Outline of a systematic quasi-geostrophic derivation

Basic state
Define a horizontally uniform static basic state

$$\frac{\partial \Phi}{\partial \rho} = - \frac{RT}{\rho} \quad \Phi(p), T(p)$$

Let \[ \Phi_{\text{full}} = \Phi + \Phi \]

\[ T_{\text{free}} = T + T \]

Preserve coordinate equations of motion are then

$$\frac{d}{dt} \begin{bmatrix} \Phi \\ T \end{bmatrix} + \begin{bmatrix} 0 & \nabla \times \mathbf{v} \\ \nabla \Phi \end{bmatrix} + \begin{bmatrix} \nabla \times \mathbf{v} \times \mathbf{v} \\ \nabla \times \nabla \Phi \end{bmatrix} = 0$$

$$\frac{\partial \Phi}{\partial \rho} = - \frac{RT}{\rho}$$

$$\nabla \times \mathbf{v} + \frac{\partial \omega}{\partial \rho} = 0$$

$$\frac{\partial T}{\partial t} + \nabla \cdot \left( \mathbf{v} \frac{\partial T}{\partial \rho} \right) + \omega \frac{\partial T}{\partial \rho} - \kappa \frac{T}{\rho} \omega = 0$$

Define a measure of static stability (as before)

$$\frac{\partial T}{\partial \rho} - \kappa \frac{T}{\rho} = - \frac{H^2}{R} \frac{N^2}{\rho}$$

$$N^2 = \text{Brunt frequency}$$

$$H_s = \frac{RT}{g}$$

This is consistent with

$$N^2 = \frac{g}{T} \left( \frac{\partial T}{\partial \rho} + \frac{g}{\rho} \right)$$

(geometric coordinate)

Note: \( N^2 = N^2(p) \) not function of \( x, y, t \).

Coriolis parameter - use \( \beta \)-plane.

$$f = 2\pi \sin \phi = 2\pi \left[ \sin \phi_0 + \frac{d \sin \phi}{d \phi} (\phi - \phi_0) + \ldots \right]$$

but \( a(\phi - \phi_0) = y \) for local cartesian system.

\[ f = 2\pi \sin \phi_0 + \frac{2\pi \cos \phi_0}{a} y \equiv f_0 + \beta y \]
Scaled equations on paper:

Introduce dimensionless variables with hats. It is intended that these variables are $O(1)$.

\[
\hat{y} = \frac{y}{L}, \quad \hat{z} = \frac{z}{L}, \quad \hat{x}, \hat{y} = L(x, y)
\]

\[
\omega = \frac{\rho_0}{\rho} \hat{\omega}, \quad \hat{p} = \frac{\rho_0}{\rho} \hat{p}, \quad t = \frac{L}{\nu} \hat{t}
\]

\[
\Theta = f_0 U L \hat{\Theta}, \quad f = f_0 (1 + \Theta^2), \quad \beta = \frac{\beta L}{f_0}
\]

Substitute

\[
\frac{\nu}{L} \frac{d^2 \hat{\omega}}{dt^2} + f_0 U \nabla^2 \hat{\omega} + f_0 U \hat{\nabla} \cdot \hat{\Theta} = 0
\]

\[
\frac{f_0 U L}{\rho_0} \frac{\partial \hat{\Theta}}{\partial \hat{p}} = -\frac{\Delta T}{\rho_0} \frac{\hat{p}}{\hat{p}}
\]

\[
\hat{\nabla} \cdot \hat{\omega} + \frac{\partial \Theta}{\partial \hat{p}} = 0
\]

\[
\frac{U}{L} \frac{\Delta T}{\rho_0} \frac{d \hat{p}}{dt} - H_s N_s^2 \frac{R_u}{\rho_0 L} \frac{\hat{p}}{\hat{p}} = 0
\]

Hydrostatic eq. $\Rightarrow$ \( R \Delta T = f_0 U L \)

