THEORY AND ANALYSIS OF INTERPLANETARY SCINTILLATIONS

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THEORY AND ANALYSIS OF INTERPLANETARY
SCINTILLATIONS

A Thesis
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Doctor of Philosophy

by
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ABSTRACT

The thesis investigates several problems of the theory of interplanetary scintillations and the analysis of observations. The propagation of a wave deep into a medium having random spatial variations of index of refraction is discussed with geometrical optics and a wave optics model. It is shown with geometrical optics that nearby rays typically spread apart exponentially with propagation distance. Consideration of the wave propagation takes into account the exponential spreading of rays. The wave optics model consists of an arbitrary number of thin parallel phase changing screens and it permits exact evaluation of certain correlation functions. The two approaches are compatible and both indicate that a deep medium is equivalent in several respects, but not all, to a thin phase changing screen.

Observations of weak interplanetary scintillations are described which indicate a power-law wavenumber spectrum of electron density irregularities. The theory of scattering and scintillations for power law spectra is worked out. In contrast with cases where there is only a single dominant irregularity size, two new effects occur: first, the position of a radio source may vary randomly with time simultaneous with angular broadening and weak or strong scintil-
lations. Secondly a sequence of narrow equi-spaced pulses received through a medium with a power law spectrum may show a random variation in arrival time. The power law theory is compared with observations of interplanetary scattering and weak and strong scintillations. There is good quantitative agreement on all the points examined.

A theory connecting density and magnetic field fluctuations is proposed for the interplanetary medium. The theory is compatible with magnetic field observations by the Mariner 2 space probe, and it implies some properties of the density fluctuations deduced from interplanetary scintillations. The theory suggests a close connection between the processes causing small scale (\( \sim 50 \text{ km} \)) density fluctuations observed in interplanetary scintillations and the large scale (\( > 10^4 \text{ km} \)) magnetic field variations detected by Mariner 2.

A theory of interplanetary-interstellar scintillations is developed and applied to preliminary observations of the intensity variations of the pulsar CP 0950. The limited comparison of theory with observations shows good agreement.
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LIST OF SYMBOLS

The parenthesis below, for example ( II ), indicate usage only in Part II of the thesis. No parenthesis indicates uniform usage.

\( a \) = irregularity size for case of Gaussian spectrum.
\( a_n \) = scattering amplitude ( II and IV ).
\( A \) = for type A irregularities.
\( A(x) \) = scalar wave amplitude.
\( b \) = projected baseline of interferometer ( I and II ).
\( b_j(\gamma) \) = ray optics matrix function ( III ).
\( b_m \) = scattering amplitude ( IV ).
\( B \) = for type B irregularities.
\( B_j \) = ray optics transformation matrix ( III ).
\( c \) = speed of light.
\( C \) = for type C irregularities.
\( C_1, C_2, \ldots \) etc. = numerical constants.
\( D \) = distance of observer usually from front of screen, but in Part II the distance from the center of the plasma slab.
\( e = 2.718 \ldots \)
\( e_\lambda \) = charge of \( \lambda \) th species.
\( E(x) \) = electric field.
\( f \) = frequency, usually radio, but in (II) the frequency of scintillation spectra.
\( f(x) \) = certain functions.
\( F(q) \) = wavenumber spectrum of irregularities.
\( g(x) \) = certain functions.
\( G_A(\zeta), G_M(\zeta), \ldots \) etc = dimensionless functions of dimensionless arguments.
\( h(x) = \) certain functions.
\( \mathcal{H}(x), \delta \mathcal{H}(x) = \) magnetic field of wave in (I) or of medium(II).
\( i = (-1)^{1/2} \) and also integer index.
\( I(x) = \) radio wave intensity.
\( j(x) = \) current density.
\( J_n(\xi) = \) Bessel function of order \( n \).
\( J(x) = \log \) intensity (III).
\( k = 2\pi/\lambda = \) radio wavenumber.
\( L = \) focal length of irregularities.
\( L = \) total thickness of irregular medium.
\( m = \) modulation or scintillation index.
\( m_\alpha = \) mass of \( \alpha \)th species.
\( M(q), M_F(f), etc. = \) various intensity spectral functions.
\( n = \) integer index, number of scatterings in (I).
\( n(x) = \) index of refraction.
\( N = \) total number of screens in multiple thin screen model(III).
\( N(x) = \) electron number density (I and II).
\( \sigma = \) infinitesimal.
\( O = \) operator (I).
\( O(x) = \) order of magnitude of \( x \).
\( p = \) sine( solar elongation ) (II).
\( \rho, \rho = \) ray momentum (III).
\( P_{\rho\rho} = \) cross phase term in two screen model (IV).
\( q, q, \omega = \) wavevectors, wavenumbers of modes of irregularity spectrum.
\( Q = \) wavenumber of elliptical spectrum (II).
\( r = \) magnitude of radial vector, and turbule size of interest.
\( r_e = \) classical radius of the electron.
R = radial distance from the sun (II).

s = logarithmic slope of wavenumber spectrum (I and II).

S = ray path parameter (III).

\( \mathcal{J} \) = Poynting flux (I).

\( t \) = time

\( T \) = time interval (II).

\( u, \bar{u} \) = speed, velocity, of bulk motion of plasma slab.

\( U, \bar{U} \) = energy density of wave, modes of fluctuation spectrum.

\( \bar{U} \) = velocity of distant screen in two screen model.

\( \bar{v}_\alpha \) = phase velocity of \( \alpha \)th mode (II).

\( \bar{v} \) = transverse two-dimensional vector.

\( w_n \) = discrete probability, \( n \) = integer subscript.

\( W(\bar{V}) \) = probability density for \( \bar{V} \).

\( \bar{x} = (x,y) \), \( \bar{x} = (x,y,z) \), = transverse, three-dimensional, position vectors.

\( \bar{X}(z) \) = transverse coordinate of a ray (III).

\( \bar{Y}(z) \) = transverse coordinate of a ray (III).

\( z \) = longitudinal coordinate in direction of incident wave.

\( \gamma \) = distance to near screen in two screen model (IV).

\( Z \) = distance to far screen in two screen model (IV).

\( Z(s) \) = longitudinal coordinate of a ray (III).

* * * * * * *

\( \alpha \) = dimensionless scaling parameter of wave equation (I, III).

\( \alpha \) = species label (I), and mode label (II).

\( B \) = scintillation strength parameter (I and II).

\( B_1, B_2 \) = scaling factors (IV).

\( \gamma' \) = angle (II).
\( \Gamma(\xi) \) = Gamma function.
\( \xi \) = reduced logarithmic slope of wavenumber spectrum (I, II).
\( \Delta(\xi) \) = modified ray separation (III).
\( \xi \) = dimensionless scaling parameter of wave equation (I, III).
\( \xi(\xi) \) = dielectric constant.
\( \psi \) = dimensionless variable.
\( \zeta \) = backwards z-direction (III).
\( \gamma \) = dimensionless variable.
\( \eta(\alpha) \) = cross wavelength amplitude correlation (II, III, IV).
\( \theta \) = scattering angle usually reserved for cases of diffractive scattering.
\( \Theta \) = scattering angle usually reserved for cases where geometrical optics holds.
\( \kappa \) = dimensionless scaling parameter of wave equation (I).
\( \kappa \) = ray spreading parameter (III).
\( \lambda \) = radio wavelength.
\( \mu \) = departure of index of refraction from unity.
\( \xi \) = solar elongation of a source (II).
\( \xi \) = dimensionless ray separation (III).
\( \pi \) = 3.14..., usually.
\( \varphi(\xi) \) = charge density (I).
\( \varphi(\xi) \) = mass density (II).
\( \varphi(\xi), \varphi(\xi) \) = different correlation functions (I-IV).
\( \sigma(z) \) = reduced ray separation variance (III).
\( \zeta(x) \) = functions of temporary interest.
\( \tau \) = time interval.
\( \Theta(x, y) \) = phase shift.
\( \Phi_z(q_x, q_y) \) = wavenumber spectrum of the phase.

\( \chi \) = angle between two sources (III).

\( \varphi(r) \) = r.m.s. phase difference (I and II).

\( \psi \) = eikonal function of ray optics (I and III).

\( \psi \) = phase function in (IV).

\( \omega \) = radio angular frequency.

\( \Omega \) = angular frequency of component in irregularity spectrum.
Introduction to Thesis:

In a number of situations of astronomical interest the electromagnetic waves from a distant radio source traverse an irregular medium of free electrons with density say $\delta N(x)$, where the index of refraction $n(x)$,

$$n(x) = \left(1 - \frac{\omega_p^2(x)}{\omega^2}\right)^{1/2} \quad \omega_p^2(x) = \frac{4\pi e^2 N(x)}{m}$$

($\omega$ = radio angular frequency, $r_e$ = classical radius of the electron, and $c$ = speed of light) has fine scale irregular variations over distances much smaller than the depth of the medium. These fine scale variations alternatively speed up and retard different transverse parts of an initially plane wavefront, giving a randomly phase an amplitude modulated wave. The modulation may be observed for example in the intensity variations of a radio star. These variations are termed scintillations.

The different media which may affect the radio wave from a distant source include the ionosphere, the interplanetary medium (or solar wind), and the interstellar medium. In this order the three media may be characterized by their distances from the earth, $\simeq 300$ km, $\simeq 1.5 \times 10^8$ km, $> 6 \times 10^{14}$ km; their typical electron density fluctuations, $\sim 10^4$ electrons/cm$^3$, 1 el./cm$^3$, $10^{-4}$ el/cm$^3$; the size of their irregularities important for the scintillations, $\sim 5$ km, 100 km, $> 10^6$ km; and the typical velocities of the media transverse to the line of sight,
0.2 km/sec, 400 km/sec, 50 km/sec.

The thesis is concerned with the theory and analysis of observations of interplanetary scintillations (IPS). IPS appear for example as rapid time scale of a second variations in intensity of small radio sources at meter wavelengths. These variations are caused by electron density irregularities carried past the line of sight by the solar wind.

Different observations of the radio sources are assumed made with a typical radio telescope or interferometer. An example of a telescope is that at the Arecibo Ionospheric Observatory which has an aperture of diameter 310 m. and a beam width of \( \sim 10\text{'}\) min. of arc at 430 MHz. An interferometer relevant to measurements on the interplanetary medium has a baseline \( \sim 1\) km at 40 MHz.

Interplanetary scintillations were discovered by Hewish, Scott, and Wills (1964) (see also Douglas, 1964; Cohen, 1965; Douglas and Smith, 1967) during a survey of sources at 178 MHz. Unusual fluctuations in intensity were noticed in recordings on the sources 3C-119, 3C-138, and 3C-147, whereas 85 other sources in the survey showed no fluctuations. Two of the three sources were noted to have exceedingly small angular diameters, and subsequently systematic observations of the small source 3C-48 (and others) were obtained at 178 MHz for solar elongations \( \xi \) ( = angle between source and the sun) in the range \( \rho = \sin(\xi) = 0.4 \) to 0.9 a.u. (1 a.u. = astronomical unit = 1.5x10^8 km.).
Normal ionospheric scintillations were ruled out by the fact that scintillations introduced by a layer at a distance $D$ from the earth are quenched if the intrinsic source diameter $\theta_I$ is much larger than $\frac{\lambda}{2\pi \theta_S} D$ where $\lambda$ is the radio wavelength and $\theta_S$ is the r.m.s. deflection angle of waves passing through the medium (Pisareva, 1959; Briggs, 1960). In order for a source known to have a size $\theta_I \lesssim 10''$ to not show ionospheric scintillations, necessarily $\theta_S \lesssim 10^\circ$, but in such case the apparent or observable source diameter $\varepsilon \theta_A \lesssim \theta_S \gg \theta_I$, which is a contradiction. For interplanetary scintillations, however, $D$ is very much larger, of the order of an astronomical unit, and the scintillations of a $\theta_I = 10''$ diameter source are quenched if $0.05 < \theta_S \ll \theta_I$, which is consistent.

Sometime earlier, diameter broadening of sources due to scattering in the solar corona had been observed (Hewish, 1955; Vitkevich, 1958). For example, Hewish (1955) measured a strong fall-off, at about 6 solar radii, in the fringe visibility of the Crab Nebula with interferometers at 38 MHz (49$\lambda$ baseline) and 178 MHz (105$\lambda$ baseline) as the source approached the sun. The effect was ascribed to multiple scattering in the solar corona, which would give even a point source an apparent size $\theta_A \lesssim \theta_S$; for $\theta_S$ larger than the lobe spacing and $\theta_S > \theta_I$ the visibility would be small. Hewish invoked a theoretical expression for $\theta_S$ due to Fejer (1953),
\[ \Theta_s = \frac{r_e \lambda^2 L^{1/2}}{2^{3/4} \pi^{3/4}} \left[ \frac{\delta N^2}{\alpha} \right]^{1/2} \]  

(2)

where \( L \) = the effective thickness of the scattering medium, of the order of \( p = \sin(\mathcal{E}) \) a.u.

\( \lambda \) = radio wavelength.

\( r_e \) = classical radius of electron, \( 2.82 \times 10^{-13} \text{cm} \).

\( \delta N^2 \) = mean square of electron density fluctuations.

\( a \) = electron density correlation length as defined in the correlation function \( \xi(r) = \exp(-r^2/2a^2) \).

In order for the scattering to have a significant effect on the fringe visibility (and for the validity of equation (2)), the r.m.s. phase shift suffered by the wave, \( \phi_L \), must be larger than a radian (Fejer, 1953),

\[ \phi^2_L = (2\pi)^{1/2} r_e^2 \lambda^2 \delta N^2 L \alpha > 1 \]  

(3)

For given observed \( \Theta_s \) there is thus a lower limit on the scale size \( a \) compatible with equations (2) and (3), \( a > \frac{\lambda}{(2\pi \Theta_s)} \), and Hewish found a value \( a > 0.2 \text{ km} \), corresponding to \( \Theta_s = 15' \) of arc at an elongation of 5 solar radii at 81.5 MHz. However, no definite value for the scale size and r.m.s. fluctuations \( \delta N^2 \) could be derived.

The observed source broadening increased rapidly with decreasing solar elongation and varied with wavelength appar-
ently in accord with theory, $\theta_s \propto \lambda^2$.

At the time of Hewish's work on coronal scattering, the nature of the interplanetary medium at large distances from the sun was only poorly understood. It was not known if material was falling into the sun or flowing out or even static. Parker (1960) argued that the static model of Chapman (1959) probably required an unreasonably large inward pressure from the interstellar medium, and that another possibility was that of outward and at large distances supersonic flow of material from the solar corona, that is, a solar wind. In a number of satellite experiments in 1958 - 1961 a steady outflowing plasma was observed having a speed $u \sim 400 \text{ km/sec}$ and typical density $\sim 10 \text{ el./cm}$.

Further coronal scattering studies (Hewish and Wyndham, 1963; and references therein) carried out with three leg interferometers at lower frequencies and over a wider range of elongations provided (i) dependence of the scattering angle on elongation, (ii) evidence for enhanced scattering in the equatorial plane, (iii) evidence that the irregularities are elongated by as much as two to one, longer in the radial direction (for elongations $p = 15$ to 90 solar radii), and (iv) evidence for enhancement of the scattering around the sunspot maximum of the solar cycle. Hewish and Wyndham suggested an upper limit on the scale size $\Delta$, by arguing that under conditions of strong scattering either (a) many scattered rays are received simultaneously within a cone of diameter $\theta_s$, or (b) the source
position changes by the observed scattering angle $\theta_A$ (which is less than or equal to $\theta_S$) in a receiver integration time, $t_{int}$. As discussed by Hewish and Wyndham, and later in this thesis, case (a) corresponds to observations at a distance $D > a/\theta_S$. In case (b) $D < a/\theta_S$ or equivalently $a > (\lambda D/2\pi)^{1/2}$. For case (b) $\theta_A \sim (u t_{int}/a) \theta_S$ if $u t_{int} \leq a$, which implies $u \geq \theta_A (D/t_{int})$; or $\theta_A \sim \theta_S$ if $u t_{int} \geq a$, which implies $u \geq \theta_A (D/t_{int})$ also. Case (b) was ruled out because for $t_{int} = 5$ seconds the wind speeds required are rather large, $\sim 10^4$ km/sec. For case (a), at an elongation $p = 0.5$ a.u., where an estimate of $\theta_S$ is $1''$ at 178 MHz, one finds that $a \ll 10^3$ km.

Hewish, Scott, and Wills (1964) interpreted the one second radio star twinkling in terms of plasma irregularities carried by the solar wind past the line of sight. The diffraction pattern of interplanetary irregularities was assumed to drift without significant distortion across the ground with some component of the solar wind velocity $u$.

Parameters of the diffraction pattern and in particular the scale size depend on certain characteristic of the interplanetary plasma which were unknown at the time, but from the thin screen theory of Pisareva (1959) and Mercier (1962) it was known that the basic quantities were the density scale size $a$ and the phase shift $\phi_L$, already mentioned in connection with the closely related coronal scattering. Briefly, weak scintillations occur for $\phi_L^2 \ll 1$, with
intensity variations small compared with the mean, and a
typical diffraction scale size (more precisely the e-folding
lag of the spatial intensity auto-correlation) of $\approx 2^{\frac{1}{2}} \lambda$, 
nearly independent of distance from the screen; that is,
roughly the same in Fresnel and Fraunhofer limits. Strong
scattering occurs for $\theta^2_L \gg 1$, but does not necessarily
entail strong intensity variations at a distance $D$ unless
$D$ is equal or larger than a typical focal length for the
irregularities, $\ell_L = a/\theta_0$ (Salpeter, 1967). Observe
that $D > \ell_L$ is Hewish and Wyndham's (1963) case (a) and
that $D < \ell_L$ is their case (b). The different scintillation
regimes may be summarized with the 'phase' diagram intro-
duced by Cohen et al. (1967) shown in Figure (1). In each
domain the scale of the diffraction pattern on the ground
($\xi_0$) and the fractional r.m.s. intensity variations ($m$)
are indicated. The Fresnel and Fraunhofer limits are also
pointed out.

Hewish, Scott, and Wills (1964) interpreted their
observations in terms of the theory appropriate to domain
III of Figure (1), where the diffraction scale size $\xi_0 \approx
\sqrt{2\pi} \theta_0$. The scattering angle $\theta_0$ was derived by ex tra-
polation in wavelength and elongation of the coronal measure-
ments of Hewish and Wyndham (1963), which gave $\theta = 1''$
at
178 MHz and $p = 0.5$ a.u. Assuming $u = 400$ km/sec, the scin-
tillation time scale, $\tau = \xi_0/u$, is about 0.4 seconds, in
rough accord with observations. However, at large elonga-
tions the modulation indices observed were rather small
FIGURE 1: Scintillation Regimes.
(m \lesssim 0.2) suggesting from Figure (1) diffraction in domain I or II where the scale size is $\xi \approx 2^{1/2} a$. In this case, assuming $\gamma = 1$ sec and $u = 400$ km/sec, one finds that $a \approx 200$ km.

Confirmation that the diffraction pattern drifts across the ground with speeds of 300 - 500 km/sec was first obtained in simultaneous observations at two (Hewish, Dennison, and Pilkington, 1966) and later three (Dennison and Hewish, 1967) separated sites. The experiment of Dennison and Hewish on 3C-48 at 81.5 MHz was consistent with the diffraction pattern moving without distortion, that is, with the average speed $u$ much larger than any 'thermal' velocities of the pattern. Also in the same experiment (i) the scale size of the diffraction pattern could be derived from auto-correlation functions without assuming the wind speed, and (ii) the elongation of the diffraction pattern could be determined or rather a limit set on it. The observations were in weak scattering domains (I or II) for $p > 0.5$ a.u. and there the intensity auto-correlation typically decayed to $e^{-1}$ for a lag $\gamma \approx 0.6$ sec. Thus for $u = 300 - 500$ km/sec observed, the density scale size fails in the range $a = 130-210$ km. With reference to Figure (1) note that $a = 170$ km places the diffraction near the boundary of domains I and II, with the ratio $\frac{\lambda D}{2\pi a^2} \approx 1.2$ at 81.5 MHz. Only a slight, if any, elongation, less than two to one, of the diffraction pattern parallel to the radial direction was detected.
Cohen (1965), Cohen et al. (1967), and Salpeter (1967) emphasized the power spectrum of the intensity scintillations recorded with a single antenna. There is a simple connection between frequency spectra observed and wavenumber spectra of the diffraction pattern in the case where there is a single solar wind speed, which is the case indicated by Denison and Hewish's (1967) observations. This connection is especially important for angular diameter measurements of sources, but also with certain assumptions it permits a study of the spectrum of irregularities in the solar wind.

Part II (Section A) of the thesis considers analysis of interplanetary scintillations recorded with a single antenna. It is shown that under the conditions that (i) the interplanetary turbulence is 'nearly' isotropic and (ii) a unique solar wind speed is a good approximation, then a certain Bessel transform (rather than the Fourier transform previously used) of the observed intensity correlation functions could be valuable for determining, first, solar wind speeds and, secondly, the wavenumber spectrum of the interplanetary irregularities. Analysis carried out on observations of CTA-21, though not especially encouraging for the use of the technique for wind speed measurements, was consistent with the assumption of isotropic turbulence and permitted derivation of irregularity spectra for a range of wavenumbers (q) as wide as $q^{-1} = 10$ to 200 km. On most of the occasions the wavenumber spectra were found to have a power law dependence
on wavenumber, resembling the inertial range Kolmogoroff spectrum of incompressible gas turbulence.

Part II (Section A) goes on to discuss the theory of weak interplanetary scintillations for cases of power-law irregularity spectra. The existence of such a spectrum in the interplanetary medium does not appear to contradict observations even in a quantitative comparison. Furthermore, for a power law spectrum, the apparent diffraction scale size is about one third of the Fresnel radius, that is, $1/3(\lambda D)^{1/2}$ ($\approx 160$ km for $\lambda = 169$ cm or $81.5$ MHz and $D = 0.86$ a.u.).

