i E_j^\ast \rangle = \langle E(X_i) E^\ast(X_j) \rangle$$

Simpler notation: $\Gamma(b)$, where

$$b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

For small fields of view, the "w" coordinate can be ignored if the $z$-direction $\approx$ the reference direction.

(Assumption breaks down for very wide-field imaging.)
Interferometric Imaging

The visibility function then depends on the "u" and "v" coordinates: Visibility function \( \Gamma(u, v) \).

\[
I(\theta) = I(x, y) = \int \int du \, dv \, e^{2\pi i (ux + vy)} \Gamma(u, v)
\]

(The Van Cittert-Zernike theorem.)

Note: The primary antenna pattern \( P_1(\theta) \) is implicit.

Interferometry:
- Measure Fourier components of \( I(\theta) \)
- \( \equiv \) Visibilities \( \Gamma(b, \tau) \).
- Transform to get image estimate \( \hat{I}(\theta) \).

VLA: 27 antennas \( \Rightarrow 351 \) unique pairs
Interferometric Imaging

\[ I(\theta) \equiv K_\lambda \int db e^{i k b \theta} \Gamma(b; \tau) \]

\[ \rightarrow \int \int du dv e^{2 \pi i (u \theta_x + v \theta_y)} \Gamma(u, v; \tau) \]

After sampling the Fourier components, transform to get:

\[ \hat{I}(\theta) = \sum_{u,v} e^{2 \pi i (u \theta_x + v \theta_y)} \Gamma(u, v; \tau) \]

where \( u, v, \theta \) are discrete.

Complications in Interferometric Imaging

- Control phases are based on a reference direction (the “phase center”), as in beam forming.
- Phases are corrupted by:
  - Intervening media (differential refraction, scattering, ...);
  - Imperfect knowledge of coordinate system (Earth rotation);
  - Imperfect knowledge of antenna locations (tectonics; geodesy);
  - Imperfect clocks and oscillators;
  - Variable phases in electronics (temperature).
Interferometry: Summary

- Correlation function of voltages is related to mutual coherence of the radiation field.
- Cross-correlations → Stokes parameter visibilities.
- Coherence time is limited by LOs, atmosphere, ...
- Compared to single dish observations, advantages:
  - Spatial filtering, RFI rejection, flexibility in image estimation.
- But:
  - Limited collecting area, computation intensive.
  - Larger scales are filtered out! Zero spacing flux density.
- Smearing effects due to finite bandwidth and integration time (i.e., averaging over time and frequency).
- Consider: calibration, imaging, sky mapping schemes.

Image formation from Interferometry: I

How do we measure $\Gamma$, the visibility function?

- $\Gamma(u, v)$ relates to the cross-correlation function of voltages in two antenna/receiver systems.
- Calibration: antenna gains, phase errors.
  - Closure phase: $\phi_{1,2} + \phi_{2,3} + \phi_{3,1} = 0$.
  - Closure amplitude: $\Gamma(1, 2).\Gamma(3, 4) = \Gamma(1, 3).\Gamma(2, 4)$
- Self calibration and hybrid mapping.
- Field of view issues: primary beam and delay beam.
How do we go from $\Gamma$ to the sky brightness distribution $B$?

- Fourier relationship between $\Gamma \leftrightarrow B$.
- Relationship is approximate in practice.
- In some cases, better to use alternatives to go from $\Gamma \rightarrow B$
  
  e.g., Maximum Entropy Method: Use an *a priori* model of the sky and maximize a pre-defined statistic.  
  (But where do you get the model? A blank sky works.)

Constructing an image from a set of visibilities:

- FFT + CLEAN, or MEM.
- Sampling in $uv$ plane is incomplete.
- Dealing with missing data: MEM, CLEAN take different approaches (smooth extended structure vs. collection of point sources).
- Calibration errors lead to image degradation.
- But we can use knowledge about the sky to improve calibration: self-calibration and hybrid mapping.

