Heat equation for general fluid (not ideal gas)

Want $T, p$ as the thermodynamic variables. Thus we wish to express

$$T \, ds = dq$$

(energy cons.)

in terms of $dT$ and $dp$.

$$T \, ds = T \left. \frac{\partial s}{\partial T} \right|_p \, dT + T \left. \frac{\partial s}{\partial p} \right|_T \, dp$$

$$\uparrow$$

$$c_p$$ ?

To evaluate $c_p$ need some $p, T$ cross derivatives. Use

$$T \, ds = de + p \, ds$$

Construct the combination that gives $p, T$ differentials.

$$d \left[ e - T \alpha + \alpha \rho \right] = d \rho - \alpha \, dT$$

Gibbs function

$$\frac{\partial S}{\partial p} \, dp + \frac{\partial S}{\partial T} \, dT$$

Since $\frac{\partial^2 S}{\partial p \partial T} = 0$, get

$$\left. \frac{\partial \alpha}{\partial T} \right|_p = - \left. \frac{\partial \alpha}{\partial p} \right|_T$$

This gives $c_p$ in the $T \, ds$ equation:

$$T \, ds = c_p \, dT - T \left. \frac{\partial \alpha}{\partial T} \right|_p \, dp$$
Example: Sound waves

Subscripts denote differentiation

\( \rho \) homogeneous medium

Density, pressure

\[ \rho_{\text{full}} = \bar{\rho} + \rho \]

\[ P_{\text{full}} = \bar{P} + P \]

Small amplitude \( \rho \ll \bar{\rho}, P \ll \bar{P} \)

Adiabatic, inviscid, \( \frac{\partial \bar{P}}{\partial \rho} \mid_{\text{adabat}} = \frac{\kappa \bar{P}}{\bar{\rho}} = \frac{c^2}{\rho} \)

2D motion \( \mathbf{v} = (u, w) \), \( \nabla \cdot \mathbf{v} = \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \)

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= \frac{1}{\bar{\rho}} \frac{\partial \bar{P}}{\partial x} = 0 \\
\rho_t + \bar{\rho} \nabla \cdot \mathbf{v} &= 0 \\
\rho_t - c^2 \rho_t &= 0
\end{align*}
\]

2, 3 \( \Rightarrow \) \( \rho_t = -c^2 \rho \frac{\partial \mathbf{v}}{\partial x} \), with (1) \( \Rightarrow \)

\[
\rho \frac{\partial \mathbf{v}}{\partial x} = 0
\]

Energy \( \mathbf{v} \cdot (1) \) and (2)

\[
\left( \frac{u^2 + w^2}{2} \right)_t + \mathbf{v} \cdot \frac{1}{\bar{\rho}} \nabla \bar{P} = 0
\]

KE per mass \( \Rightarrow \) KE creation rate of increase by pressure gradients
\[(2) + (3)\]
\[\frac{\partial}{\partial t} \left( \frac{p^2}{2} \right) + \rho c^2 \frac{\partial}{\partial x} p v = 0\]

Use divergence theorem and
\[\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \frac{\rho c^2}{2} \right) + \nabla \cdot \left( \rho c^2 p \right) - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} p \right) = 0\]

Pressure + kinetic energies
Energy flux cancels against KE production when added \(PE + KE\)

Magnitudes KE, pressure E
\[\frac{P}{K} \sim \frac{\rho c^2}{u^2}\]

Use \(e^{ikx-\omega t}\) plus dispersion relation and
\[\frac{P}{K} \sim 1\]
Atmospheric Waves

Interpretation of fundamental balances
Keep sufficient physics to retain all wave classes.

Assume a homogeneous medium: $p_{\text{full}} = p_0 + p$,
$p_{\text{full}} = p_0 + p$.

Local cartesian $\beta$-plane,
Conservation $f = 2\Omega \sin \phi = f_0$,
$f = \frac{df}{dy} = \frac{2\Omega}{a} \cos \phi$

Linearize: small amplitude motions,
No dissipation or forcing.
Classes + examples

Acoustic - use $\bigcirc$ terms. Valid for large $\frac{2}{c^2}$ and $\bigcirc$. Substitution of plane wave $\exp (k x - \omega t)$ mode:

\[
\frac{\partial^2 \rho}{\partial t^2} - c^2 \nabla^2 \rho = 0
\]

\[
\Rightarrow \frac{\omega^2}{k^2} = c^2
\]

Gravity - use $\bigcirc$ terms. Valid for $\omega \approx N$.

\[
\frac{\partial}{\partial t} \left( \rho g \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial x} \left( \rho \frac{\partial w}{\partial x} \right) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \rho g \frac{\partial w}{\partial x} = 0
\]

\[
\Rightarrow \frac{\omega^2}{N^2} = \frac{k^2}{k^2 + m^2}
\]

(exp $\exp (k x - m z - \omega t)$)

Rossby use $\bigcirc$ terms. First establish that geostrophy holds in $x, y$ plane. Then take curl of $x, y$ mom. to make vorticity.

\[
\begin{align*}
\text{rot } u &= \frac{1}{\rho_0} \frac{\partial \phi}{\partial y} \\
\text{rot } v &= -\frac{1}{\rho_0} \frac{\partial \phi}{\partial x}
\end{align*}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) + \beta \frac{\partial \phi}{\partial x} = 0
\]

\[
\omega = -\frac{\beta k}{k^2 + l^2}
\]