This offers an explanation for why observed scintillations often happen to fall on the Fresnel-Fraunhofer boundary, $\frac{\lambda D}{2\pi a^2} = 1$.

Part II (Section A) also develops a theory of the density and magnetic field fluctuations in the interplanetary medium. The theory implies a number of properties of the magnetic field variations observed by the Mariner 2 space probe, and in addition some of the observed properties of the density fluctuations deduced from IPS. The theory suggests a close connection between the processes causing small scale density fluctuations (size $\sim 50$ km) detected with IPS and large scale field fluctuations (size $> 10^4$ km) observed by Mariner 2.

Hewish, Scott, and Wills pointed out that interplanetary scintillations offered a valuable technique for estimating radio source diameters. Important observational
work on IPS has been concerned with determining diameters of radio sources, and two techniques have evolved for this purpose: one developed at Cambridge (Hewish and Okoye, 1965; Little and Hewish, 1966) involves observing a source at a number of say decreasing elongations. The increase of the presumed independently known $\theta_s(\rho)$ is assumed to quench the scintillations at an elongation where $\theta_s > \frac{\lambda}{2\pi \theta_k D}$ according to the thin screen theory of Pisareva(1959) and Briggs(1961). A second method developed at Arecibo (by Cohen et al., 1966; Cohen et al., 1967; and Salpeter, 1967) utilizes the power spectrum of the intensity scintillations, on which a finite source acts as a low-pass filter suppressing Fourier components higher than a certain frequency which is inversely proportional to the source size $\theta_s$, also according to the thin screen theory.

Cambridge studies of IPS and ionospheric scintillations and more recently IPS studies at Arecibo have inspired theoretical work on scintillations and has even given rise to a small controversy over one aspect of scattering theory. Most of the published work has been concerned with either of two models of the medium of irregularities: that of a thin medium of inhomogeneities, the thin screen; and the other extreme, the thick screen. A thin screen changes only the incident wave phase, whereas a thick screen is sufficiently deep, a distance of propagation $L$, for both phase and amplitude variations to develop within it. A
subdivision of these models is between weak and strong scintillations where the mean square phase variations \( \phi_L^2 \) suffered by the wave in passing through the medium (a distance \( L \)) is much less or much greater than a radian. Alternatively, a division between single and multiple scattering.

Most aspects of the theory have been worked out for the thin screen for the case of a single dominant irregularity size (\( a \)). The angular distribution of radiation was studied by Fejer (1953) and Radcliffe (1956) for weak and strong scattering; the mean-square intensity variations and correlations, by Pisareva (1959) both analytically and numerically for a large range of distances, \( D \leq 2\pi a^2/\lambda \), from the screen in weak and strong scintillation; the statistics of the intensity by Mercier (1962) for weak and strong scintillation in the Fraunhofer limit, \( D \gg 2\pi a^2/\lambda \); and all cases are summarized in a review by Salpeter (1967) where in addition the focal spot domain IV of Figure (1) is explored. Many others have contributed to the thin screen theory of which we mention: Budden (1965), who derives some weak scintillation formulae important in Part II of the thesis; Budden (1965) and Uscinski (1965), who treat wave amplitude correlations between two wavelengths; and Little (1968), who treats effects of a finite bandpass on intensity variations in several limits.

Part I of the thesis sets a foundation for treatment later in Part I of angle scattering and in Part II of
of scintillations caused by a thin screen of irregularities having an arbitrary power law wavenumber spectrum. In the main the analysis shows that all of the scattering and scintillation effects known to occur for the case of a single irregularity size also occur for power law spectra. However, the scattering and scintillation parameters generally have different dependences on radio wavelength and other physical parameters, to an extent determined by the logarithmic slope of the wavenumber spectrum. This fact has been noted in previous studies of scattering and scintillations for cases of power law spectra; for example, by Chernov (1960), Tatarski (1961), and Salpeter (1969). However, in addition to the known effects there are (at least) two new effects which we discuss: In Part I it is shown that for spectra having a certain range of logarithmic slope the position of a source may undergo random variations while at the same time exhibiting angular broadening and weak or strong scintillations. In Part II (Section B) it is shown that a sequence of narrow pulses received through a power law screen may show random variation in arrival times. These two effects are distinctive features of power law irregularity spectra.

Part III of the thesis is a treatment of wave propagation through a thick screen. In most of this analysis it is assumed that a scale size (a) exists. If the phase fluctuations are small, $\theta_L^2 \ll 1$, there is only weak scintillation, $m^2 \ll 1$, and the first Born approximation is applicable. For $\theta_L > 1$, strong intensity variations are possible, but these
may not occur within the medium. If \( \frac{\phi^2}{L} \gg 1 \), geometrical optics is applicable and initially parallel rays of an incident plane wave may be deflected in angle by a sufficient amount to cross within the medium. Rays may cross if the thickness \( L \) is larger than a typical focal length, \( \ell_L \equiv a / \theta_S \) (Salpeter, 1967). \( L > \ell_L \) implies that strong intensity variations may occur within the medium. In the following by a thick screen we mean \( L > \ell_L \) and \( \frac{\phi^2}{L} \gg 1 \).

An investigation of the thick screen was stimulated by the fact that the interplanetary medium was expected to become thick at long wavelengths and at small elongations, the latter because \( \ell_L = \frac{a(p)}{\Theta_S(p)} \) was estimated to decrease with elongation more rapidly than the screen thickness, \( L \approx \left( \frac{2}{\sqrt[3]{3}} \right) \sin(\xi) \) a.u. The lack of an analytical wave solution for the thick screen and the difficulty in obtaining these evoked conjectures about the nature of scintillation phenomena in this limit and in particular the behaviour of very small source scintillations. One suggestion (Cohen et al., 1967) was that the finite diameter cone of rays which develops inside the medium as the wave progresses, even for a point source, would act, when viewed through the remainder of the layer, as a finite source and this it was noted could quench some of the wave fluctuations. Exactly such an effect does occur (see Part IV) but the total fluctuations in the wave do not decrease with increasing \( L / \ell_L \) because the contribution of the far side of the screen
compensates the loss of fluctuations from the near side.

In Part III of the thesis two complementary approaches to the thick screen are developed: one is based on geometrical optics and the second on a wave optics model for a thick screen. The geometrical optics or ray tracing theory considers typical paths of rays and is based on the statistical ray theory of Salpeter(1967) and Chandrasekhar(1952). It is shown that nearby parallel rays typically spread apart exponentially with increasing propagation distance. From the ray picture a heuristic expression for the wave amplitude is constructed. Although the ray optics does not give any exact results, and rests on an assumption of the kind Keller(1962) terms 'dishonest'; it is valuable in not concealing complexity of the wave propagation.

The wave optics model consists of an arbitrary number N of parallel thin phase changing screens (each not restricted to weak or strong scattering) separated by equal distances \( L/(N-1) \). The model is not equivalent to a thick screen, but for sufficiently large N, the ray optics results characteristic of the thick screen also occur in the model. The model leads to certain exact solutions for different amplitude correlation functions. The derivations, although 'honest' (Keller, 1962), are formal and do not provide insight to the propagation phenomena. Nevertheless, the model was valuable in providing theoretical invalidation of several early conjectures obtained from the ray theory.
With ray-optics and the wave-optics model we consider: (a) the dependence of the emergent wave on the angular size of the incident wave; (b) time dependence of the wave due to motion of plasma irregularities; and (c) wavelength dependence of the emergent wave. The results are surprising: The behaviour of (a)-(c) is essentially the same as that for the case where the thick medium is collapsed to a layer, a thin screen, with the same phase fluctuations $\phi_L$ (for observations on the far boundary of the thick screen and a distance $L$ from the thin screen). Previously, Cole and Hewish (1967) (Hewish, 1969) concluded that effect (a) for a thick screen was not appreciably different from the case of a thin screen. However, the theory in the two cases is quite different and agreement of (a)-(c) does not indicate equivalence. To emphasize this an experiment is suggested in Part III which could show a strong effect unique to a thick screen.

Part IV of the thesis is a treatment of the scintillations caused by two thin phase changing screens. Exact analytic formulae are obtained for different intensity and amplitude correlation functions in a certain Fraunhofer limit. Some features of the two screen model discussed by Scheuer (1968) are confirmed. Application of the theory to interplanetary-interstellar scintillations is considered.

The fact is not overlooked that much theoretical work on random wave propagation problems in thick media has been done by several Russian authors. There is considerable work
based on the Rylov Method of Smooth Disturbances (MSD) (see Tatarski, 1961; or Chernov, 1960) in which an attempt is made to find an approximate solution to the non-linear inhomogeneous equation for the logarithm of the wave amplitude. Only rather recently have correct limits been set on the validity of MSD, establishing that it holds only under conditions of weak scintillations (Tatarski, 1966). Tatarski (1966) (see also DeWolf, 1969) by taking into account the non-linear term ignored in MSD is able to extend the method and account for saturation of light fluctuations observed beyond a certain long distance in the atmosphere. But his method involves many approximations which are difficult to assess and we do not pursue it further.

Uscinski (1968) has proposed a complete analytic solution to the thick screen problem, but actually has derived only the probability density for angular scattering as a function of distance into the medium. The 'complete' solution is obtained by assuming without justification that the complex wave amplitude has a joint Gaussian distribution. This assumption would imply equivalence of a thick screen to a thin screen with the same r.m.s. phase fluctuations. However, it is shown in Part III of the thesis that a thick screen is not equivalent in all respects to a thin screen.
Synopsis:

Part I of the thesis derives a wave equation for propagation in an irregular medium; lists different approximations to this equation; and discusses facets of the thin screen approximation. Of practical interest for interpretation of observations is the definition, in Section B, of different explicit models for the spectrum of irregularities; and, in Section D, the discussion of the theory and observations of angular scattering for the different irregularity spectra.

Part II develops the thin screen theory and applies it to interplanetary scintillations. Section A is concerned mainly with weak interplanetary scintillations; practical formulae are derived for a variety of effects including the different models for the irregularities defined in Part I. Section B discusses the strong scintillations caused by a thin screen for the different irregularities.

Part III continues discussion of approximations of the wave equation of Part I, and then focuses attention on application first of ray optics and then a wave optics model to the problem of a wave propagating deep into an irregular medium. In the main the results of this Part are not of practical interest.

Part IV is a fairly thorough examination of the scattering and scintillations caused by two thin screens. Many practical formulae are derived. Consideration is given to application of the theory to interplanetary—interstellar
scintillations, and in particular to some preliminary observations of the intensity variations of the pulsar CP 0950.
PART I

Aspects of
Geometrical and Wave Optics
of Thick and Thin Screens
Introduction to Part I:

This Part of the thesis deals with basic aspects of wave propagation in random media. In Section A (§ 1) the wave equation for a medium with spatially and temporally varying index of refraction is derived. Section A (§ 2) discusses statistical and physical assumptions regarding the index variations, which are used in the rest of the thesis. Section B defines two dimensionless parameters of the wave equation which permit distinction between thick and thin screens and the first Born (and Rylov's) approximation and strong scattering. Section C mainly develops a mathematical characterization of different phase functions which are used later in the thesis. Section D reviews and extends the theory of angular scattering by a thin screen.
A. Generalities:

§1. The Wave Equation:

In a number of problems of random wave propagation the scattering medium may be conveniently represented by a slab model where the irregularities are entirely contained in a layer bounded by \( z = 0 \) and \( z = L \). The theory is concerned mainly with an incident plane wave (originating from \( z < 0 \)) traveling in the \(+z\) direction through and beyond the layer. The radio wave in most of the thesis is assumed quasi-monochromatic (arbitrarily narrow frequency spread) with angular frequency \( \omega = 2\pi f \). We assume that the plasma within the layer is sufficiently tenuous for collisions to be rare; therefore, the Vlasov equation describes the plasma. The electromagnetic wave propagating within the slab is described by the Maxwell-Vlasov equations:

\[
\begin{align*}
\nabla \times \mathbf{E} & = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{H} \quad ; \quad \nabla \cdot \mathbf{E} = 4\pi \mathcal{Q} \\
\nabla \times \mathbf{H} & = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi}{c} \mathcal{J} \quad ; \quad \nabla \cdot \mathbf{H} = 0
\end{align*}
\]

\[
\mathcal{Q} = \sum_\alpha e_\alpha \int d^3 \mathbf{v} \, f_\alpha (\mathbf{v}, x, t) \quad ; \quad \mathcal{J} = \sum_\alpha \int d^3 \mathbf{v} \, \mathbf{v} f_\alpha (\mathbf{v}, x, t)
\]

\[
\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{e_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{H}) \cdot \frac{\partial}{\partial \mathbf{v}} f_\alpha = 0
\]

\((\alpha = 1, 2, \ldots)\)

where \((\alpha)\) denotes the charge species (charge \( e_\alpha \), mass \( m_\alpha \))
electrons, protons, etc. \( S, \ j \) are the charge, current, density. \( f_\alpha \) is the velocity distribution function for the \( \alpha \)th species.

The plasma in the absence of the electromagnetic wave is assumed charge neutral over distance scales of interest (\( > \text{Debye length} \)). Let \( f_\alpha = f_\alpha^0(v, x, t) \), \( E = 0 \), \( H = H_0 \), \( S = 0 \), and \( j = 0 \) denote the plasma variables without the wave, where \( H_0 \) is assumed a constant uniform external magnetic field of arbitrary direction. \( f_\alpha^0 \) for the moment is assumed a given function of space and time. Further, \( f_\alpha^0 \) is assumed gyrotropic and of zero mean velocity.

With the wave present write \( H = H_0 + \delta H \), \( E = \delta E, f_\alpha = f_\alpha^0 + \delta f_\alpha \), where \( |\delta H| \ll |\delta E| \ll |\delta f_\alpha| \) and \( |\delta f_\alpha| \ll f_\alpha^0 \). However, perturbations accumulated by the wave are not assumed small compared with the incident wave amplitude. Retaining only first order terms in smallness \( |\delta f_\alpha| \) in the Vlasov equation one obtains:

\[
\left\{ \frac{3}{\partial t} + v \cdot \nabla + \frac{e_\alpha}{m_\alpha} (v \times H_0) \cdot \frac{\partial}{\partial v} \right\} \delta f_\alpha^0 = -\frac{e_\alpha}{m_\alpha} \delta E \cdot \frac{\partial}{\partial v} f_\alpha^0 \tag{5}
\]

We may generally write \( \delta E(x, t) \) as a Fourier composition of terms of the form \( \delta E(k, \omega) \exp(i k \cdot x - i \omega t) \) where \( k \) is the wavevector and \( \omega \) the frequency of the electromagnetic wave. Furthermore, let us expand \( f_\alpha^0(v, x, t) \) in terms of components \( f_\alpha^0(v, q, \Omega) \exp(i q \cdot x - i \Omega t) \), where \( (q, \Omega) \) is the wavevector, frequency, of one component of the number density of the \( \alpha \)th species. Assume that the amplitudes \( f_\alpha^0(v, q, \Omega) = \)
\[ \hat{f}_0^*(\nu_i, q_j, \Omega) \] which guarantees that \( f_0^\alpha \) is real. (Asterisk superscripts always denote complex conjugation.) Inserting these expansions in equation (5), one finds that if 
\[ \delta f^\alpha (\nu, x, t) \] is written as a composition of terms 
\[ \delta f^\alpha (\nu, k', \omega') \cdot \exp(ik' \cdot x - i\omega' t) \], then 
\[ k' = k + q \quad \text{and} \quad \omega' = \omega + \Omega \].

The perturbation of the distribution function is:

\[ \delta f^\alpha (\nu, x, t) = \frac{e_c}{i m_c \omega} \frac{\delta E(x, t)}{\delta x} \int_0^\infty d^3 q \int d \Omega \ e^{i(q \cdot x - \Omega t)} \frac{\partial}{\partial \nu} \ f_0^\alpha (\nu, q, \Omega) \]

\[ \mathcal{O} \equiv \left\{ \frac{1}{\omega} - \frac{k \cdot \nu}{\omega} - \frac{q \cdot \nu}{\omega} - \frac{e_c}{i \omega m_c} \left( \frac{\nu \times \mathcal{H}}{\nu} \right) \frac{\partial}{\partial \nu} + i \sigma \right\}^{-1} \] (6)

where \( \sigma \) in the operator \( \mathcal{O} \) is a positive infinitesimal introduced to define the contour to be used in evaluating the integral. From expression (6) one can obtain the charge and current densities. Then from Maxwell's equations an equation for \( \delta E(x, t) \) may be derived. We shall do this for the case where \( \mathcal{H}_0 = 0 \), which means neglect of Faraday rotation.

For a number of plasmas of interest in random wave propagation problems one may reasonably assume that the spread in frequency and wavenumber of the zero order distribution \( f_0^\alpha (\nu, q, \Omega) \) is much smaller than the radio frequency \( \omega \) and wavevector \( k \) : \( \Omega \ll \omega \) and \( |q| \ll |k| \). Furthermore, assume that the velocity spread of \( f_0^\alpha \) is much less than the speed of light \( c \). For the interplanetary plasma
for the fluctuations of $f_\alpha^c$ of importance and at meter
radio wavelengths, $\Omega/\omega \sim 10^{-8}$, $|q|/|k| \sim 10^{-5}$, and
$v/c \sim 10^{-3.5}$. The smallness of these ratios permits expansion of the $O$ operator:

$$O = 1 - \frac{\Omega}{\omega} + \frac{k \cdot v}{\omega} + \frac{q \cdot v}{\omega} - 2 \frac{\Omega}{\omega} \frac{k \cdot v}{\omega} \ldots \quad (7)$$

where only the terms which contribute in zero or first
order in $\Omega/\omega$ or $|q|/|k|$ to the current or charge density
have been retained. For example, terms of the order of $v^2/c^2$ times the lowest order contribution to the current
density have been neglected mainly for simplicity of the
ensuing formulae. The expansion (7) inserted in equation (6)
may be manipulated into convenient forms. Terms propor-
tional to $\Omega$ (or $q$) become partial time derivatives
(or gradients) of $f_\alpha^c$. $\int f_\alpha^c$ integrated over velocity
becomes the number density of the $\alpha$th species. The
charge and current density obtained in this way are:

$$\mathcal{E} = \frac{1}{4\pi} (\varepsilon - 1) \nabla \cdot \mathbf{E} - \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \cdot \nabla (\varepsilon - 1) - \frac{1}{2\pi} \frac{\partial \mathbf{F}}{\partial \mathbf{E}} \left( \frac{\varepsilon - 1}{\mathbf{E}} \right)$$

$$\mathcal{J} = \frac{1}{4\pi} (\varepsilon - 1) (\partial \mathbf{E})_t - \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} (\varepsilon - 1)_t$$

where $(\ldots)_t$ denotes a partial time derivative; $\mathcal{E}(x,t) = 1 - \omega_p^2(x,t)/\omega^2$, and $\omega_p^2(x,t) = 4\pi e^2 \delta N(x,t)/m$ is the
electron plasma frequency with $\delta N(x,t) = \int d^3v f_\alpha^{el}(v,x,t)$.
the number density of electrons. Only electrons contribute significantly to \( \mathbf{j} \) and \( \mathbf{J} \) because of the smallness of the electron mass. The first of equations (8) may be simplified by using Maxwell's equation \( \nabla \cdot \mathbf{E} = 4\pi \mathbf{J} \) and the fact that in (8) only zero and first space and time derivatives of \( \mathbf{E} \) are retained: one finds \( \mathbf{j} = -\frac{1}{4\pi} \frac{\delta \mathbf{E} \cdot \nabla \ln n_e}{\mathbf{E}} \).

It is easily verified that equations (8) conserve charge accurate to first derivatives of \( \mathbf{E} \) : \( (\mathbf{j})_t = -\frac{1}{4\pi} \frac{\mathbf{E} \cdot \nabla \ln E}{\mathbf{E}} \) and \( \nabla \cdot \mathbf{j} = -\frac{1}{4\pi} (\mathbf{E} \cdot \nabla \ln E) = \frac{1}{4\pi} (\delta \mathbf{E} \cdot \nabla \ln E) = 0. \) Notice that in deriving (8), it has not been assumed that the ratio \( \omega_p/\omega \) is small.

Introducing expressions (8) into Maxwell's equations, operating on the first Maxwell equation (4) with curl to obtain a wave equation, and discarding higher order than first derivatives of \( \mathbf{E} \) gives,

\[
\left\{ \frac{\nabla^2}{c^2} - \frac{\mathcal{E}(x,t) \frac{\partial^2}{\partial t^2}}{c^2} \right\} \delta \mathbf{E}_j(x,t) = -\frac{3}{4\pi} (\nabla_j \delta \mathcal{E}(x,t)) \frac{\partial \mathcal{E}(x,t)}{\partial t} \]

\( j = 1, 2, 3 \)

\( \mathcal{E}(x,t) \) is thus the dielectric constant, \( n = \sqrt{\mathcal{E}} \) is the index of refraction. The frequency \( \omega \) which enters \( \mathcal{E} \) explicitly is that which the wave would have in the absence of the plasma. A perfectly monochromatic incident wave does of course acquire a frequency spread at least \( \sim \int \omega \ll \omega \) within the plasma. These small frequency shifts, caused by locally moving plasma blob, are accounted for in equation (9).
The wave equation (9) describes the wave within the plasma slab, $0 \leq z \leq L$. For larger distances, $z > L$, the propagation is in vacuo, $\varepsilon = 1$.

