Problem 1

Tritium ($^3\text{H}$) is a radioactive isotope of hydrogen. The nucleus decays (by emitting an electron and an antineutrino), changing from a triton (one proton and two neutrons) to a $^3\text{He}$ nucleus (two protons and one neutron). This changes the charge of the nucleus from $e$ to $2e$. For this problem, assume that the change is instantaneous. That is, the orbital electron in a tritium atom sees the charge at the nucleus suddenly change from $e$ to $2e$, and is unaffected by the emitted electron which leaves the atom quickly. Also, since an electron is much less massive than a proton, neglect the change in the reduced mass of the atom.

{a} If the orbital electron in the original tritium atom is in the ground state, what is the probability that it is in the ground state of the $^3\text{He}$ ion immediately after the decay?

**HINT:** The electron wave function immediately after the decay is the ground state wave function of a hydrogen atom, because that is what it was immediately before the decay. And the ground state wave function of a hydrogen atom in polar coordinates is $(\pi a^3)^{-1/2} e^{-r/a}$, where $a = a_0/Z$, $a_0 =$ Bohr radius, $Z =$ nuclear charge.

{b} Calculate the mean energy radiated by the ion after the decay, when the electron makes transitions to the ground state if it was left in an excited state.

**HINT:**

\[
\text{Mean energy radiated} = (\text{prob. } n = 1) \times (E_1 - E_1) + (\text{prob. } n = 2) \times (E_2 - E_1) + (\text{prob. } n = 3) \times (E_3 - E_1) + \cdots
\]

Problem 2

A one-dimensional harmonic oscillator is constructed in such a way that the spring constant may be adjusted. The oscillator is in its lowest energy state when suddenly at $t = 0$ the spring constant is reduced to zero, without changing the wave function. What is $\Psi(x, t)$ for $t > 0$?

**HINT:** You may find the following integral useful:

\[
\int_{-\infty}^{\infty} e^{-ak^2+bk} dk = \sqrt{\frac{\pi}{a}} e^{b^2/4a}
\]

Problem 3

**NOTE:** For this problem, the following may be useful:

\[
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p}_x \right)
\]

\[
\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i}{m\omega} \hat{p}_x \right)
\]

\[
\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle, \quad \hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle
\]

At $t = 0$ the state of a 1-d harmonic oscillator has the following properties:
(i) An energy measurement will yield only the results $\frac{1}{2}h\omega$ or $\frac{3}{2}h\omega$ with equal probability.

(ii) A measurement of position will give a mean value that is as large a positive value as is consistent with condition (i).

{a} Find $|\Psi(t = 0)\rangle$
{b} Find $|\Psi(t)\rangle$
{c} At $t = T$, a measurement of momentum gives the largest possible positive mean value. What is the smallest value of $T$ for which this is true?

**Problem 4**
Find the probability that the electron in a hydrogen atom is at a distance from the nucleus greater than its energy would permit in classical mechanics. Assume the electron is in the ground state.

**Problem 5**
The Hamiltonian for the hyperfine splitting in hydrogen contains a term that is proportional to the spins of the electron and the proton:

$$H = cs_1 \cdot s_2$$

where $c$ is a real constant.

{a} Find the energy eigenvalues of this operator by working in the basis of eigenstates of the total spin, $S = s_1 + s_2$, namely $|s m\rangle$.

Hint: Consider $S^2$.

{b} Find the energy eigenvalues again, working in the basis $|m_1 m_2\rangle$. Get the $4 \times 4$ Hamiltonian matrix and find its eigenvalues explicitly.

{c} Since $H$ is independent of time, one can write down the general solution for $|\psi(t)\rangle$ as a superposition of the energy eigenstates times suitable time-dependent factors. Do so, using the eigenstates from part (a). Your result should have 4 arbitrary constants.

{d} Suppose that at $t = 0$ particle 1 has spin up along the $z$-axis, while particle 2 is spin down. Find $|\psi(t)\rangle$.

{e} As a function of time, compute the probability that both particles have spin up along $z$, and also that particle 1 has spin up while particle 2 has spin down.

**Problem 6**
Use perturbation theory to find the shift in eV of the ground-state energy of the hydrogen atom because the nucleus is not really a point charge. Assume the proton is a uniformly distributed spherical charge cloud of radius $R = 1$ fermi. Neglect fine structure.

Hint: Find the potential both inside and outside $R$ and subtract the Coulomb potential to get $H'$. You can simplify the subsequent calculation by noting that $R \ll a_0$.

**Problem 7**
A 3-dimensional rotator with moment of inertia $I$ and electric dipole moment $d$ along the rotation axis is placed in a uniform electric field $E$ that makes an angle $\theta$ with $d$. Evaluate the first nonvanishing correction to the ground-state energy of the rotator, treating $E$ as a perturbation. Show that such a treatment is valid provided $E \ll \hbar^2 / dI$. 

Problem 8

When an atom is placed in a uniform external electric field, the energy levels are shifted—the so-called Stark effect. Consider a hydrogen atom in its ground state \( n = 1 \) which is immersed in a uniform electric field \( \mathcal{E} \) in the \( z \) direction. Neglect fine structure effects.

\{a\} What is the Hamiltonian due to the electric field? Give the expression in cartesian and in spherical polar coordinates.

\{b\} Show that the first-order correction to the ground-state energy is zero.

\{c\} The second-order change in the ground-state energy is \( \Delta E = \frac{1}{2}\alpha \mathcal{E}^2 \), where \( \alpha \) is the polarizability of the hydrogen atom (not to be confused with the fine structure constant). Show that

\[
\alpha = 2e^2 \sum_{n=2}^{\infty} \frac{|z_{n1}|^2}{E_1 - E_n}
\]

and write down an expression for \( z_{n1} \) in terms of an integral involving radial hydrogen wavefunctions. Do not try to evaluate the integral.

\{d\} Show by dimensional analysis from eqn. (1) that \( \alpha \approx -a_0^3 \) \( (a_0 = \text{Bohr radius}) \). (It can be shown that the exact evaluation of eqn. (1) gives \( \alpha = -\frac{9}{4}a_0^3 \).

Problem 9

When an atom is placed in a uniform external electric field, the energy levels are shifted—the so-called Stark effect. The Stark effect is usually quadratic in the applied electric field. However, if there are degenerate energy levels of opposite parity, there is a linear Stark effect. Calculate the splitting of the 4 degenerate \( n = 2 \) levels of hydrogen using degenerate perturbation theory. Use symmetry arguments as much as possible to decide which matrix elements must be zero.