Let

\[
Ro = \frac{U}{f_0 L} \quad (a \, \text{number, } \ll 1) \]

\[
ri = \frac{H_s N_s^2}{U^2}
\]

Keep it as \( ri (p) \) then \( N_s, H_s \)

\[
(1+65)
\]

\[
Ro \frac{d \hat{\omega}}{dt} + \left( \frac{1}{\partial \hat{p}} \right) \left( \nabla \times \hat{\omega} \right) + \hat{\nabla} \cdot \hat{\Theta} = 0
\]

\[
\frac{\partial \hat{\Theta}}{\partial \hat{p}} = -\frac{1}{\hat{p}}
\]

\[
\hat{\nabla} \cdot \hat{\omega} + \frac{\partial \Theta}{\partial \hat{p}} = 0
\]

\[
Ro \frac{d \hat{p}}{dt} + \hat{p} - Ro^2 \hat{ri} \frac{\partial \hat{\Theta}}{\partial \hat{p}} = 0
\]
Sizes. \( \beta \) midlatitude long waves, cyclones
\[
R_i \sim 0.15
\]
\[
b \sim 0.1
\]
The Richardson \# does not depend on \( L \), only on \( U \) and the stratification. \( \beta \) troposphere
\[
R_i \sim 50
\]
Notice \( R_i \beta \sim 1 \). Will return to interpretation of this.
Assume \( R_i^2 \beta \sim O(1) \) in what follows.

Expansion in \( R_i \)

Let
\[
\mathbf{k} = \mathbf{k}^{(0)} + R_i \mathbf{k}^{(1)} + R_i^2 \mathbf{k}^{(2)} + \ldots
\]
and same for other variables.

Leading order: assume \( b = O(R_i) \)
\[
\nabla \times \mathbf{v}^{(0)} + \nabla \times \mathbf{k}^{(0)} = 0
\]
\[
\frac{\partial \mathbf{k}^{(0)}}{\partial t} = - \frac{\mathbf{k}^{(0)}}{P}
\]
\[
\nabla \cdot \mathbf{k}^{(0)} + \frac{\partial \mathbf{w}^{(0)}}{\partial P} = 0
\]
\[
\mathbf{w}^{(0)} = 0 \quad \text{(heat eq.)}
\]

Geostrophy! Note \( \mathbf{w}^{(0)} = 0 \) is consistent with \( \nabla \times \mathbf{w}^{(0)} = 0 \)!

Holton calls these the geostrophic velocity etc. (Eq. 6.7).

Note \( f = f_0 \) in the balance. No latitude variation of \( f \).
Next order

\[ R_0 \frac{\partial \Phi^{(0)}}{\partial t} + b \frac{\hat{v} \times \hat{\omega}^{(0)}}{R_0} + R_0 \hat{v} \times \hat{\omega}^{(1)} + R_0 \hat{v} \cdot \hat{\Phi}^{(1)} = 0 \]

\[ R_0 \frac{\partial \Phi^{(1)}}{\partial p} = R_0 \frac{\hat{v}^{(1)}}{\hat{p}} \]

\[ \nabla \cdot \hat{\omega}^{(0)} + \frac{\partial \hat{\Phi}^{(0)}}{\partial p} = 0 \]

\[ R_0 \frac{\partial \Phi^{(0)}}{\partial t} + (R_0 \hat{v} \cdot \hat{R}) R_0 \hat{\omega}^{(0)} = 0 \]

Let \( b = \frac{1}{R_0} = O(1) \). Then this is a consistent set.