If Faraday rotation, a constant uniform $H_0$, is included as a small additional term in expansion (7), one finds an additional contribution to the current density, which necessitates inclusion of the term

$$\frac{1}{c^2} \left( \varepsilon(x,t) - 1 \right) \left[ \frac{eH_0}{mc} \times \frac{\partial}{\partial t} E(x,t) \right] \mathbf{j} \quad (10)$$

on the right side of the wave equation (9). We neglect term (10) in all of this thesis, except for the considerations in Part I (D; 6.9).

An energy conservation equation for the wave may be obtained from the following identity for Maxwell's equations:

$$\frac{c}{4\pi} \nabla \cdot \left( \delta E \times \delta H^* + \delta E^* \times \delta H \right) + \frac{1}{4\pi} \frac{\partial}{\partial t} \left( \delta E \cdot \delta E^* + \delta H \cdot \delta H^* \right) =$$

$$- \left( \delta E \cdot \mathbf{j}^* + \delta E^* \cdot \mathbf{j} \right) \quad (11)$$

where complex conjugate fields have been used to suppress the rapid time variations $\exp(-i\omega t)$. If the wave incident on the plasma were of only one frequency $\omega_0$, then the right side of (11) would be $\frac{1}{2\pi} \delta E \cdot \delta E^* (E - 1) \mathbf{j}$. However, in most situations this is unrealistic; one usually has waves with relatively large frequency spreads $\Delta \omega \gg \omega$, but
still such that $\Delta \omega \ll \omega$. Under these conditions the radio frequency dependence of $\mathcal{E}$ in the expression for the current (8) must be accounted for. The fields (and currents) are written as $\delta E(x, t) = \delta E_0(x, t) \cdot \exp(-i\omega t)$, where $\delta E_0(x, t)$ is slowly varying function of time (frequencies $\sim \Delta \omega$) and $\omega_0$ is the center frequency of the wave. Then following methods discussed by Landau and Lifshitz (1960, page 255) and Sudan (1970) one may obtain:

$$\dot{\delta E}_0(t) e^{i\omega_0 t} = -\frac{i\omega_0}{4\pi} (\varepsilon_{\omega_0} - 1) \delta E_0 + \frac{1}{4\pi} \frac{d}{d\omega} \left\{ \omega (\varepsilon_{\omega} - 1) \right\} \frac{2}{\omega_0} \frac{\partial}{\partial t} \delta E_0$$

$$- \frac{1}{4\pi} \delta E_0 (\varepsilon_{\omega_0} - 1) t$$

(12)

Introducing (12) in (11) one obtains the conservation equation,

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \mathcal{U} = \frac{1}{8\pi} |\delta E|^2 (\varepsilon - 1)_t = 0$$

$$\mathcal{J} \equiv \frac{c}{8\pi} \left( \delta E \times \delta H^* + \delta E^* \times \delta H \right)$$

$$\mathcal{U} \equiv \frac{1}{8\pi} \left( |\delta E|^2 \frac{d}{d\omega} (\omega \varepsilon) + |\delta H|^2 \right)$$

(13)

where $\mathcal{J}$ is the Poynting flux and $\mathcal{U}$ is the energy density of the wave. The right side of equation (13), $\propto (\varepsilon - 1)_t$, is of the order of $\Delta \omega / \Delta \omega \ll 1$ smaller than the terms on the left, and for practical purposes may be neglected.
§2. Irregularities:

Electron density fluctuations in the layer $\delta N(x,t) = \int d^3v f_o^el(y, x, t)$ are idealized as being statistically homogeneous (in space) and stationary (in time). For most purposes of the scintillation theory, only two moments of the density $\delta N(x,t)$ are needed: The first, average of $\delta N(x,t)$ in space or time or of an ensemble is assumed zero: $\langle \delta N(x,t) \rangle = 0$. The second moment is the autocorrelation,

$$\overline{\delta N^2} \mathcal{S}_N(x-x', t-t') \equiv \langle \delta N(x,t) \delta N(x',t') \rangle \quad (14)$$

with $\mathcal{S}_N(0,0) = 1$, and where $\overline{\delta N^2}$ (= constant) is the total mean square density fluctuations of all periods and scale sizes. (For no particular reason the bar rather than brackets denote the average.) The angle brackets almost always denote an ensemble average which is assumed equivalent to spatial or temporal averages over sufficiently large space or time intervals. Dependence of the autocorrelation $\mathcal{S}_N$ on only the difference coordinates ($x-x', t-t'$) follows from the homogeneous-stationary assumption. Also this implies $\mathcal{S}_N$ a symmetric function of space and time arguments: $\mathcal{S}_N(x-x', t-t') = \mathcal{S}_N(x'-x, t'-t)$. The wavenumber power spectrum of $\delta N(x,t)$ is:
\[ F(\mathbf{q}, \mathbf{\Omega}) = \int_{-\infty}^{\infty} \frac{d^3 \mathbf{r}}{(2\pi)^3} e^{-i(\mathbf{q} \cdot \mathbf{r} - \mathbf{\Omega} \cdot \mathbf{r})} \frac{\delta N^2}{\delta N} \phi^*(\mathbf{r}, \mathbf{r}) \] (15)

where \( \mathbf{q} = (q_x, q_y, q_z) \) and \( \mathbf{\Omega} \) is the wavevector and frequency of one component of the fluctuating density.

In deriving the wave equation (9) it was assumed that (i) the radio frequency is much larger than frequencies \( \mathbf{\Omega} \) of strong fluctuations of the spectrum, and (ii) that the radio wavelength is much smaller than dimensions \( q^{-1} \) of strong fluctuations. From (i) the wave energy is conserved. Even further than (i) we usually take \( \delta N(\mathbf{x}, t) = \delta N(\mathbf{x}) \) independent of time in an appropriate reference frame, that in which the average \( 0 = \int d^3 v \mathbf{v} \langle \phi^*(\mathbf{v}, \mathbf{x}, t) \rangle \).

The extent to which the latter is a good approximation depends on the scintillation problem; some results of the theory hold independent of time variations of \( \delta N(\mathbf{x}, t) \) and others are sensitively dependent on these. Effects of time variations of \( \delta N(\mathbf{x}, t) \) are considered in Part II (Section B; §7) of the thesis.

The wavenumber spectrum of the time-independent density \( \delta N(\mathbf{x}) \) is denoted \( F(q) \), and the auto-correlation function \( \phi_N(\mathbf{x}) \).
\[
\overline{\delta N^2} \varphi_N(r) \equiv \langle \delta N(x) \delta N(x+r) \rangle ; \varphi_N(0) = 1
\] 

\[
F(q) \equiv \int \int \int d^3r \frac{e^{-i q \cdot r}}{(2\pi)^3} \overline{\delta N^2} \varphi_N(r)
\]

Condition (ii) above later appears a necessary condition for small angle scattering, but of immediate concern here is the fact that at a radio wavelength \( \lambda \) such that \( q \ll 1 \), the right side of the wave equation (9) is small to the extent \( q \lambda \) relative to terms on the left side and therefore may be discarded. With this term absent the vector wave equation becomes three uncoupled scalar wave-equations for \( \delta E_x, \delta E_y, \) and \( \delta E_z \). If the incident wave is plane polarized, say of the form \( \delta E_{x0} \exp(ikz - i\omega t) \), then subsequent angular scattering will give rise to a z-component to the electric field. However, for the very small scattering angles assumed here (generally in this thesis) the z-component is negligibly small and only the scalar wave equation for say \( \delta E_x(x,t) = A(x) \exp(-i\omega t) \) need be considered. \( A(x) \) is termed the **scalar wave amplitude**.

Two essentially different kinds of isotropic wave-number spectra are assumed in subsequent work: One is the standard Gaussian:
$$F_G(q) = \frac{\delta N^2}{(\sqrt{2\pi})^3} \exp\left\{ -\frac{1}{2} q^2 a^2 \right\}; \quad q^2 = q_x^2 + q_y^2 + q_z^2 \tag{17}$$

$$s_N(r) = \exp\left\{ -\frac{1}{2} \frac{r^2}{a^2} \right\}; \quad r^2 = r_x^2 + r_y^2 + r_z^2 \}

\text{where } \alpha \text{ is identified as a typical turbulence (or blob) size. } F_G(q) \text{ is chosen as representative of a wide class of spectra for which there is effectively only a single irregularity size. It includes for an extreme example } F_G \propto \delta(q - a^{-1}).$$

A second class of spectra may be typified by the isotropic power law spectrum,

$$F_K(q) = K_N q^{-s}; \quad K_N, s = \text{constants}; \quad q^2 = q_x^2 + q_y^2 + q_z^2 \tag{18}$$

This is suggested by observations of interplanetary scintillations discussed in Part II (A; $\S 3$) of the thesis (typical value $s \approx 3.6$), by studies by Coleman (1966, 1968) of large scale variations of the interplanetary magnetic field (dimensions $> 10^4$ km where $s \approx 3.2$) and Schubert and Colman's (1968) data on the large scale density variations. Also, but less relevant here, this form of spectrum is suggested by the Kolmogoroff inertial range spectrum of gas turbulence ($s = 11/3$). $F_K(q)$ is assumed a power law only
for a range \( q_1 < q < q_2 \) of wavenumbers. For larger \( q < q_1 \) or smaller \( q > q_2 \) irregularities the actual spectrum is where necessary assumed to increase much less rapidly or decrease much more rapidly than (18), respectively. That is, \( q_1^{-1} \) is assumed to be the outer scale and \( q_2^{-1} \) the inner scale of the inertial range (18). Spectra of the form of \( F_K(q) \), in contrast with the Gaussian spectrum, may show different scale sizes in different scintillation experiments, and these may depend on radio wavelength and other parameters in a way depending on \( s \). Our interest in the different dependences lies in part in the possibilities offered by scintillation observations for deducing the form of the irregularity spectrum. Chernov(1960) and Tatarski (1961) also treat in detail a number of electromagnetic (and acoustic) wave scattering and scintillation problems for media with power law spectra, but the overlap with our discussion is small.

At sufficiently high radio frequency, refractive index fluctuations \( \mu(x) = n(x) - 1 \) are by equation (1) proportional to the density fluctuations,

\[
\mu(x) = \frac{-1}{2\pi} \frac{\lambda^2}{r} \delta N(x); \quad \omega \gg \omega_p
\]

For interplanetary scintillations the radio frequencies are very high; typical values of \( \mu \) encountered are of the order of \( 10^{-9} \). The mean square of \( \mu \) is denoted \( \mu^2 \) in departure from the usual notation. \( \mu^2 \) is assumed constant.
B. Propagation within the Slab:

Consider a plane wave, \( \exp(ikz - i\omega t) \), incident normally on a plasma slab which is sufficiently thin for there not to be large transverse deviations in the wave direction through the slab thickness \( L \). One then expects that the scalar amplitude \( A(x) \) within the slab can be written in the form

\[
A(x) = \exp(i\psi) ; \quad \psi^0 = kz + \phi(x) \tag{20}
\]

(with the time dependence \( \exp(-i\omega t) \) suppressed). The amplitude must satisfy the Helmholtz equation,

\[
\left\{ \nabla^2 + k^2(1 + \mu(x))^2 \right\} A(x) = 0 \tag{21}
\]

Substitution gives an equation for \( \psi^0 \),

\[
\frac{1}{i} \nabla^2 \psi + (\nabla \psi)^2 - k^2(1 + \mu)^2 = 0 \tag{22}
\]

The complexity of (22) reflects the different facets of the phenomenon of a wave propagating into a random medium. And we approach the overall problem of wave propagation within the slab by examining different limits of this equation for \( \psi \) or alternatively that for the phase \( \phi \), which is is zero if there are no fluctuations (\( \mu = 0 \)).
\[
\frac{1}{i} \nabla^2 \phi + (\nabla \phi)^2 - k^2 \mu^2 + 2k \left( \frac{\partial \phi}{\partial z} - k \mu \right) = 0 \quad (23)
\]

(a) \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (b) \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (c)

The terms are labeled for later convenience; term (a) describes amplitude variations and diffraction and shows that the wave cannot remain of unit amplitude ( \( |A|^2 = 1 \)) with increasing distance \( z \). Terms (b) and (c) have to do with the wave phase.

Let us discuss, in general terms, different kinds of approximations to (23), to the extent possible without precise estimates for the various derivatives and without taking into account the effect integration has on the solutions. For this purpose introduce the following dimensionless scaling factors,

\[
\mu = \epsilon \tilde{\mu} ; \quad k = \frac{1}{\alpha} \tilde{k} ; \quad \phi = \epsilon \tilde{k} \tilde{\phi} \quad (24)
\]

where \( \epsilon \) is related to the strength of the irregularities and \( \alpha \) is proportional to radio wavelength. Then equation (23) may be written as,
\[ \alpha \frac{1}{i \kappa} \nabla^2 \tilde{\phi} + \varepsilon \left\{ \left( \nabla \tilde{\phi} \right)^2 - \mu^2 \right\} + 2 \left( \frac{\partial \tilde{\phi}}{\partial z} - \tilde{\mu} \right) = 0 \tag{25} \]

\( (a) \hspace{2cm} (b) \hspace{2cm} (c) \)

§ 1. The Thin Screen:

For a first example of an approximation to (25), assume very weak irregularities (\( \varepsilon \rightarrow 0 \)) and even weaker amplitude variations (\( \alpha \ll \varepsilon \)), with the following order relation \( 1 \gg \varepsilon \gg \alpha \). The second part of the double inequality is from (24) a requirement that the phase be sufficiently large. Neglecting terms (b) and (a) gives the approximate wave amplitude,

\[ A = \exp \{ i k z + i \phi \}; \phi = \kappa \int_0^z d\zeta \mu(x, y, z') \tag{26} \]

For reasons more evident later (Part III; A; § 1) the limit giving (26) is termed the thin screen approximation.

§ 2. Geometrical Optics:

Geometrical optics is well known as the limit of very short wavelengths (\( \alpha \rightarrow 0 \)) and is more general than the thin screen approximation in that \( \varepsilon \) does not have to be small compared with unity. The appropriate order of the scaling parameters is: \( \varepsilon \gg \alpha \ll 1 \). Accordingly neglect of term (a) gives the eikonal equation of geometrical optics,
\[(\nabla \psi)^2 = k^2 (1 + \mu)^2 \tag{27} \]

for the eikonal \(\psi\). Geometrical optics is developed later in Part III (A) as an approach to the thick screen.

§3. Rylov's Method of Smooth Disturbances:

Another suggestion for an approximation to equation (25) apparently originated by Rylov (see for example Tatarski, 1961) may be summarized in our notation by the double inequality \(\alpha \gg \epsilon \ll 1\), which indicates very weak irregularities but non-negligible diffraction (\(\alpha\)). The relation (24) shows that the phase is very small in this limit. Neglect, in equation (25), of term (b) gives,

\[ \frac{1}{i} \nabla^2 \phi + 2k \frac{\partial \phi}{\partial z} - 2k^2 \mu = 0 \tag{28} \]

or introduction of \(\phi = \phi_* \exp(-ikz)\) gives

\[ \{\nabla^2 + k^2\} \phi_* = 2i k^2 \mu e^{ikz} \tag{29} \]

an inhomogeneous Helmholtz equation for \(\phi_*\).

§4. Born Approximation:

The Born approximation is obtained by assuming
\[ A(x) = \exp(ikz) + u(x) \] (30)

with \(|u| \ll 1\). Inserting (30) in the Helmholtz equation (21) gives,

\( \left\{ \nabla^2 + k^2 (1 + \mu)^2 \right\} u = -2k^2 \mu \left(1 + \frac{i}{2} \mu\right) e^{ikz} \) (31)

or

\( \kappa \left\{ \nabla^2 + \frac{k^2}{\kappa^2} (1 + \epsilon \mu)^2 \right\} \tilde{u} = -2 \frac{\epsilon}{\kappa^2} k^2 \mu \left(1 + \frac{i}{2} \epsilon \mu\right) e^{ikz} \) (32)

where \( u = \kappa \tilde{u} \). In order to have \(|u| \ll 1\), either one or the other of two conditions is necessary: which ever of the two terms on the left side of (32) is the larger when set equal to the right side must be small compared with unity. This gives \( \kappa \epsilon \ll 1 \) for \( \alpha \gg 1 \) and \( \kappa \ll \frac{\epsilon}{\alpha^2} \ll 1 \), for \( \alpha \ll 1 \). Thus in either case \( \alpha \gg \epsilon \ll 1 \). Hence \( u \) obeys the equation

\( \left\{ \nabla^2 + k^2 \right\} u = -2k^2 \mu e^{ikz} \) (33)

It is evident that equation (20) is of the form of (30) if, as is true from (24), \( \phi \ll 1 \): \( A = \exp(ikz) + i\phi(x) \exp(ikz) \) and \( u = i\phi \exp(ikz) = i\phi_x \). Apparently Rytov's approximation is equivalent to the Born approximation.
Investigation of the different approximations is continued in Part III (A, §4). There the parameters $\alpha$ and $\varepsilon$ are identified with physical parameters of the medium. The next Section (C) develops certain aspects of the thin screen theory which are used in Section (D) and in Part II.
C. The Phase Functions (thin screen);

The thin screen phase,

$$\phi(x, y, z) = k \int_{-z}^{z} dz' \mu(x, y, z')$$

(34)

is fundamental in scintillation problems even when the thin screen solution (26) does not hold. The average value of $\phi$ is zero because $\langle \mu \rangle = 0$. For illustration of the nature of $\phi$ note, that for index variations $\mu$ in the form of blobs of size $a \ll z$ that there are roughly $N \approx z/a \gg 1$ blobs along a distance $z$ and that each contributes a phase dealy of about $\pm k\mu a$ radian; the sum of many such delays combined in a random walk gives a typical total delay of $\phi \approx N^{1/2} k\mu a$ which increases with the square root of the distance $z$. Statistical formulae describing $\Theta(x, y, z)$ for different irregularity spectra are derived below.

First define a phase auto-correlation function,

$$\phi_z^2 \equiv \langle \phi(x, y, z) \phi(x+r_x, y+r_y, z) \rangle; \phi_z(0, 0) = 1$$

(35)

where $\phi_z$ is the r.m.s. phase shift. Its two dimensional wavenumber spectrum termed the phase spectrum is:
\[
\Phi_Z(q) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dr}{2\pi} \int_{-\infty}^{\infty} \frac{dr}{2\pi} e^{-iq \cdot r} \phi_z^2 \rho_z(r) \quad (36)
\]

where \( q = (q_x, q_y) \) and \( r = (r_x, r_y) \). (Squiggly lines under variables always denotes a transvers x-y vector.) Using the definition (16) and equation (19),

\[
\phi_z^2 \rho_z(r_x, r_y) = \lambda^2 r_x^2 \int_{0}^{Z} dz' dz'' \delta N^2 \rho_N(r_x, r_y, z'-z'') \quad (37)
\]

\[
= 2 \lambda^2 r_x^2 \int_{0}^{Z} dz' (z-z') \delta N^2 \rho_N(r_x, r_y, z') \quad (38)
\]

where the step from (37) to (38) is an identity for any symmetric function of \((z'-z'')\).

For the Gaussian spectrum \( F_G \), \( \rho_Z(r_x, r_y) = \exp(-(r_x^2 + r_y^2)/2a^2) \) is independent of distance, and from equation (38) \( \phi_z^2 \) may be obtained,

\[
\phi_z^2 = 2 \lambda^2 r_x^2 Z \delta N^2 \int_{0}^{Z} dz' (z-z') e^{-\frac{z'^2}{2a^2}} \quad (39)
\]

\[
\approx \sqrt{2\pi} \lambda^2 r_x^2 \delta N^2 Z^2 \left( \frac{4 - \pi}{2a^2} \right) ; \quad z \ll a \quad (40)
\]

\[
\approx \sqrt{2\pi} \lambda^2 r_x^2 \delta N^2 Z a \left\{ 1 - \sqrt{\frac{\pi a}{Z}} \ldots \right\} ; \quad z \gg a \quad (41)
\]
Thus for \( z \gg a \), the r.m.s. phase shift \( \phi_z \) has the dependence indicated by elementary arguments. For \( z = L \), equation (3) is obtained.

Irregularity spectra such as \( F_K \) have a wide range of turbule or eddy sizes, and consequently the electron density fluctuations show very long range correlation. With \( F_K(q) \) specified only for a range of wavenumbers \( q_1 < q < q_2 \), the correlation function \( C_N \) in equation (38) is not obviously defined. To circumvent this, temporarily make the replacement \( F_K \rightarrow F_K G \), where \( G(q) \) is some isotropic function which decreases rapidly enough to suppress the singularity of \( F_K(q) \) for \( q \ll q_1 \). The three dimensional correlation function which corresponds to \( F_K G \) is then well-behaved. From (38) one obtains,

\[
\overline{\Phi}_z (q_x, q_y) = 2\pi \lambda^2 \rho_0^2 Z \int_{-\infty}^{\infty} dq_z \frac{G(q_x, q_y, q_z) F_K(q_x, q_y, q_z) q(q_z)}{q(q_z)}
\]

\[
q(q_z) = \frac{2 \sin^2 \left( \frac{1}{2} q_z Z \right)}{\pi (q_z Z)^2} ; \int_{-\infty}^{\infty} dq_z \frac{q(q_z)}{q_z Z} = 1
\]

Note that if \( q_x^2 + q_y^2 > q_1^2 \) then \( q^2 > q_1^2 \) so that the \( G \) factor is superfluous in (42), \( G \delta_1 \); also that \( F_K \) falls off monotonically with \( |q_z| \) increasing from zero, with a characteristic scale \( = (q_x^2 + q_y^2)^{1/2} \). The latter fact allows \( F_K(q_x, q_y, q_z = 0) \) to be pulled from under the integral in (42) for \( z \cdot (q_x^2 + q_y^2)^{1/2} \gg 1 \) (or \( z \gg a \) for \( F_G \)). This
gives a fairly general relation which is important later:

\[ \Phi_z(q_x, q_y) = 2\pi z (\lambda r_e)^2 F_k(q_x, q_y, q_z) \] (43)

Evidently index variations in the direction of the wave propagation are projected out. Later it is further evident that the detail of different scintillation phenomena depends on transverse index variations while longitudinal variations have only a secondary role.