Dynamic range: Dirty maps $\rightarrow 10:1$

<table>
<thead>
<tr>
<th>CLEAN</th>
<th>$\rightarrow 100s:1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Self-cal</td>
<td>$\rightarrow 10^4:1$</td>
</tr>
<tr>
<td>SKA goal</td>
<td>$\rightarrow 10^6 - 10^7$.</td>
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</tbody>
</table>
Any continuous function can be built up as a superposition of sinusoids.

This could represent, e.g., a sky brightness distribution.

2 parameters: \[ A \sin(\omega t + \phi) = (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t \]
The “Dirty Beam”

Beam for LGM0600+45 in Plan (18.1cm)

Amplitude vs UV dist for VLB 0600+45.UVDATA.1 Source:LGM06+45

Ants * - * Stokes LL IF# 1 Chan# 1 - 16
Freq = 1.6650 GHz, Bw = 8.000 MHz

Visibility amplitudes vs baseline – uncalibrated and calibrated.

Calibration

Visibility amplitudes vs baseline – uncalibrated and calibrated.
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Deconvolution with CLEAN

Deconvolution with CLEAN
Deconvolution with CLEAN

Deconvolution with CLEAN
Notes

A variant usage of \( \text{array} \)

 Rather than sampling output in real time, process pairs \((x, y)\) separately and do all the filtering offline in software.

\[ = \] Fourier transform vs. Correlation (secant).

Schematically we have:

\[ \text{array} (x) \rightarrow \text{less correlated pairs} \rightarrow F \rightarrow \text{corr. with fig} \rightarrow \text{corr. with fig} \rightarrow \text{corr. with fig} \rightarrow \text{corr. with fig} \\]

\[ \mathcal{I}(\theta) = \int dp \: \frac{\nu^m(x, \theta)}{E} \cdot E(x) \]

\[ \begin{align*}
\mathcal{I}(\theta) &= \int dp \: \frac{\nu^m(x, \theta)}{E} \cdot E(x) \\
&= \mathcal{I}_1(x, \theta)
\end{align*} \]
Interferometry = measure Fourier component of $I_\hat{i}$

$\sum_{i} I_{i}(\hat{z})$

Primary beam \times fringe pattern of 2-lobed int.

\[
I(\hat{z}) = \sum_{i} e^{i(kx_i+\nu_i)} \Gamma_n(\hat{z})
\]

VLA: 27 antennas $\Rightarrow$ 351 unique pairs

$\hat{I}(\hat{z}) = \sum_{i} e^{i(kx_i+\nu_i)} \Gamma_n(\hat{z})$

\[
\Gamma_n(\hat{z}) = e^{i(kx_i+\nu_i)}
\]

\[
\Gamma_n(\hat{z}) = \int_{D} e^{i(kx_i+\nu_i)} dA
\]

Monochromatic case

Suppose $E(x,y) = e^{i(kx+\nu_0)}$

\[
\langle E(x,y) E(x,y) \rangle = \int_{D} \Gamma(k,x)
\]

In vacuum $k = \frac{1}{c}$

\[
k = \frac{1}{c_b} = \frac{1}{c_0} c_0
\]

\[
\Gamma(k,x) = e^{i \omega \left[ \frac{1}{2} (x - v_0) \right]}
\]

Temporally incoherent case

Note that $\Gamma$ is a second moment.

If we see a temporally incoherent source, then we sum terms, not products: we expect that these implies the statistically independent.
i.e. the model for \( E(x,t) \) is \( (3.11) \) for a point source.

\[
E(x,t) = \int_0^L E(w) e^{i(kx-wt)} \, dw
\]

incident \( \Rightarrow \) \( E(w) \) is independent of \( E(v) \), \( w \neq v \).

Can be expressed as \( \langle E(w)E(w') \rangle \propto \delta(w-w') \).

**Rectangular Filter**

Suppose that we select (with a reason) only a portion of the spectrum; \( \int_{-a}^{a} \int_{-b}^{b} \).

i.e., we measure \( E_k(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(w) e^{i(kx-wt)} \, dw \).