Since \( \Phi^{(0)} = 0 \), hence

\[ \frac{\partial \Phi^{(1)}}{\partial t} = \frac{\partial \hat{\Phi}^{(1)}}{\partial x} \frac{\partial \hat{\Phi}^{(1)}}{\partial x} + \frac{\partial \hat{\Phi}^{(1)}}{\partial y} \frac{\partial \hat{\Phi}^{(1)}}{\partial y} \quad \text{no vertical derivative!} \]

Simplify

\[ \frac{\partial u^0}{\partial t} + \frac{\partial u^1}{\partial x} + \frac{\partial u^1}{\partial y} - \frac{b}{R_0} \frac{\partial \hat{v}^{(0)}}{\partial x} - \frac{\partial \hat{v}^{(0)}}{\partial y} + \frac{\partial \hat{v}^{(1)}}{\partial x} = 0 \]

Make vorticity

\[ \frac{\partial}{\partial t} \left[ \frac{\partial \hat{v}^{(0)} - \partial \hat{u}^{(0)}}{\partial x - \partial y} \right] + \frac{\partial \hat{v}^{(0)} - \partial \hat{u}^{(0)}}{\partial x - \partial y} - \frac{\partial \hat{v}^{(1)}}{\partial y} + \frac{\partial \hat{v}^{(1)}}{\partial x} \]

\[ \frac{\partial}{\partial t} \left[ \frac{\partial \hat{v}^{(0)} - \partial \hat{u}^{(0)}}{\partial x - \partial y} \right] + \frac{\partial \hat{v}^{(1)}}{\partial x} + \frac{\partial \hat{v}^{(1)}}{\partial y} = 0 \]

Solve heat eq. for \( \Phi^{(0)} \), substitute.

Notice: \( 1^{st} \) order terms gone! Closes for \( 0^{th} \) order terms.
Pull out $\frac{d}{dt}$ from $R_i$ term. Extra terms:

$$\frac{d}{dt} \frac{\partial}{\partial p} \left[ \right] = \frac{\partial}{\partial p} \frac{d}{dt} \left[ \right] - \left[ \frac{\partial u}{\partial p} \frac{\partial T}{\partial x} + \frac{\partial v}{\partial p} \frac{\partial T}{\partial y} \right] \left[ \text{other terms} \right]$$

Thus

$$d^{(0)} \left[ \frac{\partial \hat{v}^0}{\partial x} - \frac{\partial \hat{u}^0}{\partial y} + \hat{v} \hat{u} \hat{p} + \frac{1}{\beta} \left( \frac{\partial^2 \Phi^0}{\partial R^2} \frac{\partial \hat{p}}{\partial p} \right) \right] = 0$$

Return to dimensional variables. Use

$$v = \frac{1}{f} \frac{\partial \Phi}{\partial x}$$

$$u = -\frac{1}{f} \frac{\partial \Phi}{\partial y}$$

$$-\frac{\partial}{\partial t} \left[ \nabla \Phi + \beta t \hat{v} \right] + \frac{1}{\beta} \left( \frac{\partial^2 \Phi}{\partial \hat{p}} \left( \frac{f_0^2}{N^2 H^2} \hat{p} \frac{\partial \hat{p}}{\partial \tilde{p}} \right) \right] = 0$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \frac{1}{f_0} \frac{\partial \Phi}{\partial y} \frac{\partial}{\partial x} + \frac{1}{f_0} \frac{\partial \Phi}{\partial x} \frac{\partial}{\partial y}$$

This is the quasigeostrophic approximation for atmospheric dynamics.

Comments

1) $L$ emerges spontaneously in baroclinic instability. Max growth rate for $\nabla^2 \Phi$ same order as $\frac{\partial}{\partial \tilde{p}}$ term

$$\frac{L}{L} \sim \frac{f_0^2}{N^2 H^2} \left( \text{radius of deformation} \right)$$

2) $\beta$ is a distinct physical parameter. Not by coincidence it is same order as $R_0$.

3) The conserved quantity is called the quasigeostrophic potential vorticity.

Burger number
Ernst's potential vorticity

Back to full vorticity equation, no approximations.