In IPS problems \( \varnothing(x,y,z) \) as a function of \( (x,y) \) has usually been taken to be a Gaussian random variable, when there are a large number of turbules along the wave path, \( \eta = z/a \gg 1 \), for cases of Gaussian irregularity spectra; that is, the central limit theorem is invoked. If \( W(\varnothing)d\varnothing \) denotes the probability of \( \varnothing \) occurring in the interval \( \varnothing \) to \( \varnothing + d\varnothing \), then

\[ W(\varnothing)d\varnothing = \frac{d\varnothing}{\sqrt{2\pi}\varnothing_z} \exp \left\{-\frac{\varnothing^2}{2\varnothing_z^2} \right\} \quad \int_{-\infty}^{\infty} d\varnothing W(\varnothing) = 1 \] (44)

A handy property of a Gaussian variable, or any linear combination of such variables, say \( h \), is that

\[ \langle \exp(ih) \rangle = \exp \left( -\frac{1}{2} \langle h^2 \rangle \right) \] (45)

The relation (45) is used frequently in later work, and as an illustration, it allows evaluation of an auto-correlation function of the thin-screen amplitude (equation 26) which
is useful later,

\[ \left\langle A(x, y, z) A^*(x+r_x, y+r_y, z) \right\rangle = \exp \left\{ -\frac{\phi_z^2}{2} \left( 1 - s_z(r_x, r_y) \right) \right\} \] (46)

where \( \phi_z \) denotes the phase shift as given by

\[ \phi_z^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 \mathbf{q} \ \tilde{\phi}_z(q_x, q_y) = 2\pi z (\lambda_r)^2 \int_{-\infty}^{\infty} d^2 \mathbf{q} \ F(q_x, q_y, 0) \] (47)

Of course, evaluation of (47) gives (41) for the case \( F = F_G \).

With a specified Gaussian-like correlation function \( \phi_z \) and the assumption that \( \phi \) is Gaussian random, one has a complete statistical description of the thin screen amplitude \( A(x, y, z) \).

For power law spectra \( F_k(q) \) the situation is different. There is a wide range of turbulence sizes; the largest with size \( \sim q_k^{-1} \) have the largest electron density variations (across distances \( \sim q_k^{-1} \)); smaller turbules of size \( q_k^{-1} \) (\( q > q_k \)) are superposed on the larger and have smaller density variations across distances \( q_k^{-1} \). The smallest turbules of size \( \sim q_2^{-1} \) contribute the smallest density variations. Because of the relation (43) the phase \( \phi(x, y, z) \) as a function of \( (x, y) \) has similar properties; the largest phase variations occur over distances \( \sim q_1^{-1} \); the smallest over distances \( \sim q_2^{-1} \); etc. For the power law spectra considered here, the mean-square phase shift is determined primarily by the largest turbules. The phase \( \phi \) may or may not be a Gaussian random variable, depending on whether \( q_1 z \gg 1 \) or \( q_1 z < 1 \), respectively. On the other hand the transverse
gradient of the phase $\nabla \phi(x,y,z)$ (which is later seen to be important to scintillations) may be determined mainly by either the largest or the smallest turbules, depending on the logarithmic slope of the spectrum (s).

We first investigate power law spectra having a restricted range of logarithmic slope (s) with the property that the phase difference, defined by

$$\Delta_{l} \phi(x, r) \equiv \phi(x + r_{/2}, z) - \phi(x - r_{/2}, z), \quad (48)$$

depends mainly on the intensity of turbules of size $\sim |r_{/2}|$.

Let $\rho_{Z}(r_{x}, r_{y})$ denote the r.m.s. phase difference. Then,

$$\rho_{Z}^{2}(r) = 4\pi \int_{0}^{\infty} q \Phi_{Z}(q) \left[ 1 - \frac{J_{0}(q r)}{q r} \right] \frac{d q}{q} \quad (49)$$

where $r = (r_{x}^{2} + r_{y}^{2})^{1/2}$ and $q = (q_{x}^{2} + q_{y}^{2})^{1/2}$, and $J_{0}$ is the Bessel function of zero order. The phase difference, compared with the phase (equation 47) strongly suppresses the contribution of turbules with dimensions larger than the separation $r$ as may be seen in the factor $(1 - J_{0}(qr))$ in the integrand in equation (49) which is proportional to $q^{2}$ for $q \ll r^{-1}$. A variety of other quadratic moments of $\Delta_{l} \phi$ may be expressed in terms of weighted integrals of $\Phi_{Z}(q)$ all with the same small $q$ dependence of the weight factor. For example, the correlation $\Delta_{l} \phi$ at different points $r$ and $r'$;
\[
\langle \Delta_{\hat{\xi}} \phi(x, x') \Delta_{\hat{\xi}} \phi(x, x') \rangle = \frac{1}{2} \left( \varphi_{z}^{2}(1|x|) + \frac{1}{2} \varphi_{z}^{2}(1|x'|) \right) - \frac{1}{2} \varphi_{z}^{2}(1|x-x'|) \tag{50}
\]

This relation shows that there is good correlation of the phase difference between points such that \((x - x')^2 \ll |x| |x'|\).

A relation inverse to equation (44) is:

\[
\Phi_{Z}(q) = \frac{1}{2\pi q} \int_{0}^{\infty} r dr \int_{1}^{\infty} (q, r) \frac{d}{dr} \left\{ \varphi_{z}^{2}(r) \right\} \tag{51}
\]

Thus a first order Bessel transform provides a unique connection between the phase difference and the phase spectrum.

For \(\Phi_{Z} \propto q^{-s}\) the integral (49) converges at the lower limit for \(s < 4\) and at the upper limit if \(s > 2\).

Here and later the range \(2 < s < 4\) defines what we call type A spectra. For this kind of spectra the gradient of the phase is determined mainly by the smallest irregularities; that is, the mean-square gradient depends on turbulence of size \(q_{z}^{-1}\). Type A spectra include for example the Kolmogoroff inertial range spectrum (\(s = 11/3\)). Scintillation and scattering problems for this kind of spectrum are treated by Tatarski (1961) and Chernov (1960), where the r.m.s. phase difference (squared) is termed the phase structure function. In fact Tatarski and Chernov treat only type A spectra.
With \( \Phi_z(q_x, q_y) = 2\pi z (\lambda r_e)^2 F(q_x, q_y, 0) \) and \( F_k(q) = K_N q^{-s} \) it follows from equation (49) that,

\[
\frac{\varphi}{\varphi_z}(r) = C_1 \beta \left( \frac{r}{r_0} \right)^{\delta} \; ; \; r_{a2}^{-1} < r < r_1^{-1}
\]

\[ \delta = (s-2)/2 \; ; \; 0 < \delta < 1 \]

\[
\beta^2 = 16\pi K_N (\lambda r_e)^2 z \cdot (r_o^2/2\pi)^{\delta} \cdot c_2^{-2}
\]

\[
c_1^2 = (\pi a_2^{\delta}) \left[ \Gamma(1+\delta) \cos \left( \frac{\pi}{2} \delta \right) \right]^{-1}
\]

\[
c_2^2 = 4 \delta \left[ \pi \Gamma(1-\delta) \cos \left( \frac{\pi}{2} \delta \right) \right]^{-1}
\]

The parameter \( \beta \), although at this point arbitrary because \( r_o \) is, is later defined (Part II; B; \( \delta \)) in such a way that it is a measure of the strength of the scintillations. With this later choice, \( \beta^2 \ll 1 \) is weak scintillations, and \( \beta^2 \gg 1 \) strong.

For type A spectra \( \varphi_z(r) \) is a measure of the r.m.s. phase deviations contributed by turbules of size \( q^{-1} \sim r \). Very roughly \( \varphi_z(r) = (r^{-2} \Phi_z(r^{-1}))^{1/2} \). Therefore, if \( z \gg r \), the phase difference is a Gaussian random variable,

\[
W(\Delta_1 \phi) = \frac{1}{\sqrt{2\pi} \varphi_z^{(1/2)}} \exp \left\{ -\frac{(\Delta_1 \phi)^2}{2\varphi_z^{(1/2)}} \right\}
\]

\[
\int_{-\infty}^{\infty} \text{d}(\Delta_1 \phi) W(\Delta_1 \phi) = 1
\]
and the identity (45) holds for \( \Delta \phi \).

Relations (52) and (53) are sufficient description of the phase statistics for solving a number of scintillation and scattering problems.

Another physically reasonable kind of power law spectra are those having logarithmic slope \((s)\) greater than four. Such spectra are steeper and have relatively stronger larger turbules than type A. Similar to type A, the phase at any point is contributed mainly by the largest turbules; however, in contrast, the gradient of the phase also depends on the largest turbules. Proceeding by analogy, consider spectra of slope \((s)\) such that the second-phase-difference,

\[
\Delta^2 \phi(x, r) = (\Delta_1 \Delta_1) \phi(x) = \phi(x + r, z) + \phi(x - r, z) - 2 \phi(x, z)
\]

(54)
depends mainly on the intensity of turbules of size \( \sim |r| \).

Let \( \mathcal{P}(r_x, r_y) \) denote the r.m.s. second phase difference. Then,

\[
\mathcal{P}(r) = 4 \pi \int_0^\infty \mathcal{Q}(q) \left[ 3 - 4 J_0(qr) + J_0(2qr) \right] \, dq
\]

(55)

where \( r = (r_x^2 + r_y^2)^{1/2} \) and \( q = (q_x^2 + q_y^2)^{1/2} \). It is evident from comparison of (55) and (49) that the second difference even further suppresses the contribution of small \( q \) (large) irregularities. For \( \mathcal{Q}(q) \propto q^{-s} \), the integral (55) converges for \( 2 < s < 6 \). Spectra with logarithmic slope \((s)\) in the range \( 4 < s < 6 \) are termed type-B spectra. The only
treatment of scintillations for type B spectra in the literature known to the author is that by Salpeter (1969), which is concerned only with this type of spectrum. The special case \( s = 4 \) is termed type C.

Evaluation of equation (55) for \( \phi \propto F_K = k_N q^{-s} \)
gives

\[
\phi_{Z_2}(r) = C_3 \beta \left( \frac{r}{r_0} \right)^{\delta}; \quad q_2^{-1} < r < q_1^{-1}
\]

\( \delta = \frac{(s-2)}{2}; \quad 0 < \delta < 2 \)

\( r_0 \) = arbitrary distance as in equation (52).

\( \beta^2 \) = scintillation strength parameter as in equation (52).

\[
C_3^2 = \frac{\pi}{2} (2\pi)^{\delta} C_2^2 C_4^2
\]

\( C_2^2 \) = given in equation (52); \( = 8/\pi^2 \) for \( \delta = 1 \).

\[
C_4^2 = \frac{1}{2^\delta} \left[ 2^\delta (1-\delta) - 1 \right] \frac{\Gamma'(1-\delta)}{\Gamma(1+\delta)}; \quad 0 < \delta < 1
\]

\[
= \ln(2) \quad ; \quad \delta = 1
\]

\[
= \frac{1}{2^\delta} \left[ 1 - 2^{1-2(\delta-1)} \right] \frac{(\delta-1)\Gamma(2-\delta)}{\Gamma'(1+\delta)}; \quad 1 < \delta < 2
\]

For type B spectra (and type A) \( \phi_{Z_2}(r) \) is a measure of the strength of turbulence of size \( \sim r \). For the same choice of \( r_0 \) mentioned after equation (52), \( \beta \) is later shown to be a measure of the scintillation strength; \( \beta^2 \ll 1 \) weak and \( \beta^2 \gg 1 \) strong scintillations.
As discussed for $\varphi_z(r)$ for type A spectra, it is reasonable to assume that $\varphi_z(r)$ has a Gaussian probability density for $r \ll z$.

In contrast to the behaviour exhibited by equation (52) or (56) is that of Gaussian-like spectra for which the phase difference is,

$$\varphi_z(r) = \sqrt{2} \phi_z \sqrt{1 - \rho_z(r)} \quad (57)$$

For $F(q) = F_G(q)$, one finds $\varphi_z \sim (r/a)\phi_z$ for $r \ll a$, and $\varphi_z \sim 2^{3/2} \phi_z$ at the other extreme. The phase difference in this case may not be a good measure of the strength of phase fluctuations of size $\sim r$; the actual strength of small irregularities of size $r \ll a$ is much smaller than indicated by (57), roughly $(r^{-2} \phi_z (r^{-1}))^{1/2}$. Previously, Hewish (1951) discussed the phase difference, but only for Gaussian irregularity spectra, where it is of no essential value.

Figure (2) shows hand sketches of $\vartheta(x,y)$ along one dimension for cases of Gaussian, type A, and type B spectra.
FIGURE 2: Model Signals.
D. Angle Probability Density (thin screen):

Introduction:

Consider a wave incident on a thin screen with wave vector $\mathbf{k} = (0, 0, k)$. At a distance $z$ within ($0 \leq z \leq L$) or beyond ($Z > L$) the screen the wave amplitude has a spatial Fourier composition with wavevector components $(k_x, k_y, k_z)$. It is always assumed in this thesis that $k_x, k_y \ll k_z$ and $k_z \approx k$. In this Section we discuss the probability density of transvers wavevectors $(k_x, k_y)$ or, equivalently, the scattering angles $\theta_x = k_x/k$ and $\theta_y = k_y/k$. Previously, for Gaussian-like irregularity spectra Fejer(1953) (and also Radcliffe, 1956) discussed the angle probability density for weak ($\varphi_z \ll 1$) and strong ($\varphi_z \gg 1$) scattering. Here the Fejer results are summarized (§1) and compared with the angle density for cases of type A power law irregularity spectra (§2). Later the angle density for type B (§3) and type C (§5) spectra are treated necessarily as is shown by taking into account the distance of the observer from the screen. A hybrid spectrum of type A and type B is discussed qualitatively in (§8). Source position variations occur for type B (§4) and type C (§6) spectra, and position variations are discussed in general in (§7). The possibility of measuring a longitudinal magnetic field in the screen from radiation scattered from an incoherent unpolarized source is discussed in (§9).
In this Section we reserve the term **strong** (or **weak** ) scattering to describe situations where the fraction of the energy not scattered at all is much less than unity (or comparable to unity). Strong scattering it will be seen does not necessarily imply strong intensity variations, that is, strong scintillations.

A typical scattering angle for either Gaussian or type A spectra may be estimated by elementary considerations utilizing the wave uncertainty relation \( \delta x \delta k_x \gg 1 \), where \( \delta x \) represents the transverse distance over which the amplitude \( A(x,y,z) \) changes appreciably, and \( \delta k_x \) the corresponding change in the wave vector. For the thin screen, variations in \( A(x,y,z) \) arise from changes in the phase \( \phi(x,y,z) \) alone; \( A = \exp(i\phi) \). Thus \( \delta x \) is roughly the transverse distance over which \( \phi \) typically changes by a radian, which is given by \( \frac{\lambda}{Q_z(\delta x)} = 1 \), where we assume \( Q_z(q^{-1}) \gg 1 \) and \( Q_z(q^{-1}) \ll 1 \). \( \delta x \) is characteristic of the screen alone and is termed the diffraction length. From either equation (52) or (57) the typical scattering angle may be estimated as \( \theta_5 \sim \frac{\lambda}{2\pi \delta x} \).

The probability \( W \) for a wave vector deflected into the interval \( \theta_x \) to \( \theta_x + d\theta_x \), \( \theta_y \) to \( \theta_y + d\theta_y \) is given by
\[ W(\theta, x) = \left( \frac{\iota}{2\pi} \right)^2 \int_{-\infty}^{\infty} dr \ e^{-\iota r \cdot \theta} S_A(x, r) \]

\[ S_A(x, r) = \langle A(x + \frac{r}{2}, z) A^*(x - \frac{r}{2}, z) \rangle \]

\[ \int_{-\infty}^{\infty} d\theta_x d\theta_y W(\theta, x) = 1 \quad ; \quad \iota \equiv \frac{2\pi}{\lambda} \]

(Wigner, 1932). The amplitude correlation \[ S_A \] for Gaussian or power law (type A) is \[ \exp\left( -\frac{1}{2} \frac{\iota^2}{\lambda^2} (r_x, r_y) \right) \] and is independent of \( x \); therefore, \[ W(\theta, x) = W(\theta) \]. For \( 0 < z < L \) the angle density depends on \( z \) through the dependence of the phase difference \( \frac{\iota^2}{\lambda^2} \) on \( z \). However, in general for \( z > L \) there is no additional scattering and the angle density \[ W(\theta) \] is independent of \( z \). The phase difference (and phase) is then evaluated at \( z = L \): \[ \frac{\iota}{\lambda} = \frac{\iota}{L} \).

The angle probability (58) combines the probability for diffraction and the probability arising from the randomness of the phase \( \varphi(x, y, z) \) which determines \( A(x, y, z) \). The second effect may be suppressed temporarily by considering a one-dimensional sinusoidal phase screen first examined by Hewish (1951). The phase \( \varphi(x) = \varphi(a) \cdot \sin(x/a) \) is determinant. The corresponding angle density is:
\[ W(\theta) = \sum_{n=-\infty}^{\infty} w_n \delta^{(1)}(\theta - \frac{n}{ka}) \]

\[ w_n \equiv \int_n^2 (\varphi(a)) ; \sum_{n=-\infty}^{\infty} w_n \equiv 1 \]

The scattered energy is contained in a number of discrete sidebands, each occurring with probability \( w_n \). The index \( n \) is interpreted as the number of scatterings. In weak scattering, \( \varphi^2(a) \ll 1 \), \( w_0 \equiv 1 - \frac{1}{2} \varphi^2(a) \) is the probability of no scattering (or diffraction) and \( w_1 = \frac{1}{4} \varphi^2(a) \) is the probability of diffraction into either of the two sidebands \( n = \pm 1, \theta = \pm (ka)^{-1} \). For strong scattering \( \varphi^2(a) \gg 1 \). Then the classical or geometrical optics probability density is: \( W(\theta) = (\pi \sqrt{\varphi^2(a) - \theta^2})^{-1} \) for \( -\theta_0 < \theta < \theta_0 \) and \( W \equiv 0 \) for \( |\theta| > \theta_o \), where \( \theta_0 = \varphi(a) \cdot (ka)^{-1} \). Equation (59) resembles the classical density closely, but some characteristic features of diffraction remain. For \( n \ll \varphi(a) \), the probability \( w_n \sim (\pi \varphi(a))^{-1} \) is very small, but for \( n \) increasing to \( n = \varphi(a) \), the probability increases to a sharp peak of height \( \sim (\varphi(a))^{-2/3} \) and width \( \sim (\varphi(a))^{1/3} \). This corresponds to the singularity of the classical density at \( \theta = \theta_0 \). For \( n \gg \varphi(a) \) (where the classical density is zero), \( w_n \sim (2\pi n)^{-1} \exp(-2n \cdot \ln e(2n/e \varphi(a))) \).
It is evident from Hewish's sinusoidal screen that there are at least two ways in which very large scattering angles may arise: for \( \varphi^2(a) \ll 1 \), the diffraction angle \( \theta = (ka)^{-1} \) may be made large by decreasing \( a \), but the probability, \( \varphi^2(a) \ll 1 \), is then also small. For \( \varphi^2(a) \gg 1 \), a large number of scatterings \( n \gg \varphi(a) \) may also give very large deflection angles, but with small probability. The first possibility is found to apply for type A spectra, whereas the second holds for Gaussian-like spectra.

Another kind of one-dimensional sinusoidal screen was found independently by the author. The phase \( \varphi(x) \) is not determinant, but its spectrum is sharply peaked so that phase variations are nearly all of one spatial wavelength. As a mathematical approximation,

\[
\Phi_L(q) = \frac{1}{2} \varphi^2 \left\{ \delta(q - \frac{1}{a}) + \delta(q + \frac{1}{a}) \right\}
\]

\[
\varphi^2(r) = 2 \varphi^2 \left\{ 1 - \cos \left( \frac{r}{a} \right) \right\}; \quad s_L(r) = \cos \left( \frac{r}{a} \right)
\]

If the finite width of the spectral lines of \( \Phi_L(q) \), say \( \delta q \), were included, the idealized correlation \( s_L(r) = \cos(r/a) \) would decrease to zero for \( r \gg (\delta q)^{-1} \gg a \). The width \( \delta q \) of the actual lines is important because we assume \( \varphi(x) \) to be a Gaussian random variable. However, the effects of taking \( \delta q = 0 \) may usually be deduced afterwards. The angle density obtained from equation (58) and (45) is:
\[ W(\theta) = \sum_{n=-\infty}^{\infty} W_n \delta(\theta - \frac{n}{ka}) \]

\[ W_n = I_n(\phi_L^2) \cdot e^{-\phi_L^2} ; \quad \sum_{n=-\infty}^{\infty} W_n = 1 \]  

(61)

where \( I_n \) is a Bessel function of imaginary argument. If the finite width of the \( \Phi_L(q) \) lines is included, then the side bands of the angle density are not infinitely sharp, but have a width \( \Delta\theta \sim \frac{\delta q}{k} \). In weak scattering equation (61) is similar to Hewish's sinusoidal screen. However, for \( \phi_L^2 \gg 1 \), \( W_n \sim (\sqrt{2\pi}\phi_L)^{-1} \exp(-n^2/2\phi_L^2) \). This corresponds to a smoothed angle density,

\[ W(\theta) = (\sqrt{2\pi}\theta_o)^{-1} \cdot \exp\left\{-\frac{\theta^2}{2\theta_o^2}\right\} \]

\( \theta_o \equiv \frac{1}{ka} \phi_L ; \quad \int_{-\infty}^{\infty} \theta W(\theta) = 1 \)  

(62)

From this example and another discussed below, it is clear that the angle density is a Gaussian function for a wide class of irregularity or phase spectra \( \Phi_L(q) \) (those termed Gaussian-like) when the phase shift \( \phi_L^2 \gg 1 \).
§1. Gaussian Spectrum:

Evaluation of equation (58) for the Gaussian spectrum \( F_G \) gives,

\[
W(\theta) = W_0 \delta^{(2)}(\theta) + \sum_{n=1,2} W_n \frac{(ka)^2}{2\pi n} e^{-\frac{(ka\theta)^2}{2n}}
\]

\[
W_n \equiv \frac{1}{n!} (\phi_z^2)^n e^{-\phi_z^2} ; \sum_{n=0,l,...} W_n \equiv 1
\]

The different terms of equation (63), \( n = 0,1,2,... \) are interpreted as zero, single, double, etc., scattering. Each has a probability of occurring of \( W_n \) and an a r.m.s. scattering angle \( (2n)^\frac{1}{2}(ka)^{-1} \). Observe that \( W_n \) is the probability of a Poisson process for the random number of scatterings \( n \). Therefore, the average number of scatterings is generally \( \bar{n} = \phi_z^2 \), and the mean square is \( n^2 = \bar{n}(\bar{n}+1) \).