Now we consider

\[
\Pi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(w) e^{i(kx-wt)} \, dw
\]

Since \( \Pi(w) \) is real for \( \Pi(w) = \Pi(-w) \), we can sum over frequency to summate the linear response:

\[
\Pi(x,t) = \frac{i}{\omega} \int \left( \int \frac{\Pi(w') \hat{E}(w')}{2\pi} e^{i(kx-wt)} \, dw' \right) e^{i(kx-wt)} \, dw
\]

\[
= \frac{i}{\omega} \int \int \frac{\Pi(w') \hat{E}(w')}{2\pi} e^{i(kx-wt)} \, dw'
\]

**Model for Homogeneously Incoherent Noise**

\[
X(t) = \text{white noise (white = flat power spectrum over all bands)}
\]

\[
\langle X(t) X(t+\tau) \rangle = \langle X \rangle^2 + \sigma^2 \delta(\tau)
\]

\[
X(t) \leftrightarrow \tilde{X}(\nu)
\]

\[
\langle \tilde{X}(\nu) \tilde{X}(\nu') \rangle \propto |\tilde{X}(\nu)|^2 \delta(\nu-\nu')
\]
\[ \mathcal{E}(t) = \text{constant} = 1 \quad \text{(while noisy)} \]

\[ \Gamma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \frac{\omega}{2} (t + \frac{1}{t})} d\omega \]

\[ \Gamma(t) = e^{i \frac{\omega}{2} (t + \frac{1}{t})} \frac{\sin \left[ \frac{\omega}{2} (t - \frac{1}{t}) \right]}{\frac{\omega}{2} (t - \frac{1}{t})} \]

\[ \text{The point is that the response } \Gamma(t) \text{ properly, one must find the peaks of the } \mathcal{E}(t) \text{ envelope.} \]

For a general background shape

\[ E_0(r, t) = \int d\omega \, H(\omega, r) E_0(\omega) e^{i \omega t} \]

\[ \Gamma(t) = \int d\omega \, |H(\omega, r)|^2 |E_0(\omega)|^2 e^{i \omega t} \]

\[ |E_0(\omega)|^2 = 1 \]

\[ \text{let } \omega' = \omega - \omega_0 \]

\[ \Gamma(t) = e^{i \frac{\omega'}{2} (t + \frac{1}{t})} \left( \int d\omega' \, |H(\omega', r)|^2 e^{i \omega' t} \right) \]

\[ \text{Fourier transform } \mathcal{F} \]

\[ \mathcal{F}[\mathcal{E}] \text{ sidewise } \sim \frac{1}{\sqrt{2\pi \omega_0}} \]

\[ \mathcal{F}[\mathcal{E}(t)] \text{ sidewise } \sim \frac{1}{\sqrt{2\pi \omega_0^2}} (t - \frac{1}{t}) \]

\[ \text{Fourier transform with } \delta \text{ point source in } \text{direction } \theta \]

\[ \text{Spatial part} \]
Extra

How to generate principal angles

C.F. Reciprocation Thm.

\[ \begin{align*}
\theta &= \arctan \left( \frac{y}{x} \right) \\
\phi &= \arctan \left( \frac{y}{x} \right)
\end{align*} \]

Some notes:

- \( \theta \) and \( \phi \) are in the same quadrant.
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See Ref. 1.5 for reciprocation from a dope sheet.

The dope sheet:

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\end{align*} \]
\[ V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(u) e^{iut} du \]

\[ V(u) = \text{Fourier transform of } V(t) \]

\[ V(u) = \frac{|V(t)|}{u} e^{i\phi(t)} \]

\[ \text{where } \phi(t) = \text{phase of } V(t) \]

\[ \text{Output } V(t) \text{ as} \]

1. Suppose \( \hat{E}_n \neq \hat{X} \) or \( \hat{E}_n \neq \hat{Y} \). The get a random radiation field.
2. To get a complex radiation, need the complexification:
   \[ V(u) = \hat{E}_n \]
   \[ \hat{E}_n = \hat{X} + \hat{Y} \]
   \[ \hat{E}_n = \hat{X} \hat{Y} \]

independent = wise guessers