Let

\[ \omega_a = 2 \Omega + \omega \]

absolute vorticity

\[
\frac{d}{dt} \omega_a = - \omega_a \nabla \cdot \mathbf{v} + (\omega_a \cdot \nabla) \mathbf{v} - \nabla \left( \frac{1}{\rho} \right) \times \nabla \rho - \nabla \times \mathbf{D}
\]

Ideal gas

\[ \Theta = T \left( \frac{f_0}{\rho} \right)^k = \frac{T}{\rho R} \left( \frac{f_0}{\rho} \right)^k = \Theta (\rho \rho) \quad \rho = \rho R T
\]

\[
\frac{d\Theta}{dt} = - \frac{\Theta Q}{\rho c_p T}
\]

Dot vorticity eq. with \( \Theta \). Because \( \Theta \) is the sum of two terms || \( \Theta \rho \) and \( \Theta \rho \), it is perpendicular to \( \nabla \left( \frac{1}{\rho} \right) \times \nabla \rho \).

Theorem goes through because this operation picks out a component of the vorticity that is not altered by density gradients.

\[
\nabla \Theta \cdot \frac{d\omega_a}{dt} = -\left( \nabla \Theta \cdot \omega_a \right) \nabla \cdot \mathbf{v} + \nabla \Theta \cdot (\omega_a \cdot \nabla) \mathbf{v} - \nabla \Theta \cdot \nabla \times \mathbf{D}
\]

\[
\frac{d}{dt} (\omega_a \cdot \nabla \Theta) = \omega_a \cdot \frac{d}{dt} \nabla \Theta = \nabla \Theta \cdot \omega_a \frac{d\rho}{dt} + \nabla \Theta \cdot \mathbf{v} - \nabla \Theta \cdot \nabla \times \mathbf{D}
\]

\[
\frac{d}{dt} (\omega_a \cdot \nabla \Theta) = \nabla \Theta \cdot \omega_a \frac{d\rho}{dt} = (\omega_a \cdot \nabla) \frac{d\Theta}{dt} - (\omega_a \cdot \nabla) \nabla \cdot \Theta + \nabla \Theta (\omega_a \cdot \nabla) \mathbf{v}
\]

\[
+ \nabla \Theta \cdot \omega_a \frac{d(\rho)}{dt}
\]

\[
\therefore \frac{d}{dt} \frac{\omega_a \cdot \nabla \Theta}{\rho} = \frac{1}{\rho} \left( \omega_a \cdot \nabla \left( \frac{\Theta Q}{\rho c_p T} \right) - \nabla \Theta \cdot \nabla \times \mathbf{D} \right)
\]

No approximations!

In absence of \( Q \) or \( D \), \( \omega_a \cdot \nabla \Theta \) is conceived following motion.
Interpretation

\[ \int_{\partial A} \mathbf{F} \cdot d\mathbf{A} = \oint_{C} \mathbf{F} \cdot d\mathbf{l} \text{ circulation.} \]

\[ \frac{d\mathbf{r}}{dt} = \mathbf{F}(\mathbf{r}) \times \mathbf{v} \text{ if } \mathbf{F} = \mathbf{F}(\mathbf{r}) \text{ and } \mathbf{r} = \mathbf{r}(p) \text{ this integrates to zero around loop.} \]

In \( \Theta = \text{const. surface} \), \( \mathbf{F} = \mathbf{F}(p) \)
\( \nabla \Theta \cdot \mathbf{w} \) picks out \( \mathbf{w} \perp \text{const } \Theta \text{ surface} \).

Comment

Although conceptually important, Ertekin's PV theorem is not directly useful because it does not give a useful eq. of motion

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left[ \frac{(2 \nabla \cdot \mathbf{v}) \cdot \nabla \Theta}{\rho} \right] = 0 \text{ inviscid, adiabatic} \]

No simple expressions for \( \mathbf{v}, \Theta, \rho \ldots \).

\[ \rightarrow \text{ Quasi-geostrophic approximation accomplishes this.} \]