For large \( \phi_z^2 \gg 1 \), \( W_n \approx (2\pi \bar{n})^{-\frac{1}{2}} \exp\left(-\frac{(n-\bar{n})^2}{2\bar{n}}\right) \), and thus the wave is scattered nearly exactly \( \bar{n} = \phi_z^2 \) times.

In contrast with the one-dimensional models discussed above (for phase shifts much larger than unity) there are typically many more scatterings, and related to this the characteristic deflection angle of the nth sideband is proportional to \( n^{\frac{1}{2}} \), which reflects a random addition of individual scattering angles, each of order \( \pm (ka)^{-1} \). In the \( \phi_z^2 \gg 1 \) limit equation (63) may be approximated by the single term \( n = \phi_z^2 \).
\[ W(\Theta) = \left( \frac{2\pi \Theta^2_s}{\omega} \right)^{-1} \exp\left\{ -\frac{\Theta^2}{2\Theta^2_s} \right\} \]

Equation (64) is in analytic form the same as the density for the case where there is only a narrow range of irregularity sizes, and a wide distribution of the number of scatterings, equation (62).

Hewish and Wyndham (1963) discuss different cases which may arise in experiments designed for observing the angle density of equation (64). For simplicity consider just one kind of observation of interplanetary scintillations: a multiplying amplitude interferometer with baseline \( \mathbf{b} = (b_x, b_y) \) in the x-y plane of an observer at \( z = D + L \), a distance \( D \) beyond the front of the screen. For no scattering the visibility, or product of amplitudes from the two elements of the interferometer, as a function of time is of the form \( V_0(t) = \cos(kt \mathbf{\Omega} \cdot \mathbf{b}) \), where \( \mathbf{\Omega} \) is a constant vector to account for rotation of the earth.

The Hewish and Wyndham case (a) is: \( \Theta_s D \gg a \); the observer receives simultaneously many rays from different irregularities on the screen. The instantaneous visibility
has an angular size \( \sim \theta_s \). The visibility averaged over a transverse distance of \( \delta x \sim \frac{\lambda}{2\pi} \theta_s \) (or a time interval of \( \delta x/|u| \), where \( u \) is the solar wind velocity) is just the amplitude correlation of equation (58), 
\[
\exp\left( -\frac{1}{2} \rho_L^2 \right)
\]
times the no-scattering visibility \( V_0(t) \). This is the familiar Fourier transform relation (equation 58) between the average source brightness distribution \( W(\theta) \) and the visibility.

The Hewish and Wyndham case (b) is: \( \theta_s D \ll a \). At any instant only a single ray or beam is received from a single irregularity on the screen. The instantaneous visibility is: 
\[
V(t) = \cos(kt \mathbf{n} \cdot \mathbf{b} + k \mathbf{\theta}(t) \cdot \mathbf{b})
\]
where \( \mathbf{\theta}(t) \) is the random angle with respect to the z-axis of the received beam. If \( \mathbf{\theta}(x) \) denotes the deflection angle of a geometrical optics ray on the screen at a point \( x = (x,y) \), then approximately \( \mathbf{\theta}(t) = \mathbf{\theta}(x = ut) \), where \( u = (u_x, u_y) \) is the solar wind velocity in the plane perpendicular to the line of sight. The geometrical optics angle \( \mathbf{\theta}(x) \) is discussed in detail later (Part III; A; §1); however, the probability density of \( \mathbf{\theta}(x) \) is that given by equation (64). And \( \mathbf{\theta}(x) \) typically changes by itself for a change in \( x \) of the order of \( a \). Thus \( \mathbf{\theta}(t) \) varies randomly with a time scale of \( \sim a/|u| \).

With a suitable variable delay line the known phase \( kt \mathbf{n} \cdot \mathbf{b} \) may be eliminated and the random phase \( \mathbf{\phi}(t) = k \mathbf{\theta}(t) \cdot \mathbf{b} \) derived. \( \mathbf{\phi}(t) \) may be characterized by the amount it changes in a time interval \( \tau \): 
\[
\Delta \mathbf{\phi}(\tau) = \mathbf{\phi}(t+\tau) - \mathbf{\phi}(t)
\]
The mean-square of \( \Delta \mathbf{\phi}(\tau) \) is obtained as,
\[ \langle \{ \Delta \mathcal{P}(\tau) \}^2 \rangle = k^2 \langle \Theta^2(t) \rangle b^2 \left( 1 - \frac{\nabla^2 \rho_L(\tau \tau)}{\nabla^2 \rho_L(0)} \right) \] (66)

where \( \rho_L(x) \) is defined by equation (35) for \( z = L \) and \( \nabla^2 \) is the transverse Laplacian. \( \langle \Delta \mathcal{P}^2(\tau) \rangle \) increases linearly with \( \tau \) up to a maximum \( k \langle \Theta^2 \rangle / b \) for \( \tau > a \mu \). The visibility which remains after removal of the known phase is of the form \( \cos(\mathcal{P}(t)) \), which averages to \( \exp(-\frac{1}{2} \mathcal{P}^2(t)) \) over periods \( \tau > a \mu \). This is the same as the reduction in visibility in Hewish and Wyndham's case (a).
§2. Type A Spectrum

The angle density for type A irregularity spectra $F_k$ is also given by equation (58),

$$W(\bar{\theta}) = \frac{k^2}{2\pi} \int_0^\infty r \, dr \, J_0(k|\theta|r) e^{-\frac{1}{2} \frac{C_1^2 \beta^2 (r/r_0)^2}{\theta^2}}$$

$$W(\bar{\theta}) = (2\pi \theta_S^2) \theta_S^{-1} O_A \left( \frac{1}{\theta} \right) ; \quad q_1 < |k\theta| < q_2$$

$$\theta_S \equiv \left( C_1 \frac{1}{\beta^2} \right)^{1/2} B^{1/2} \left( k \frac{r_0}{\lambda} \right)^{-1}$$

$$G_A(\eta) \equiv \int_0^\infty \eta \, d\xi \int_0^\infty \eta \xi e^{-\xi^2 \eta} ; \quad \int_0^\infty \eta \, d\eta \, G_A(\eta) = 1 ; \quad G_A > 0$$

$$S \equiv (S-2)/2 ; \quad C_1 = \text{given in equation (52)}.$$ 

$\theta_S$ is the characteristic scattering angle. $G_A(\eta)$ is a dimensionless function of a dimensionless argument. Equation (67) is well defined only for type A spectra, $0 < \delta < 1$.

The explicit dependence of $\theta_S$ on different parameters is:

$$\theta_S \sim K \gamma_N^{-1/2} \gamma_e^{1/2} \lambda^{1+\delta} \leq \lambda^{1/2\delta}$$

Observe that the radio wavelength dependence and that on the propagation distance through the medium, $\delta$, differ markedly from the Gaussian case (equation 65) when $\delta$ is significantly less than unity.

$G_A(\eta)$ may be evaluated as an elementary function for a logarithmic slope of $s = 3$ or $\delta = \frac{1}{2}$.
\[ G_A(\eta) = (1 + \eta^2)^{-3/2} ; \quad \delta = \frac{1}{2} \]  \hfill (69)

For other values of the slope, asymptotic expressions in different limits are:

\[ G_A(\eta) \approx \frac{1}{2\delta} \Gamma\left(\frac{1}{\delta}\right) - \eta^2 \frac{\Gamma\left(\frac{2}{\delta}\right)}{8\delta} ; \quad \eta \ll 1 \]  \hfill (70)

\[ G_A(\eta) \approx \delta \eta^{1+2\delta} \frac{\Gamma\left(1+\delta\right)}{\Gamma\left(1-\delta\right)} \eta^{-2-2\delta} ; \quad \eta \gg 1 \]  \hfill (71)

Note that a mean-square scattering angle is not defined for the angle density arising from type A spectra. However, this is of no matter at least for IPS observations because for the cases usually assumed of a Gaussian angle density with \( \theta_L^2 \gg 1 \), the conventionally tabulated angle (Hewish, 1958) is the e-folding radius of \( W(\theta) \): \( \Theta_e = 2^{1/2} \theta_5 \). This angle is related to the baseline \( b_e \) for which the visibility is reduced to \( e^{-1} \) by \( \Theta_e = \frac{\lambda}{\pi b_e} \).

If the last formula is used to define \( \Theta_e \) for type A spectra, then \( \Theta_e = 2 \theta_5 \), where \( \theta_5 \) is given in equation (67).

Two facets of the angle density for type A power law spectra are noteworthy: one is that for large angles \( G_A \) falls off relatively slowly, as a power law, in contrast with the Gaussian fall-off in the Gaussian case. Another is
that apparently no fraction of the incident wave energy is unscattered; that is, there is no delta function component in the density $G_A$. It is interesting to trace the different behaviour to differences of the irregularity spectra. For the moment consider a situation where the intensity of turbules with dimensions of the order of the diffraction length $\delta x$ with $\delta x < a$ (where $\mathcal{P}(\delta x) = 1$ and $\phi_z^2 > 1$) are roughly equal for the Gaussian $F_G$ and power law spectrum $F_K$. Then either much larger or much smaller turbules are stronger in the $F_K$ spectrum. However as noted turbules of size $\delta x$ determine the characteristic scattering angle $\Theta_s = \frac{\lambda}{2\pi} \delta x$. Thus these angles are equal in the two cases.

Now examine scattering by phase fluctuations of spatial wavelength or scale $r \ll \delta x$ and $r \gg \delta x$, in the way suggested by Salpeter (1969). If there were only fluctuations of one wavelength $r \ll \delta x$ with amplitude $\mathcal{P}_z(r)$, then equation (63) would hold and the associated scattering (or more precisely diffraction) angle would be $\Theta_r \approx \frac{\lambda}{2\pi r} \mathcal{P}_z(r)$.

If there were only wavelengths around $r \gg \delta x$, then the scattering angle would be $\Theta_r \approx \frac{\lambda}{2\pi r} \mathcal{P}_z(r)$. Summarizing,

$$\Theta_r \approx \frac{\lambda}{2\pi r} \gg \Theta_s ; \mathcal{P}_z(r) \ll 1 ; r \ll \delta x \left(72\right)$$

$$\Theta_r \approx \frac{\lambda}{2\pi r} \mathcal{P}_z(r) \ll \Theta_s ; \mathcal{P}_z(r) \gg 1 ; r \gg \delta x$$

Thus small irregularities cause large angle diffraction and correspond to weak scattering. If $\mathcal{P}_z(r)$ and $r$ for the power law case are identified with $\phi_z$ and $a$ of the
Gaussian case, then the probability of diffraction through a large angle \( \theta_r \gg \theta_0 \) is small. In fact, this probability that \( \theta > \theta_r \) is simply \( w = \varphi^2 \left( \frac{\lambda}{2 \pi \theta_r} \right) \). This is identical to equation (71) aside from a numerical factor. Of course, for \( r < q_1^{-1} \), \( \varphi_Z(r) \) decreases more rapidly than indicated by equation (52) and therefore the angle density falls off more rapidly than equation (67) for \( \theta > \lambda q_2 / 2 \pi \).

The origin of the \( \theta \gg \theta_0 \) tail of the angle density for the Gaussian spectrum \( F_G \) is distinctly different. In this case with \( \varphi_Z^2 \gg 1 \) almost always the wave is scattered \( n = \varphi_Z^2 \gg 1 \) times. The resulting angle is roughly \( \theta \sim (ka)^{-1} \sum_{l} (\pm 1) \), which has a Gaussian density. This density has a cut-off at roughly \( \theta = (ka)^{-1} \varphi_Z^2 \), which is much larger than the r.m.s. angle.

The reason there is 'no' unscattered component for type A spectra is that the amplitude of phase fluctuations of wavelength \( r \gg \delta x \), \( \varphi_Z(r) \), increases monotonically with \( r \). For the Gaussian spectrum the amplitude of components with \( r > \delta x \) saturates at a level \( \sim \varphi_Z \), and therefore each leaves a small fraction of the wave energy unscattered. For type A, the fraction unscattered by phase components \( r > \delta x \) decreases with increasing \( r \). However, if account is taken for the fact that \( \varphi_Z(r) \) ceases increase for \( r > q_1^{-1} \), then it can be shown that there is a minute unscattered component \( \sim \exp(-\frac{1}{2} \varphi_Z^2(q_1^{-1}) ) \ll 1. \)
A one-dimensional correlation function which displays properties of type A spectra and also is convenient for numerical simulation of scintillations was suggested by H. Hardebeck (1967). It is:

\[
\begin{align*}
\Phi_L(x) &= \begin{cases} 
1 - \frac{|x|}{a} & ; |x| \leq a \\
0 & ; |x| > a
\end{cases} \\
\phi_L^2(x) &= 2 \phi_L^2 \frac{|x|}{a} \\
\phi_L^2 &= \begin{cases} 
2 \phi_L^2 & ; |x| \leq a \\
0 & ; |x| > a
\end{cases} \\
\Phi_L(q) &= \frac{a}{2\pi} (\sin(qa^2)/(qa^2))^2 ; \\
\phi_L^2 &= \int_{-\infty}^{\infty} q \Phi_L(q)
\end{align*}
\tag{73}
\]

This correlation is for example that between square pulses of width \( a \). It is easily obtained numerically by taking a running mean of a white noisy signal. Note that for \( qa \gg 1 \), the sine squared averages to one-half and \( \Phi_L(q) \) is a pure power law \( \alpha \propto q^{-2} \). The correspondence of parameters \( \Phi_L \) and \( a \) with the scintillation strength \( \beta \) and distance scale \( r_o \) is: \( \beta^2 = \frac{2}{\pi} \phi_L^2(r_o/a) \) and \( \phi_L^2(r) = \pi \beta^2 \frac{|r|}{r_o} \) for \( |r| \leq a \gg r_o \). In addition, there is a qualitative correspondence of the one-dimensional \( q^{-2} \) spectrum to a two-dimensional \( q^{-3} \) spectrum.

Assuming a Gaussian random phase \( \phi(x) \), the angle density is:

\[
W(\theta) = e^{-\phi_L^2} \delta(\theta) + \frac{1}{\pi \theta_0^2} \left( 1 - e^{-\phi_L^2} \frac{\cos(k\theta a) + \phi_L^2 \sin(k\theta a)}{1 + \theta^2/\theta_0^2} \right)
\]

\[
\theta_0 = \frac{1}{k a} \phi_L^2 = \frac{\pi}{2} \frac{1}{k r_o} \beta^2 ; \quad \int_{-\infty}^{\infty} d\theta \ W(\theta) = 1
\tag{74}
\]
The limit of the density (74) which corresponds to the prior treatment of type A spectra is \( \varphi_L^2 \gg 1 \); that is, strong scattering (but not necessarily strong scintillations \( \beta \ll 1 \)). The angle density for \( \varphi_L^2 \gg 1 \) becomes simply \( W(\vartheta) = (\pi\vartheta_o)^{-1}(1 + \vartheta^2/\vartheta_o^2)^{-1} \). The scattering (and scintillations) then depend on \( \beta \) and \( r_o \), and the parameters \( \varphi_L \) and \( a \) disappear entirely from the formulae.

* * *

The Hewish and Wyndham (1963) discussion of observations of the angle density must be modified for type A spectra. First \( \vartheta_{\theta} \) has a new definition in equation (67). For observations at \( z = D + L \), a distance \( D \) beyond the front of the screen, in case (a) \( \vartheta_{\theta}^D \gg \delta x \); Many beams from different irregularities of size \( \delta x \) are received simultaneously. The instantaneous visibility has an angle size of \( \sim \vartheta_{\theta} \). The visibility averaged over a transverse distance of \( \sim \delta x \) in the observer's plane (or a time interval of \( \gamma \sim \delta x/u \) for the case of the screen moving past the line of sight with velocity \( u \)) is:

\[
V = \exp(-\frac{1}{2}(\varphi_L^2(\vartheta_{\theta}/u)),
\]

where \( b = \) the projected or x-y baseline.

However, in case (b), \( \vartheta_{\theta}^D \ll \delta x \), the instantaneous visibility is not that of a point source. This is in contrast with the Hewish and Wyndham case (b) for Gaussian irregularities where \( \vartheta_{\theta}^D \ll a \) (and \( \varphi_L^2 \gg 1 \)) where the instantaneous visibility is that of a point source. Although only
a single beam is received at any instant, it still has an angular size of roughly $\frac{\theta}{s}$ due to diffraction by small turbules. In case (b) a spatial average over a distance $\sim \delta x$ (or a time average of $\sim \delta x / |\mu|$) is required to obtain a stable average of the visibility.
§3. Type B Spectrum:

Treatment of the angle probability density for type B spectra (4 < s < 6 or 1 < \$ < 2) requires considerations rather different from those applied to type A and Gaussian spectra. In particular, we assume from the outset observations at a distance \( z = D + L \), beyond the screen, with for example an interferometer. Throughout, the second phase difference for the largest turbules is assumed much larger than unity, \( \mathcal{\Omega}_2(q_1^{-1}) \gg 1 \), and for the smallest turbules much less than unity, \( \mathcal{\Omega}_2(q_2^{-1}) \ll 1 \). The largest turbules are assumed of the order of or smaller than the screen thickness \( L \).

Previously, Salpeter(1969) gave a qualitative treatment of type B spectra scintillations in a model for pulsar amplitude variations. Some of his formulae are verified, and new results are discussed.

The dominance of large intense turbules in type B spectra makes geometrical optics applicable subject to certain restrictions. The local ray direction on the front of the screen (i.e., on \( z = L \)) is:

\[
\varphi(x) = k^{-1} \varphi(x),
\]

where \( \varphi(x) = \varphi(x,y,L) \). \( \theta(x) \) is determined by the largest turbules. It has a typical value \( \sim \theta_o = (\lambda q_1/2\pi) \cdot \) \( \mathcal{\Omega}_2(q_1^{-1}) \) (assumed much less than a radian), and \( \theta(x) \) changes by itself in a distance \( \sim q_1^{-1} \). We assume that the largest turbules considered by themselves focus well beyond the observer's plane: \( \theta_o \ll (Dq_1)^{-1} \). Then at any instant an observer receives at least one or possibly several
rays from the screen all from an area on the screen of size smaller than $q^{-1}$. Arbitrarily let $	ilde{\Theta}(\mathbf{x}_o) = k^{-1} \nabla \Theta(\mathbf{x}_o)$ denote one of the received rays, which may be used as a reference ray. The angles of other received rays at the same instant differ in angle from $\Theta(\mathbf{x}_o)$ by much less than $\Theta_o$; therefore, $\tilde{\Theta}(\mathbf{x}_o)$ may be interpreted as the source position measured with a 'low' resolution interferometer.

The change in the ray direction in a transverse distance $r$ across the screen is denoted

$$
\Delta \tilde{\Theta}(x, r) \equiv \tilde{\Theta}(x + r) - \tilde{\Theta}(x)
$$

(75)

The root-mean-square angle difference is:

$$
\langle \Delta \Theta^2(r) \rangle^{1/2} = C_5 \frac{1}{kr_o} \beta \left( \frac{r}{r_o} \right)^{\delta-1}; \frac{1}{q^{-1}} \ll r \ll q^{-1}
$$

(76)

$$
\delta = (s-2)/2; \quad 1 < \delta < 2; \quad r \equiv |r|
$$

$r_o$ = arbitrary distance scale of equation (52).

$\beta$ = scintillation strength parameter of equation (52).

$$
C_5^2 = \frac{2^{-2\delta}}{\delta-1} \frac{\Gamma(2-\delta)}{\Gamma(3)} \pi (2\pi)^{\delta} C_o^2
$$

$C_2^2$ = constant of equation (52).

A restriction on (76) is that the distance $r$ be such that $\langle \tilde{\Theta}(r) \rangle \gg 1$. This is a requirement that the phase variations determining $\Delta \Theta(x, r)$ (due to turbules of size $|r|$) be larger than a radian. This is a necessary condition for geometrical optics to hold.
The probability density for $\Delta \Theta (x, r)$ is Gaussian for $r \ll L$:

$$W(\Delta \Theta) = \left( \pi \langle \Delta \Theta^2 (r) \rangle \right)^{-\frac{1}{2}} \exp \left\{ -\frac{\Delta \Theta^2}{\langle \Delta \Theta^2 (r) \rangle} \right\}$$

$$\int_{-\infty}^{\infty} d^2(\Delta \Theta) \ W(\Delta \Theta) = 1$$

(77)

Thus each turbule of size $\sim r$ has its own angle density.

For interplanetary scintillations the motion of the screen past the line of sight to the source, with transverse velocity $u$, may lead to position variations of the source. The coarse grained position observed at a distance $D$ from the screen is $\overset{\sim}{\Theta} = \Theta (x + \overset{\sim}{\Theta} \cdot D)$, where $x$ is the coordinate of the observer and $x + \overset{\sim}{\Theta} \cdot D$ the coordinate of the area on the screen from which rays are received. Motion of the screen with velocity $u$ is equivalent to motion of the observer with opposite velocity; therefore, the observed angle as a function of time is $\overset{\sim}{\Theta}_t = \Theta (ut + \overset{\sim}{\Theta}_t \cdot D)$. The change in observed position in a time interval $\tau$ is:

$$\Delta \overset{\sim}{\Theta}_\tau = \overset{\sim}{\Theta}_{t+\tau} - \overset{\sim}{\Theta}_t \left( ut + \overset{\sim}{\Theta}_t \cdot D \right)$$

or

$$\Delta \overset{\sim}{\Theta}_\tau = \Delta \Theta (ut + \overset{\sim}{\Theta}_t \cdot D, u\tau) + \Delta \Theta (ut + \overset{\sim}{\Theta}_t \cdot D, \Delta \overset{\sim}{\Theta}_\tau \cdot D)$$

(78)

where $\Delta \Theta (x, r)$ is defined by equation (75). The root mean square of the first term on the right side of (78) is
\[ \left\langle \Delta \bar{\theta}^{2} | u \tau | \right\rangle \right\rangle^{\frac{\nu}{2}} \], which is given by (76). The second term on the right of (78) is a random function of random argument, but an estimate of its magnitude is still,

\[ \left\langle \Delta \bar{\theta}^{2} \left( | u \tau | \right)^{2} \right\rangle \right\rangle^{\frac{\nu}{2}} . \]

Hence the second term on the right of (78) may be neglected for \(| u \tau | \gg \left\langle \Delta \bar{\theta}^{2} \right\rangle \right\rangle^{\frac{\nu}{2}} . \) If this condition holds, the random position of a source is characterized by its typical change in a time interval \( \tau \) given by

\[ \left\langle \Delta \bar{\theta}^{2} \right\rangle \right\rangle^{\frac{\nu}{2}} = \left\langle \Delta \bar{\theta}^{2} (| u \tau |) \right\rangle \right\rangle^{\frac{\nu}{2}} \propto \tau^{\frac{\nu}{2}} . \]

This property of type B spectra has not been noted previously in the literature. We return to the position variations in (74).

From equation (76) it is clear that more than one ray may be received by an observer at any instant. We already assumed that a ray is received from the point \( \bar{x}_{0} \) on the screen. The geometrical condition obeyed by the other received rays from points \( \bar{x}_{0} + \bar{r}_{\alpha} \) \( (\alpha = 1, 2, \ldots) \) is:

\[ \Delta \bar{\theta} \left( \bar{x}_{0}, \bar{r}_{\alpha} \right) \right\rangle \right\rangle^{\frac{\nu}{2}} = \bar{r}_{\alpha} . \]

The condition may also be written as

\[ 0 = \nabla_{\bar{r}} \Omega \left( \bar{r}_{\alpha} \right) , \]

where

\[ \Omega \left( \bar{r}_{\alpha} \right) \equiv \frac{K}{D} \bar{r}_{\alpha}^{2} - \Sigma \left( \bar{x}_{0}, \bar{r}_{\alpha} \right) \]

\[ \Delta \bar{\theta} \left( \bar{x}_{0}, \bar{r}_{\alpha} \right) = \nabla_{\bar{r}} \Sigma \left( \bar{x}_{0}, \bar{r} \right) ; \Sigma \left( \bar{x}_{0}, 0 \right) = 0 \]

and where \( \Sigma \left( \bar{x}_{0}, \bar{r} \right) \) is a function with statistical properties similar to those of \( \Delta \bar{\theta} \left( \bar{x}, \bar{r} \right) ; \left\langle \Sigma^{2} (\bar{r}) \right\rangle \propto \bar{r}^{\nu} . \) The level points of the surface \( \Omega \left( \bar{r} \right) \) correspond to points on the screen from which rays are received at one instant. Alternatively, these points correspond to a non-
zero brightness distribution. For very large $|\bar{r}|$, $\Omega$ is typically large, positive, and increasing. However, for radii less than that for which $\langle \sum_{i}^{2}(r) \rangle \sim kr^2/2D$, the level points are fairly numerous. The critical radius is roughly $\sim r_S = r_o \left( \frac{D}{kr_0^2} \beta \right)^{\frac{1}{2-\delta}}$ and the corresponding angle is $\sim \Theta_S = r_S/D$. Some idea of the density of level points is obtained from the number of zeros of $\mathcal{N}(\bar{r})$ along a radius. For the one-dimensional case, between adjacent zeros at least one level point is expected. The number of zero crossings of $\mathcal{N}(\bar{r})$ along a radial distance from 0 to $r$, is $\sim \ln_e(q_2^r)$ for $r < r_S$ and $\sim \ln_e(r_S q_2^r) \gg 1$ for $r > r_S$, where $q_2^{-1}$ is the smallest or inner scale of the turbulence. Of course, diffraction would prevent observation of extremely fine scale structure of the brightness distribution. Later it is shown (Part II, B; § 5) that $\beta^2 \gg 1$ is a necessary condition for the prior discussion to hold. Also we have assumed $q_2^{-1} < r_S < q_1^{-1}$.

Consider now the probability density of angles $\sim \Theta_S$. Assume a large number of independent realizations or instantaneous pictures of the received scattered rays from the source. For each arbitrarily (i.e., randomly) pick one ray $\hat{\Omega}(x_{o_j})$ which intersects the point of observation. Then one may determine the probability density of $\delta \hat{\Omega}_j = \hat{\Omega}(x_{o_j}) - \hat{\Omega}(x_{o_j})$. Of course, for some realizations $\hat{\Omega}(x_{o_j})$ may be near the periphery and others near the center of the instantaneous distribution of received rays. Nevertheless, the density of $\delta \hat{\Omega}$ is a valid statistical function, even
even though it is not exactly what we would desire.

The phase $\phi(x)$ is renormalized thusly:

$$\tilde{\phi}(x, x_o) = \phi(x) - \phi(x_o) - k(x - x_o) \cdot \vec{r}(x_o)$$  \hfill (80)

From the discussion above it follows that the amplitude correlation $C_A$ of equation (58) multiplied by

$$\exp(-ik\vec{r}(x_o) \cdot \vec{r})$$

is averaged in the usual way. In addition $\phi$ is assumed a Gaussian random variable. Unlike for type A or Gaussian spectra $C_A$ depends on both the separation of the amplitude points $\vec{r}$ and their 'center of mass' $\vec{x}$:

$$C_A(x, x - x_o) = e^{ik\vec{r}(x_o) \cdot \vec{r}} e^{-\frac{1}{2} \Psi(r, x - x_o)}$$  \hfill (81)

$$\Psi(x, x - x_o) = \langle (\tilde{\phi}(x + \frac{r}{2}, x_o) - \tilde{\phi}(x - \frac{r}{2}, x_o))^2 \rangle$$

$$\Psi = \iint dq \int dq' \Phi(q) \int dq \Phi(q') \left| 2 \sin \left( \frac{q \cdot r}{\lambda} \right) - q \cdot r e^{iq \cdot (x_o - x)} \right|^2$$

The angle probability density at the screen also depends on $x$: $W = W(\phi(x) - \phi(x_o), x - x_o)$. Thus each point around the reference point $x_o$ scatters or diffracts radiation with its own angle density. The density at the point of origin of the received ray $x_o$ is:
\[ W(\delta \Theta, 0) = \frac{k^2}{(2\pi)^2} \left( \sum_{\infty} \int_{-\infty}^{\infty} e^{-i k \delta \Theta \cdot r} e^{-\frac{i}{2} \delta \gamma(x, 0)} \right) \]

\[ = \frac{k^2}{2\pi} \int_{0}^{\infty} r \, dr \, J_0(k |\delta \Theta| r) e^{-\frac{1}{2} C_6 \beta^2 (\frac{r}{\delta \Theta})} \]

\[ = \left(2\pi \theta_s^2 \right)^{-1} G_B \left( \frac{\delta \Theta}{\theta_s} \right) \]

\[ \theta_s = \left( \frac{\theta_s^2}{\theta_s^2} \right)^{\frac{1}{2}} \frac{1}{k r_0} \beta \]

\[ G_B(\eta) = \int_{0}^{\infty} \xi^2 d\xi \, J_0(\xi g) e^{-\xi^2 \delta \gamma} ; \int_{0}^{\infty} \eta \, d\eta \, G_B(\eta) = 1 ; G_B(\eta) > 0 \]

\[ \psi(r_0) = \frac{C_6^2}{\beta} \left( \frac{r_0}{\delta \Theta} \right)^{2\delta} \]

\[ C_6 = \frac{\pi}{2} \left( 2\pi \right)^6 C_2 \int_{0}^{\infty} \xi^2 d\xi \left[ 1 - J_0(\xi) + \frac{1}{4} \xi^2 - \xi^4 J_1(\xi) \right] \]

(82)

\[ \theta_s \] is the characteristic diffraction angle for type B spectra. It is roughly given by \( \theta_s \sim \frac{\lambda}{2\pi \delta x} \) where \( \delta x \), given by \( \mathcal{N}_2(\delta x) = 1 \), is the diffraction length for type B spectra. It is clear that diffraction has the effect of smearing the points satisfying the geometrical optics condition \( \nabla^2 \mathcal{N}(r) = 0 \). Equation (82) holds without restrictions on \( \beta \).

The probability density of radiation received by an observer a distance \( D \) from the screen is: \( W(\delta \Theta, \delta \Theta D) \), which is denoted \( W_B(\delta \Theta) \). This follows from the geometry. It is clear that we cannot expect \( W_B \) to be exactly normalized to unity.
$W_B$ may be evaluated approximately for $|\delta \Theta| \gg (\lambda/D)^{1/3}$. ($\lambda/D)^{1/3}$ is the characteristic diffraction angle of a straight edge at a distance $D$. This condition on $|\delta \Theta|$ is later (Part II; B; §5) shown appropriate when $\beta^2 \gg 1$, that is, under conditions of strong scintillations. For $\beta^2 \ll 1$, $W_B(\delta \Theta) \approx W(\delta \Theta, 0)$ as given by equation (82). And for $\beta^2 \ll 1$, the scattering is determined by diffraction. For $\beta^2 \gg 1$, scattering angles are determined by geometrical optics. In the latter case assuming $|\delta \Theta| \gg (\lambda/D)^{1/3}$, the dominant contribution to $W_B$ is from $|r| \ll |\delta \Theta| D$, and this permits approximating $\mathcal{V}$ thusly:

$$\mathcal{V}(x, \delta \Theta, D) \approx \left[ C_7^2 + C_8^2 \cos(2\alpha) \right] \beta^2 \frac{(|r|)^2}{r_0^2} \left( \frac{|\delta \Theta| D}{r_0} \right)^{2\delta^2}$$

$$C_7^2 = \frac{1}{2} C_5^2 ; \quad C_8^2 = \frac{5-1}{2} C_7^2$$

where $\alpha$ is the angle between $\delta \Theta$ and $r$. From equation (58), the $\delta \Theta$ probability density is:

$$W_B(\delta \Theta) = \frac{1}{2\pi \sqrt{3s-1}} \frac{1}{\delta + 1} \exp\left\{ -\frac{|\delta \Theta| / \Theta_5}{\Theta_5^{4-2\delta} |\delta \Theta|^{2\delta-2}} \right\} \left[ 2(C_7^2+C_8^2) \right]^{4-2\delta} \beta^{2-5} (kr_0)^{-2-\delta} (D/r_0)^{5-1\delta}$$

$$\Theta_5 \equiv \left[ 2(C_7^2+C_8^2) \right]^{4-2\delta} \beta^{2-5} (kr_0)^{-2-\delta} (D/r_0)^{5-1\delta}$$

$$\sum_{-\infty}^{\infty} d^2(\delta \Theta) W_B(\delta \Theta) = \frac{1}{2(2\delta-1) \sqrt{3s-1}}$$
(85) is the characteristic geometrical optics scattering angle observable at a distance $D$ from the screen. Because for (84) it is assumed $\beta^2 \gg 1$, there is the order $\Theta_S \gg \Theta_S$ and by an independent assumption $\Theta_S \ll \Theta_o$. The angle $\Theta_S$ is interpreted as the instantaneous source size. This angle multiplied by $D$ is of the order of the maximum radius $r_S$ for which level points of the surface $\mathcal{N}$ are probable.

The explicit parameter dependence of $\Theta_S$ is:

$$
\Theta_S \sim K\frac{1}{N} \frac{1}{e^{2-\delta}} \frac{1}{\lambda^{2-\delta}} \frac{2}{L^{2-\delta}} D^{s-1} \frac{\Sigma S}{2-\delta} \quad (85)
$$

Thus the dependence of $\Theta_S$ for type B spectra agrees with that of $\Theta_S$ for type A when $\delta = 1$ (equation 68), which is a case not included in either type B or A. It is also interesting to note that the analytic form of the angle density for type B spectra is similar to that of the visibility for type A under the replacement $\delta \rightarrow 2-\delta$.

Previously, Salpeter (1969) obtained an estimate similar to (85) from the constraint $\Delta \Theta(r) D = r$. In fact, the density for $\Delta \Theta(x, r)$ (equation 77) subject to this constraint is nearly identical to the density $W_B(\delta \Theta)$. (The difference is that the constant $C_8$ of equation (84) is lacking.) It is clear that scattering angles $\Delta \Theta \sim \Theta_S$ arise from turbules of size $r \sim \Theta_S D \sim \Delta \Theta^2(r) \frac{\Sigma S}{2-\delta} D$. Larger angles $\Delta \Theta \gg \Theta_S$ are much less probable because the turbine sizes required, $r \sim \Delta \Theta D \gg \langle \Delta \Theta(r) \rangle \frac{\Sigma S}{2-\delta} D$, have typical
scattering angles much smaller than $|\delta \theta|$.

Because of the proliferation of inequalities in discussing scattering by type B irregularities, it is useful to summarize these as well as the important parameters:

\[ \Theta_0 \approx \frac{\lambda q_1}{2\pi} \rho(q_1^1) = \text{ray deflection angle due to the largest irregularities.} \]

\[ \Theta_S \approx \frac{r_S}{D} = \text{observable angular spread of instantaneous rays (when applicable).} \]

\[ \Theta_5 \approx \frac{\lambda}{2\pi \delta x} = \text{diffraction angle, observable instantaneous source size in weak scintillations.} \]

$q_2^{-1}, q_1^{-1}$ smallest, largest, scales, respectively.

$r_S = \text{size of irregularities determining } \Theta_S$.

$\delta x = \text{size of irregularities giving phase variations of the order of unity, which determine the diffraction angle } \Theta_5$.

Type B assumes in general $q_1 << q_2$ and $\rho(q_2^{-1}) << 1$

$\rho(q_1^{-1}) >> 1$.

For strong scintillations, $\beta^2 >> 1$:

\[ q_2^{-1} \gg \delta x < r_S < q_1^{-1} \]

\[ \Theta_S < \Theta_S < \Theta_0 \]

the angle density is: $W_B(\delta \theta)$.

For weak scintillations, $\beta^2 << 1$:

\[ q_2^{-1} \gg r_S (\text{not really defined}) < \delta x < q_1^{-1} \]

\[ \Theta_S (\text{not defined}) < \Theta_S < \Theta_0 \]

the angle density is: $W(\delta \theta, 0)$.
§4. Position Variations Type B

The typical change in angular position of a source in a time interval $\tau$ previously derived in (§3) for type B spectra may be expressed simply in terms of $\Theta_S$ (equation 84) or $\Theta_S$ (equation 82) and the scintillation time scales $\tau_D \approx \frac{1}{2\pi \Theta_S |u|}$ for $\beta^2 \gg 1$ (see Part II; B; §5) or $\tau_D \approx r_0 |u|$ for $\beta^2 \ll 1$ (Part II; A; §4). If the time interval $\tau$ is measured in multiples of the time scale $\tau_D$, that is, $\tau = n \tau_D$, then,

$$
\left< \frac{\Delta \Theta^2(|u|)}{\Theta_S} \right>^{1/2} \sim m^{-1} \frac{2(5-1)}{2-5} \beta^2 \quad (n \gg 1)
$$

$$
\left< \frac{\Delta \Theta^2(|u|)}{\Theta_S} \right>^{1/2} \sim m^{-1} \beta^{-5/6} \quad (n < 1)
$$

At a given radio wavelength the appropriate ratio of angles must be larger than unity to permit detection of the source motion. Therefore, as small a $\beta > 1$ within practical limits is desirable. Small $\beta$ values require high radio frequency and thus high angle resolution because $\Theta_S$ or $\Theta_S$ is small.

Observation of a source with intrinsic diameter $\Theta_I > \text{Max}(\Theta_S, \Theta_S)$ obviously requires a longer time interval for any position change to be detectible ($\gg \Theta_I$). Observation of two nearby sources with angular separation $\Theta_{sep}$, each of intrinsic diameter $\Theta_I \ll \Theta_{sep}$, may be expected to show relative motion over time intervals $\tau$, if $\Theta_{sep} > \text{Max}(\Theta_S, \Theta_S)$, $\text{Max}(\Theta_I, \Theta_S, \Theta_S) \cdot D < |\nu| \tau < \Theta_{sep} \cdot D$, and
\[ \Theta_{\text{sep}}^D < q_1^{-1} . \] For \( |u\mathcal{C}| > \Theta_{\text{sep}}^D \), the two sources would move in unison.

The maximum position variations of a single source over time intervals \( \mathcal{C} \sim (q_1 |u|)^{-1} \) are of the order of \( \Theta_\theta \). Detection of a position change in an interval \( \mathcal{C} \) would provide a measurement of the intensity of turbules of size \( |u\mathcal{C}| \) (with \( r_S < |u\mathcal{C}| < q_1^{-1} \) for \( \beta^2 > 1 \), or \( \delta x < |u\mathcal{C}| < q_1^{-1} \) for \( \beta^2 < 1 \)) which have no affect on ordinary scintillations (or intensity variations). A sketch of source position as a function of time is shown in Figure (3) for \( \delta \sim 3/2 \).

Wavelength dependence of position: At a wavelength \( \lambda \) the observed coarse grained (smoothed over \( \Theta_S \) or \( \Theta_S \)) source position is given by the implicit relation \( \Theta_\lambda = \Theta_\lambda + \mathcal{H}_\lambda (x + \mathcal{H}_\lambda D) \), where \( x \) is the transverse coordinate of the observer and \( x + \mathcal{H}_\lambda D \) the coordinate on the screen of the area from which rays are received. The \( \lambda \) subscript denotes the explicit wavelength dependence of \( \Theta_\lambda \). Let \( \Delta \Theta_\lambda \) denote the change in observed source position at one point of observation, say \( \mathcal{X} = 0 \), due to a small change in wavelength \( \delta \lambda \ll \lambda \). Then

\[
\Delta \Theta_\lambda = \Theta_\lambda + \delta \lambda \left( \Theta_\lambda + \delta \lambda D \right) - \Theta_\lambda \left( \Theta_\lambda D \right)
\]

\[
= \left\{ \Theta_\lambda + \delta \lambda \left( \Theta_\lambda D + \Delta \Theta_\lambda D \right) - \Theta_\lambda \left( \Theta_\lambda D + \Delta \Theta_\lambda D \right) \right\} + \left\{ \Theta_\lambda \left( \Theta_\lambda D + \Delta \Theta_\lambda D \right) - \Theta_\lambda \left( \Theta_\lambda D \right) \right\}
\]
Angular position $\theta$ at time $t$.

$\Delta \theta(t) \sim \langle \Delta \theta^2(t) \rangle^{1/2} \propto \delta^{-1}$

Position at time $(t+\tau)$.

Maximum change in position $\sim \theta_0$.

Instantaneous source size $= \text{Max}(\theta_s, \alpha_s) \ll \theta_0$.

FIGURE 3: Source Position, Type B Spectrum.
where the notation implies \( \Theta_\lambda = \Theta_\lambda (\Theta_\lambda D) \) if not otherwise defined. In the first term on the right of equation (87) we have isolated the explicit lambda dependence, and in the second on the right that caused by lambda dependence of the spatial coordinate. The second term on the right of (87) has a typical value \( \langle \Delta \Theta_\lambda^2 (|\Delta \Theta_\lambda^D|) \rangle \) given by (76); for \( |\Delta \Theta_\lambda| = \Theta_5 \) this term is of the order of \( \Theta_5 \); for larger \( |\Delta \Theta_\lambda| > \Theta_5 \), the term is smaller than the first term on the right of (87) and may be neglected. Thus we have \( \Delta \Theta_\lambda = \delta \lambda \frac{\partial}{\partial \lambda} \Theta_\lambda = 2 \frac{\delta \lambda}{\lambda} \Theta_\lambda \) for \( |\Delta \Theta_\lambda| > \Theta_5 \), where we have used the fact that \( \Theta_\lambda = k^{-1} \Theta \propto \lambda^2 \). However, there is an additional requirement: in order for (87) to hold \( |\Delta \Theta_\lambda^D| > \delta \lambda \), which is a condition for applicability of geometrical optics. This condition is equivalent to \( |\Delta \Theta_\lambda| > \Theta_5 \). To summarize, \( \Delta \Theta_\lambda = 2 \frac{\delta \lambda}{\lambda} \Theta_\lambda \) if \( \Theta_5 > \Theta_5 \) and \( |\Delta \Theta_\lambda| > \Theta_5 \); or if \( \Theta_5 > \Theta_5 \) and \( |\Delta \Theta_\lambda| > \Theta_5 \).

Consider now the source position at two widely different wavelengths \( \lambda_1 < \lambda_2 \) at \( \lambda_2 \) and \( \lambda_1 / \lambda_2 \) at \( \lambda_1 \) from the above discussion. The difference in position at the two wavelengths is \( = (1 - \lambda_2 / \lambda_1) \Theta_\lambda \). We have \( |\Theta_\lambda^\lambda_2| \sim \Theta_\lambda \) and \( \Theta_\lambda / \Theta_5 \sim (q_1 r_5)^{-\left(\frac{\delta}{-1}\right)} \) for
\[ r_s \ll q_1^{-1} \text{ and } \Theta_s > \Theta_s'. \text{ Or } \Theta_0 / \Theta_s \sim (q_1 \delta x)^{-1} (\delta^{-1}) \]

for \( \delta x \ll q_1^{-1} \) and \( \Theta_s > \Theta_s' \). The angle \( \Theta_0/\lambda \) is thus relatively large. A further fortuitous advantage for observations is that the wavelength dependence of \( \Theta_0/\lambda \) is one power of lambda stronger than the resolution of a fixed baseline interferometer. It is noted that observation of the wavelength dependence of \( \Theta_0/\lambda \) might be possible with an interferometer operated simultaneously at a wavelength \( \lambda \) and \( \lambda/2 \) for measurement at both wavelengths of the angle between two nearby sources (the latter in order to reduce the effect of the large ionospheric refraction). However, we have not as yet seriously investigated observational problems of detecting wavelength dependence of the source position.
§5. Type C Spectrum:

A type C spectrum \( (\xi = 4 \text{ or } \xi = 1) \) combines effects characteristic of type A and type B. We have already seen that the angle density for type A irregularities is due mainly to diffraction, whereas that for type B (with \( \beta^2 \gg 1 \)) is described by geometrical optics depending on the observer's distance beyond the screen. For a qualitative treatment of type C irregularities, assume observations in a plane at \( z = D + L \), a distance \( D \) beyond the front of the screen. Assume phase shifts of the largest turbules \( \varphi_{L_d}(q_1^{-1}) \gg 1 \) and of the smallest, \( \varphi_{L_d}(q_2^{-1}) \ll 1 \). The diffraction length \( \delta x(q_2^{-1}) \) is given by \( \varphi(\delta x) = 1 \) and the diffraction angle by \( \Theta_5 = \lambda/2\pi \delta x \).

The remarkable property of type C irregularities is that the geometrical optics deflection angles due to turbules (or phase components) of size \( r > \delta x \), \( \Theta_r = \frac{\lambda}{2\pi r} \varphi_{L_d}(r) \) (equation 72) is independent of \( r \) and roughly equal to \( \Theta_5 \). At first it might appear that each phase component with \( r > \delta x \) would separately contribute a deflection angle \( \sim \Theta_5 \), and that the observable source size, \( = \Theta_5 \), would be a random walk combination of these different contributions. However, for observations at any instant it is impossible to 'see' phase variations over transverse distances larger than \( \Theta_5 D \), where \( \Theta_5 \) is now more precisely identified as the typical instantaneous source size (the same as for type B spectra). Hence an estimate for \( \Theta_5 \) may be obtained by
summing the angles contributed by turbules of sizes in the range \( \delta x \) to \( \Theta_S D \). We assume \( \Theta_S D > \delta x \); if this were the other way around the observable instantaneous source size would be \( \Theta_S \) and the corresponding scintillations would be weak. The mean-square geometrical optics scattering angle expressed as a (random walk) sum of the contributions of different turbules each of size \( d^2 \) is:

\[
\langle \Theta^2 \rangle \approx \frac{2\pi}{R^2} \int_{q_a}^{q_b} q^2 dq \left[ q^2 \phi_L(q) \right] \tag{88}
\]

From the discussion, the limits of integration on (88) are 
\( q_a = (\Theta_S D)^{-1} \) to \( q_b = (\delta x)^{-1} \). Hence

\[
\begin{align*}
\xi_\xi &= \ln_e(\xi) + \ln_e(R) \\
\xi &= \Theta_S / \Theta_o ; \quad \Theta_o &= \frac{4\pi^2 k_N (r_e \lambda)^2 L}{k^2} \\
R &\approx \Theta_o D / \delta x
\end{align*}
\tag{89}
\]

where \( \Theta_o \) is approximately equal to \( \Theta_S \approx \frac{\lambda}{2\pi \delta x} \), and thus \( R \approx k \Theta_o^2 D \). In order for there to be solutions \( \xi \) to equation (89), \( R \geq \sqrt{2} e^{\frac{1}{2}}. \) And for \( R > \sqrt{2} e^{\frac{1}{2}} \) there are two solutions: one \( \xi > 1/\sqrt{2} \) and the other \( 0 < \xi < 1/\sqrt{2} \). The latter is unacceptable and is discarded, because \( \Theta_S \) must be an increasing function of \( D \). For \( R = \sqrt{2} e^{\frac{1}{2}} \), \( \xi = 1/\sqrt{2} \); for larger \( R \) (at longer wavelengths for example) say \( R = 100, \xi = 2.33; \) finally for very large \( R \), \( \xi \approx (\ln_e R)^{\frac{1}{2}} \).

Because \( R \approx k_3 \), the radio wavelength dependence of \( \Theta_S \) is slightly stronger than \( \xi^2 \) for any \( R > \sqrt{2} e^{\frac{1}{2}} \).
§6. Position Variations Type C:

Turbules with sizes in the range \( \Theta_S D < r < \Theta_1^{-1} \) for a type C spectrum determine the instantaneous source position \( \sim \Theta_0 \), where the definition of \( \Theta_0 \) is analogous to that for the case of type B spectra. From equation (88), one may obtain the estimate,

\[
\Theta_0 \sim \Theta_0 \left\{ \ln \left( \frac{1}{q_1 \Theta_S D} \right) \right\}^{\frac{1}{2}} ; \delta x \ll \Theta_0 D < q_1^{-1}
\]

(90)

In order for \( \Theta_0 \) to be observable, one needs \( \Theta_0 > \Theta_S \) at a given wavelength. For \( R \gtrsim 1 \), this condition is weak, \( q_1^{-1} > \Theta_0 D \); whereas for \( R \gg 1 \), the requirement is \( q_1^{-1} > \Theta_0 D \cdot R \sim \Theta_0 D^{2} R (\ln R)^{1/2} \).

The change in position of a source in a time interval \( \tau \) due to motion of the screen past the line of sight with velocity \( \ddot{u} \) is determined by turbules with sizes in the range \( \Theta_S D < r < |\ddot{u} \tau| \) for \( |\ddot{u} \tau| < q_1^{-1} \). Thus from equation (88) one obtains,

\[
\left\langle \Delta \Theta^2 (|\ddot{u} \tau|) \right\rangle^{1/2} \sim \Theta_0 \left\{ \ln \left( \frac{|\ddot{u} \tau|}{\Theta_S D} \right) \right\}^{1/2} ; \delta x \ll \Theta_S D < |\ddot{u} \tau| (91)
\]

(91)

In order for the position change to be detectible, one needs \( \left\langle \Delta \Theta^2 (|\ddot{u} \tau|) \right\rangle^{1/2} > \Theta_S \). This holds if \( |\ddot{u} \tau| > \Theta_S D \cdot R \), which is possible if \( \Theta_0 > \Theta_S \). On the other hand the maximum position change is \( \sim \Theta_0 \) for \( \tau \sim (q_1 |\ddot{u}|)^{-1} \).
§7. Source Position Variations in General:

It is noted that only for type A spectra or Gaussian with $\Theta_5 D > a$ is the position of a radio source not dependent on time and wavelength. A simple condition for occurrence of both scintillations and position variations with time and wavelength may be stated for the case where there are just two dominant irregularity sizes: $r_1 \ll r_2$. Assume that the small turblues ($r_1$) cause strong scattering and scintillations and that the large turblues cause only weak scintillations and scatter weakly or strongly, for observations at a distance $z = D + L$. If $\varphi(r_1)$ denotes the r.m.s. phase shift due to the small turblues and $\varphi(r_2)$ that due to the large, then $\Theta(r_1) = \frac{\lambda}{2\pi r_1} \varphi(r_1)$ with $\varphi(r_1)$ larger than unity for strong scattering by the small irregularities, and $\Theta(r_2) = \frac{\lambda}{2\pi r_2} \varphi(r_2)$ for $\varphi(r_2) > 1$, that is for the large turblues strong scattering, or $\Theta(r_2) = \frac{\lambda}{2\pi r_2}$ for $\varphi(r_2) < 1$, weak scattering ($r_2$) turblues. The small turblues give strong scintillations if $D \Theta(r_1) > r_1$ (see Part II; B; §1) and the large turblues weak scintillations if $D \Theta(r_2)$ or $\Theta(r_2) < r_2$.

In order for there to be observable position variations, one needs $\Theta(r_2)$ or $\Theta(r_2)$ larger than $\Theta(r_1)$. This is impossible if $\varphi(r_2) < 1$; therefore, the large turblues must be strong scattering. From equation (43) the condition $\Theta(r_2) > \Theta(r_1)$ translates to $\Delta N(r_2)/r_2^{3\frac{3}{2}} > \Delta N(r_1)/r_1^{3\frac{3}{2}}$, where $\Delta N(r_2)$, $\Delta N(r_1)$, denote the electron
density fluctuations associated with the large, small, turbules, respectively. The conditions \( \frac{\Delta N(r_2)}{r_2} < 1 \) and \( \frac{\Delta N(r_1)}{r_1} > 1 \) give the single condition on \( \Delta N(r) \):

\[
\frac{\Delta N(r_2)}{r_2^{3/2}} < \frac{\Delta N(r_1)}{r_1^{3/2}}.
\]

This and the above inequality on \( \Delta N(r) \) are satisfied if \( r_1 \) and \( r_2 > r_1 \) are turbule sizes of a type B spectrum. Whereas for a type C spectrum, \( \frac{\Delta N(r_2)}{r_2^{1/2}} = \frac{\Delta N(r_1)}{r_1^{1/2}} \); and \( \frac{\Delta N(r_2)}{r_2^{3/2}} < \frac{\Delta N(r_1)}{r_1^{3/2}} \) is simply \( 1/r_2 < 1/r_1 \).

A heuristic expression for general spectra for the turbule size important in determining \( \Theta_o \), which is an upper limit on the source position change, is: \( \max_r \left( \frac{\Delta N(r)}{r^{1/2}} \right) \). Thus \( \Theta_o \) may be determined by turbules much smaller than the largest, \( q^{-1} \), even though the largest give the biggest phase shifts, at least for all of the explicit models assumed here ( \( s > 2 \) ). A case where \( \Theta_o \) is determined by turbules of size much less than the outer scale \( q^{-1} \) is discussed subsequently in ( \( \S 8 \) ).

If scattering occurs in several intervening layers between observor and source, then \( \Theta_o \) is given by \( \Theta_o \sim \max_{r, \alpha} \left( \frac{\Delta N(\alpha)}{L_{\alpha} r^{1/2}} \right) \), where \( \alpha \) labels the different screens involved (at different distances along the line of sight), and \( L_{\alpha} \) is the thickness of the \( \alpha \)th screen. Even though the screen thickness is important, it is not a priori evident that the screen with the largest \( L_{\alpha} \) dominates.

Observations of random position variations of radio sources are reported by Hewish (1952). For ionospheric scintillations \( \Delta \Theta(\gamma) \sim 3' \) min of arc at \( \lambda = 8 \) meters,
with significant variations of \( \phi(t) \) in time intervals of \( \tau \approx \frac{1}{2} \) minute, at the same rate as the relatively weak intensity variations. For interplanetary scintillations there is apparently no positive detection of position changes, and the only data of which the author is aware is the upper limit \( \Delta \phi(\tau) \leq 2' \) min of arc at elongations 5-20 solar radii at \( \lambda \sim 6 \mu m \) over time intervals of \( \tau \sim 15 \) minutes, and a later estimate that position changes due to interplanetary irregularities are less than \( \frac{1}{2} \) minute of arc from day to day for roughly the same wavelength and elongation, both due to Hewish (1955). Almost needless to say interstellar scintillations are in too primitive a state for there to be even limits on \( \Delta \phi(\tau) \). However, measurement of angular positions of sources to accuracy better than \( 10^{-2} \) sec. of arc is contemplated by very long baseline interferometry methods (Cohen et al., 1968). Assuming that contributions, if any, to position variations by different intervening plasmas can be disentangled, then the observations could provide information or at least limits on the media not available by any other method.
§8. Hybrid Spectra:

With the preceding discussion of A, B, C, and Gaussian irregularities, treatment of hybrid spectra is elementary. Below we discuss qualitatively a case we term type AB spectra: type A for \( q_1 < q < q_\star \) with logarithmic slope \( s_\star \): \( 2 < s_\star < 4 \) or \( \delta_\star \equiv (s_\star - 2)/2 \) with \( 0 < \delta_\star < 1 \); type B for \( q_\star < q < q_2 \) with logarithmic slope \( s_B \): \( 4 < s_B < 6 \) or \( \delta_B \equiv (s_B - 2)/2 \) with \( 1 < \delta_B < 2 \). Observations are assumed to be taken at a distance \( z = L + D \), beyond the screen, and we assume \( q_1 \ll q_\star \ll q_2 \) and

\[
\frac{\Phi_2}{\Phi_1} (q_2^{-1}) \ll 1 \ll \frac{\Phi_2}{\Phi_1} (q_1^{-1}).
\]

The important parameters for type AB irregularities are firstly \( q_\star^{-1} = a_\star = \) the scale size at the bend in the wavenumber spectrum and secondly \( \Phi (\frac{a_\star}{\Phi_1}) = \varphi_\star = \) the r.m.s. phase shift associated with these irregularities. Here the parameter \( \beta \) is not very useful. Observe that for \( \varphi_\star < 1 \) the scattering is determined by turbules in the A part of the spectrum: \( q_1 < q < q_\star \). The diffraction length \( \delta x \) is given by \( \Phi (\frac{\delta x}{\delta A}) = 1 \) ( \( \delta x = a_\star \varphi_\star^{-1} \delta A \) ) and is larger than \( a_\star \). The scattering (or better diffraction) angle is:

\[
\Theta_s \simeq \frac{\lambda}{2 \pi \delta x}. 
\]

The scintillations are weak or strong depending on \( \Theta_s D < \delta x \) or \( \Theta_s D > \delta x \), respectively, (see Part II; B; §3). Equivalently, \( D/\varphi_\star = \varphi_\star^{-2} \frac{\delta A}{\delta x} \) divides the weak and strong scintillation domains, where

\[
\varphi_\star \equiv k \frac{a_\star^2}{\delta x} 
\]

is the typical focal length of irregularities of size \( a_\star \) (focal lengths are discussed by Salpeter, 1967).
There are no observable position variations for $\phi_* < 1$.

For $\phi_* > 1$ the problem is slightly more complicated. The diffraction length $\delta x = a_* \phi_*^{-1/2} \beta$ is determined by the B part of the spectrum, and $\delta x$ is less than $a_*$. The diffraction angle is $\Theta_0 = \frac{\lambda}{2\pi \delta x}$ and this is the observable angular source size for $\Theta_S^D < \delta x$ or equivalently $D/\ell_* < \phi_*^{-1} \frac{2-\delta_B}{\delta_B} \Theta_0^D$, which is also a condition for weak scintillations. Nevertheless, there are position variations. The instantaneous position is of the order of $\Theta_0 \simeq \frac{\lambda}{2\pi a_*} \phi_*$ which is generally larger than the instantaneous source size $\Theta_S^D$; that is, $\Theta_0/\Theta_S^D \simeq \phi_* \frac{\delta_B^{-1}}{\delta_B} > 1$ or $1 < \Theta_0/\Theta_S^D < (\ell_*/D) \frac{\delta_B^{-1}}{\delta_B}$. The change in position in a time interval $\tau$ due to motion of the screen with velocity $u_\perp$ is $\langle \delta (u_\perp)^2 \rangle^{1/2} \simeq \frac{\lambda}{2\pi a_*} \phi_* \left( \frac{\|u_\perp\|}{a_*} \right) \delta_B^{-1}$. The minimum detectable position change is $\sim \Theta_S^D$ and occurs for $|u_\perp \tau| \sim a_* \phi^{-1/2} \beta = \delta x$, and the maximum change is $\sim \Theta_0$, which occurs for $|u_\perp \tau| \simeq a_*^2$.

For $\phi_* > 1$ and $D/\ell_* > \phi_*^{-1} \frac{2-\delta_B}{\delta_B}$, the scintillations are strong. There is a domain where the geometrical optics behaviour of type B irregularities alone occurs. The ray optics instantaneous source size for type B is: $\Theta_S \simeq r_S/D$, where $r_S \simeq a_* (D/\ell_*)^{1/2} \delta_B^{-1}$. One needs $r_S$ such that $\delta x < r_S < a_*^2$; alternatively, $1 > D/\ell_* > \phi_*^{-1} \frac{2-\delta_B}{\delta_B}$, or equivalently, $a_*^2 D > \Theta_0^D > \Theta_S > \Theta_0^D$. The instantaneous source position $\Theta_0^D$ is generally larger than the instantaneous source size $\Theta_S^D$; that is, $\Theta_0^D/\Theta_S^D \simeq (\ell_*/D) \left( \frac{\delta_B^{-1}}{2-\delta_B} \right)$ with
the limits \( 1 < \Theta_\ast / \Theta_s < \varphi_\ast \frac{\delta_B^{-1}}{\delta_B} \). The change in source position in a time interval \( \tau \) is: \( \langle \Delta \mathcal{W}^2 / u \tau \rangle \approx \frac{\lambda}{2 \pi a_\ast} \theta_\ast \cdot (|u \tau| / a_\ast) \delta_B^{-1} \). The minimum observable position change is \( \approx \Theta_s \) and occurs for \( |u \tau| \approx r_s \), and the maximum change is \( \approx \Theta_\ast \), which occurs for \( |u \tau| \approx a_\ast \). Observe that both here and above position changes with \( |u \tau| \) are found to 'hang up' on the knee of the q-spectrum rather than continue to increase with \( |u \tau| \) for irregularities larger than \( a_\ast \).

For \( \varphi_\ast > 1 \) and \( D / \ell_\ast > 1 \), the strong scattering and strong scintillations are dominated by irregularities at the bend or knee of the wavenumber spectrum. The largest deflection angles are due to irregularities of size \( a_\ast \). The instantaneous source size is \( \Theta_\ast \approx \frac{\lambda}{2 \pi a_\ast} \theta_\ast \approx \frac{\lambda}{2 \pi a_\ast} \). Because the observer's plane is more distant than a typical focal length from the screen ( \( D > \ell_\ast \) or \( \theta_\ast D > a_\ast \) ) the scattering and scintillations are as described by Salpeter(1967) for large phase shift \( \varphi_\ast > 1 \) in the Fraunhofer limit of Gaussian irregularities of size \( a_\ast \). In this domain there are no observable position variations.

The above conclusions are summarized in Figure (4).

This may be compared with the slightly different diagram in Figure (1) for the case of Gaussian irregularities. The dashed straight line in the Figure indicates what would be observed as a function of wavelength for fixed \( a_\ast \) and \( \varphi_\ast / \lambda \) (which are wavelength independent quantities). That is, \( D / \ell_\ast \propto \lambda^2 \) and \( \varphi_\ast \propto \lambda \) so that \( D / \ell_\ast \propto \varphi_\ast^2 \). For inter-
FIGURE 4: Scattering Regimes, Hybrid Spectrum.
planetary scintillations $a_\star$ and $\varnothing_\star/\lambda$ might be expected to depend on the solar elongation of the source through the dependence of $a_\star$ and that of the electron density fluctuations $\Delta N(a_\star)$ of size $a_\star$ on elongation. From equation (43) we have $\varnothing_\star \propto \Delta N(a_\star) (a_\star L)^{\frac{1}{2}}$ and $D/C_\star \propto D\varnothing_\star/a_\star^2$ at a given wavelength. If we assume simple power law dependences of $a_\star$ and $\Delta N(a_\star)$: $a_\star \propto p^n (n > 0)$ and $\Delta N(a_\star) \propto p^{-m} (m > 0)$ where $p = \text{sin}(\xi)$ a.u., and $\xi$ is the elongation of the source, and $L \propto p$, then one finds $D/C_\star \propto \varnothing_\star^{-\frac{m+\frac{3}{2}n-\frac{1}{2}}{m-\frac{1}{2}n-\frac{1}{2}}}$. Only if the exponent of $\varnothing_\star$ in the last expression is exactly two is the scattering as a function of wavelength for fixed elongation the same as that as a function of elongation for fixed wavelength. It is most likely then that the lines of fixed wavelength (variable elongation) intersect those of constant elongation (variable wavelength). Thus the two kinds of lines could be used as a coordinate system for determining the domain in Figure (4) from observations.

# See Cohen and Gundermann (1969) for a fit of observations of interplanetary scintillations to different power laws under the assumption of Gaussian irregularities.
§ 9. Magnetic Field Effect on Scattering.

The interplanetary medium contains a magnetic field of the order of a Gauss in the corona decreasing to $10^{-5}$ Gauss at $\sim 1$ a.u. It is of interest to know if any method, even if only in principle, can be found for measuring this field from the theory of radio wave scattering. Of course, with a satellite in solar orbit beeping coherent pulses of known polarization, one could measure the Faraday rotation in the usual way. The rotation is expected to be larger than that due to the ionosphere for solar elongations less than about ten solar radii (Evans and Hagfors, 1968), but still less than that of distant polarized radio sources. An unpolarized incoherent source is apparently not useful because any pair of orthogonal polarization states is also orthogonal statistically, that is, zero cross amplitude correlation. Nevertheless, there is in principle under certain conditions a method for deriving the magnetic field from scattered waves received from such a source.

Assume as before a thin screen and observations beyond the screen. Also suppose that there is a quasi-longitudinal ($z$-axis) magnetic field $\mathbf{H}(x,y,z)$. Then in place of the thin screen phase (equation 34), on the front of the screen, one may obtain from equations (9) and (10),
\[
\phi_\pm(x, y, L) = \phi(x, y, L) \pm \phi_H(x, y, L) \\
\phi_H(x, y, L) = \frac{|e|}{m c^2} \int_0^L \int_{-\infty}^{+\infty} H_z(x, y, z') \mu(x, y, z') dz' 
\]

(92)

where \(\phi(x, y, L)\) is the thin screen phase for zero magnetic field, and the \((+/-)\) subscripts on \(\phi\) denote right/left handed circularly polarized electric field phases; \(E_\pm \propto \exp(i\phi_\pm)\). For a constant uniform magnetic field along the z-axis, equation (92) is: \(\phi_\pm = \phi \cdot (1 \pm \omega_c/\omega)\), where \(\omega_c\) is the electron angular gyro frequency \(\omega_c = \frac{|e| H_z}{m c} (\sim 2 \times 10^7 \text{ sec}^{-1} \text{ in the corona})\). For most, that is, not extremely small, elongations we expect \(\omega_c/\omega \ll 1\).

Let us assume further, observations with a two element interferometer, each element being equiped to receive simultaneously right and left handed circularly polarized components. Then with suitable cables, multipliers, etc., one could obtain the right and left handed visibility functions,

\[
V_\pm(b) \equiv \left\langle E_\pm(x, D+L) E^*_\pm(x+b, D+L) \right\rangle 
\]

(93)

where \(x = (x, y)\) is the transverse coordinate of one station and \(x + b = (x+b_x, y+b_y)\) the coordinate of the other. The fields in (93) are evaluated in the plane of observation and are not given by sinusoidal functions of the phases in (92).
The condition under which the two visibility functions \( V_\perp \) could give information about the magnetic field is:

\[
\Theta_\perp \gg (\Theta_S \text{ or } \Theta_S) \quad \text{for type B, type C, or type AB spectra.}
\]

The angular position of the right hand component,

\[
\Theta_+ = \Theta_+ (x + \Theta D),
\]

is different from the left,

\[
\Theta_- = \Theta_- (x + \Theta D).
\]

The observable angle difference is:

\[
\Delta \Theta_+/- = \Theta_+ - \Theta_-.
\]

By a calculation similar to that leading to equation (87) one finds

\[
\Delta \Theta_+/- = 2 k^{-1} \nabla _\perp \Phi_H \quad \text{if } |\Delta \Theta_+/-|< 1
\]

is larger than the larger of \( \Theta_S \text{ or } \Theta_S \). This is also a condition for easy detection of \( \Delta \Theta_+/- \). For the case of a uniform magnetic field along the \( z \)-axis, \( \Delta \Theta_+/- = \frac{2\omega_c}{\omega} \Theta_o \).

The typical value of \( \Theta_o \) is \( \Theta_o \). Hence \( \omega_c/\omega \) may be derived from observations (assuming \( \Theta_o \) independently known) if \( \omega_c/\Theta_o > \text{Max}(\Theta_S, \Theta_S) \). If \( \Theta_o > \Theta_S \), we need

\[
\frac{\omega_c}{\omega} > (q_1 r_S)^{-1} \quad (q_1^{-1} > r_S) \quad \text{or if } \Theta_S > \Theta_S \text{ we need}
\]

\[
\frac{\omega_c}{\omega} > (q_1 a_x)^{-1} \quad (q_1^{-1} > a_x),
\]

for type B irregularities, or type AB with \( q_1^{-1} = a_x \). Type C irregularities are not favorable for determining \( \Delta \Theta_+/- \) because the ratio \( \Theta_o/\Theta_S \) is only logarithmically large. For a type A spectrum or Gaussian (with \( \Theta_2 D \gg a \)) no way has been found for determining the magnetic field from the scattered waves.

Finally, it is noted that for the case of a number of intervening screens, the screen and irregularity size important for \( \Delta \Theta_+/- \) is given by \( \text{Max}_{r, \alpha} (H_{2, \alpha} \Delta N_{\alpha} L_{\alpha}^{1/2} r^{-3/2}) \),

where, as in \( \frac{\rho}{7} \), \( \alpha \) denotes the screen, \( L_{\alpha} \) its thick-
ness, $H_{z\alpha}$ its magnetic field, and $\Delta N_\alpha(r)$ the magnitude of electron density fluctuations of size $r$ in the screen.
PART II

Thin Screen Theory and
Analysis of Scintillations
Introduction to Part II:

This section discusses the scintillations or intensity variations which may develop in a wave at some distance, say \( z = D + L \), beyond a thin screen. Previously, (Part I; D) we investigated angular scattering; basically it is determined by bilinear amplitude correlations, \( AA^{*} \), which formally are independent of distance from the screen D. On the other hand the intensity is proportional to amplitude squared so that its statistical properties depend on products of four amplitudes, \( AA^{*}A^{*}A^{*} \). Intensity variations thus contain information not accessible through study of angle scattering alone and vice-versa. In particular, intensity variations are strongly influenced by the angular diameter of a source (Pisareva, 1959) even when this diameter is much smaller than the instantaneous angular size of the source due to scattering by the screen. Furthermore, for cases where the screen moves across the line of sight, the perpendicular velocity of this motion may be extracted from intensity variations at two or more sites (Hewish, Dennison, and Pilkington; 1966). Under certain conditions by analysis of intensity variations at one site not only may the perpendicular velocity be derived, but also the spectrum of irregularities causing the scintillations (A; §1).

The different behaviour of intensity variations compared with the instantaneous amplitude visibility arises from the fact that intensity variations occur only if the phase differences along the different paths followed by the
received wave energy are larger than roughly a radian. This is contained in Fresnel's formula for the instantaneous monochromatic amplitude, for the case of an incident plane wave,

\[
A(x) = A(x, y, D+L) = \frac{k}{2\pi D} \int_{-\infty}^{\infty} dx' e^{-i \frac{k}{2D} \left( \frac{x-x'}{2D} \right)^2} e^{i\phi(x')}
\]

(94)

\[
\varnothing(x) = \varnothing(x, y, L)
\]

intensity = \( I(x) = I(x, y, D+L) = |A(x)|^2 \)

average intensity intensity = \( \langle I(x) \rangle = 1 \)

Formula (94) holds if scattering angles are much less than a radian, \( \lambda \ll q^{-1} \) (\( \sim \) irregularity sizes of interest) and the distance of the screen much larger than its thickness \( D \gg L \). For a discussion see Salpeter (1967).

Section A of this Part of the thesis discusses statistical properties of \( I(x) \) in the x-y plane for cases where the intensity variations are small compared with the mean-intensity: weak scintillations. The theory is applied to interplanetary scintillations. Section B treats cases where intensity variations may be comparable or much larger than the mean: strong scintillations. The parameter \( B \) which distinguishes weak and strong scintillations for cases of power law irregularity spectra is defined.
A. Weak Interplanetary Scintillations:

Introduction:

Interplanetary scintillations (IPS) of small diameter radio sources are caused by plasma density fluctuations carried across the line of sight by the solar wind. IPS are manifested for example as rapid (½ second) variations of intensity of a source observed with one antenna. IPS observations are described by Little and Hewish (1968); Cohen et al. (1967); and Cohen and Gundermann (1969). The theory of weak scintillations in addition to that given here is discussed by Salpeter (1967) and Budden (1965a).

With a single antenna, only the intensity as a function of time, $I(t)$, can be measured. Even if it were valid to represent the interplanetary medium by a physically thin screen perpendicular to the line of sight at a distance $D$ from the earth of thickness $L \ll D$, there is no a priori relation between the instantaneous intensity $I(\frac{x}{\lambda})$ of equation (94) in the observer's plane and $I(t)$. However, under conditions of frozen flow there is a simple connection, $I(t) = I(\frac{u}{\lambda})$, where $\lambda$ is the velocity of the screen perpendicular to the line of sight. Frozen flow describes cases where the spatial density irregularities move without significant distortion with the single velocity $u$; that is, very small 'thermal' velocities of irregularities. In this Section (except §5) we assume validity of the frozen flow approximation. (Section B; §7 discusses departures from
frozen flow.)

But even assuming frozen flow, the relation \( I(t) = I(\vec{ut}) \) is expected to be only an approximation for IPS because the perpendicular velocity \( \vec{u} \), for uniform radial outflow of material from the sun, varies along the line of sight (Salpeter, 1967). This geometrical smearing is examined in (§1). The smearing is expected to corrupt IPS data in a way roughly similar to that due to departures from frozen flow.

Simultaneous intensity records on three antennas analyzed by Dennison and Hewish (1967) yielded directly a mean perpendicular solar wind velocity. Their results support the frozen flow approximation and indicate that geometrical smearing is not severe. Furthermore, their data was found consistent with departures from statistical isotropy of \( I(x,y) \) of not more than a two to one elongation of irregularities in the radial direction out from the sun. Under these three conditions it is shown in (§1) that observations taken with a single antenna may be analyzed to give not only a mean solar wind velocity, but also the wavenumber spectrum of electron density irregularities.

Application of the theory of (§1) to scintillation observations on the source CTA-21 are discussed in (§2). Results of the analysis, the principle finding of which is that the spectrum of density irregularities has a power law dependence on wavenumber, are discussed in (§3). In
(§4) the observable parameters for weak scintillations are calculated for cases of power law spectra. In (§5) the possible relation of electron density spectra of (§3) to large scale magnetic field fluctuations measured with the Mariner 2 space probe is investigated.
§1. Theory

Assume as in Part I a plane wave of radio wavelength \( \lambda \) from a point source passing through a plasma slab of thickness \( L \) with spatial electron density fluctuations whose three-dimensional wavenumber spectrum is \( F(q_x, q_y, q_z) \). With \( z \) the propagation direction of the wave, the scintillations are determined by the two-dimensional phase spectrum \( \mathcal{D}_L(q_x, q_y) = 2 \pi (\lambda r_e)^2 L F(q_x, q_y, q_z=0) \) (equation 43), which is the spatial spectrum of the phase \( \varnothing(x, y) \) imparted to the radio wave.

Let \( M(r_x, r_y) \) denote the two dimensional spatial auto-correlation function of the instantaneous intensity in the \( x-y \) plane a distance \( D \) from the center of the plasma slab: \( M(r_x, r_y) = \langle I(x, y)I(x+r_x, y+r_y) \rangle - 1 \). Let \( M(q_x, q_y) \) denote the wavenumber spectrum of the intensity or equivalently the two dimensional Fourier transform of \( M(r_x, r_y) \):

\[
M(q_x, q_y) = \frac{1}{(2\pi)^2} \sum_{-\infty}^{\infty} d x d y e^{-i(q_x r_x + q_y r_y)} M(r_x, r_y)
\]

\[
M(r_x, r_y) = \sum_{-\infty}^{\infty} d q_x d q_y e^{+i(q_x r_x + q_y r_y)} M(q_x, q_y)
\]

If the phase \( \varnothing(x, y) \) is sufficiently small in a certain sense, and the slab thickness \( L \) small compared with the distance \( D \), there is a simple relation between the two dimensional phase spectrum \( \mathcal{D}_L(q_x, q_y) \) and \( M(q_x, q_y) \),
\[ \mathcal{M}(q_x, q_y) = 4 \tilde{\mathcal{F}}(q_x, q_y) \sin^2 \left( \frac{\lambda D}{4\pi} q^2 \right) \]  \hspace{1cm} (96)

where \( q = (q_x^2 + q_y^2)^{\frac{1}{2}} \). Equation (96) is essentially the first Born approximation for a thin screen. It is derived in Section B (\( \S 2 \)), but was obtained previously by Budden (1965a) and Salpeter (1967).

The density fluctuations of the interplanetary plasma falls off quite rapidly with radial distance from the sun. Thus for reasonably small values of the source's elongation \( \xi \ll 30^\circ \) or so, it is a fairly good approximation to consider the medium as a thin layer near the point of closest approach of the radio ray to the sun at a distance \( D = \cos(\xi) \) a.u. from the earth. If the slab were an arbitrarily thin layer (and also assuming frozen flow), then we would have \( I(t) = I(\thicksim) = I( x = ut, 0 ) \), where the \( x \)-axis is taken parallel to the solar wind velocity \( \mathbf{u} = (u_x, 0) \) at the point of closest approach, and the transverse coordinate of the observer is taken as \( (0, 0) \). The time autocorrelation function of the intensity \( I(t) \), denoted

\[ A(\tau) \equiv \left\langle I(t) I(t+\tau) \right\rangle_{\text{time average}} \]  \hspace{1cm} (97)

would be given by
\[ A(\tau) = M(r_x = u\tau, r_y = 0) \quad (98) \]

Departure from relation (98), with \( M(r_x, r_y) \) given by (95) and (96), due to geometrical smearing is unavoidable. However, the smearing is less at smaller elongations, and for the moment neglect it. Even with this assumption one has only \( M(r_x = u\tau, r_y = 0) \), and cannot obtain the full spectrum \( M(q_x, q_y) \) unless some symmetry properties hold for \( \Phi_L(q_x, q_y) \).

For instance, the ordinary one-dimensional Fourier transform furnishes an integral, a function of frequency \( f \),

\[ M_F(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{2\pi i f \tau} A(\tau) = \frac{1}{u} \int_{-\infty}^{\infty} dq_y M(q_x = \frac{2\pi f}{u}, q_y) \quad (99) \]

with an inverse

\[ A(\tau) = \frac{2\pi}{\infty} \int df e^{-2\pi i f \tau} M_F(f) \]

Without much loss of generality let us assume that the phase function is of the form

\[ \Phi_L(q_x, q_y) = g(Q); \quad Q = \left\{ q_x^2 + \eta^2 q_y^2 \right\}^{1/2} \quad (100) \]

with \( \eta \) a constant and \( g(Q) \) a rapidly decreasing function of its argument. Then equation (99) is explicitly,
\[ M_F(f) = \frac{4}{\pi} \int_{-\infty}^{\infty} dq_x q_y G\left(\sqrt{q_x^2 + \eta^2 q_y^2}\right) \sin^2 \left(\frac{\lambda D}{4\pi}(q_x^2 + q_y^2)\right) \]  

(101)

with \( q_x = 2\pi f / u \). If the fluctuations were 'almost' one-dimensional, with a correlation length in the \( y \)-direction much larger than that in the \( x \)-direction, then \( \eta \gg 1 \), and the integrand in equation (101) decreases rapidly for values of \( q_y \gg q_x / \eta \ll q_x \). This gives,

\[ M_F(f) = \frac{4}{\pi} \int_{-\infty}^{\infty} dq_x q_y G\left(\sqrt{q_x^2 + \eta^2 q_y^2}\right) \]  

(102)

where \( f_F = u (\lambda D)^{-\frac{1}{2}} \), termed the Fresnel frequency, is the frequency of the first zero of the sine squared function. The rest of the zeros are given by \( f = n^{\frac{1}{2}} f_F \), where \( n = 2, 3, \ldots \) etc. For values of \( \eta \) near unity or much smaller, the modulation by the sine-squared factor should not be visible in the Fourier transform because the smearing by the integral in equation (101). This is illustrated in Figure (5) for \( \eta \ll 1 \).

Another case is that of a 'nearly' isotropic fluctuation spectrum, where \( \eta \approx 1 \) and \( \Phi_L(q_x, q_y) = \tilde{g}_1\left(\sqrt{q_x^2 + q_y^2}\right) \).

In this case the Bessel transform of the observed data,
\[ M_B(f) = \frac{1}{2\pi} \int_0^\infty \tau \, d\tau \, J_0(2\pi f \tau) A(\tau) \]

with an inverse
\[ A(\tau) = \frac{1}{2\pi} \int_0^\infty f \, df \, J_0(2\pi f \tau) M_B(f) \]

is more useful. If there is no geometrical smearing one can equate \( 2\pi f = u q_x \) and obtain
\[ M_B(f) = \frac{4}{u^2} g_I\left( \frac{2\pi f}{u} \right) \sin^2\left( \pi \left[ \frac{f}{f_F} \right]^2 \right) \]

If the Bessel transform obtained from observations displays deep minima which correspond to the zeros of the sine function in equation (104) for one value of the disposable parameter \( u \), then a number of points become plausible: (a) the fluctuations must be 'almost' isotropic (see following); (b) the effects of geometrical smearing must be fairly small; (c) departure from frozen flow must be small; (d) since \( \lambda \) and \( D = \cos(\xi) \) a.u. are known, the minima of \( M_B(f) \) give the numerical value of \( u \), the solar wind speed near the point of closest approach to the sun; and (e) dividing through by the sine-squared factor gives \( g_I(2\pi f/u) \). An illustration of equation (104) and the corresponding Fourier power spectrum \( M_F(f) \) is shown in Figure (5) for \( \eta \geq 1 \).
FIGURE 5: Idealized Scintillation Spectra.
The effect of geometrical smearing on the Bessel spectrum (104) for isotropic fluctuations was investigated numerically: Because the scintillations are assumed weak, the smearing may be rigorously accounted for by averaging $M_B(f)$ along the line of sight with an appropriate weight factor. That is, if the actual screen of thickness $L$ is divided into a large number of thin layers each of thickness $\delta L \ll L$ (but $\delta L \gg q^{-1} \sim u/f$), then equation (104) holds exactly for each layer, and the observed $M_B(f)$ is the sum of the $\Delta M_B(f)$ contributed by each elementary layer. The velocity which enters in $\Delta M_B(f)$ is that perpendicular to the line of sight $= u_{\perp}$. For the explicit calculation we assumed,

$$e_I(q) \propto \delta L \cdot \langle N(R) \rangle^2 \cdot q^{-s};$$  
the power law dependence on $q$ being suggested by observer discussed later. $R, \text{ in a.u.},$ is the radial distance from the sun.

$$u(R) \propto 1 - 1/5 \ln_e(1/R); \quad u(R) = \text{solar wind speed, assumed radial and a function of } R \text{ only. The function is taken as a reason-}$$

$$(105)$$

$$\text{fit to Parker's } 10^6{\text{K}} \text{ corona for the range of } R \text{ of interest.}$$

$$N(R) \propto (R^2 u(R))^{-1}; \quad \langle N(R) \rangle \quad \text{is the mean electron number density. The formula fol-}$$

$$\text{ows by their conservation.}$$

With these expressions, a formula for $\Delta M_B(f)$ is easily obtained. For this purpose write the radial distance from
the sun as \( R = \sin(\xi) / \cos(\psi) \) a.u.; \( D = (\cos(\xi) - \tan(\psi) \cos(\xi)) \) a.u.;
\( \delta L = (\sin(\xi) / \cos^2(\psi)) \delta \psi \) a.u.; \( \psi_0 = u(R) \cos(\psi); \quad u_0 = 2\pi f \). Then the sum of the contributions by the different elementary layers is:

\[
M_B(f) \propto f^{-5/3} \left\{ \frac{(\cos \psi)^5}{\left[ 1 - \frac{1}{5} \ln \left( \frac{\cos \psi}{\tan \xi} \right) \right]^{4/5}} \sin^2 \left( \frac{\psi - \frac{3}{2} \ln \left( \frac{\cos \psi}{\tan \xi} \right)}{f_f^2 \cos^2 \psi} \right) \right\}
\]

(106)

where the curly brackets denote an average over \( -\pi/2 \leq \psi \leq +\pi/2 \); where \( f_f = u(R=\sin(\xi)) \cdot (\lambda D)^{-1/2} \) as before. The important feature of (106) is the weight factor \((\cos(\psi))^5\); it determines the thickness of interplanetary material which contributes significantly to \( M_B(f) \). Observations discussed in (103) suggest that \( s \sim 4 \) at the elongations of interest. \((\cos(\psi))^4\) falls to one-half for \( \psi = \pm 33^\circ \). Therefore, the 'effective screen' subtends a \( 66^\circ \) angle viewed from the sun. This is close to the screen thickness proposed by Cohen et al. (1967) for different reasons.

The frequency smearing of the sine-squared factor in (106) may be estimated as \( \delta f \sim f_f \cdot (1 - \cos(\psi_{\text{max}})) \sim f_f \cdot (1 - 2^{-1/2}) \). Whereas, the frequency difference between the first and second zero of the sine-squared factor in (104) is: \( (2^{1/2} - 1) f_f \). The ratio of the latter to the former is about 2.6. Thus geometrical smearing alone should not completely spoil the nulls of the sine-squared function in \( M_B(f) \) of equation (104). Numerical evaluation of (106) is
shown in Figure (6). The fact that the smearing is more severe at $\xi = 33.4^0$ than at $14.5^0$ is due to the greater importance of variations of $D$ with increasing elongation. In fact, it is obvious that the interplanetary medium is not thin in the sense $L/2 < D$ for elongations $\xi > 57^0$.

Notice in general that geometrical smearing for small elongations has the effect of simply averaging power spectra for different speeds $u$ in the range $\cos(\psi_{\text{max}})u(R)$ to $u(\sin(\xi))$. The distribution of speeds is asymmetrical in that there is a largest speed, $u(\sin(\xi))$. This is in contrast with departures from frozen flow, where the spread of 'thermal' velocities is probably symmetrical ($B_7$).

The effect of anisotropic irregularities on the Bessel spectrum was investigated numerically assuming no geometrical smearing and also assuming frozen flow. The calculation started with the elliptical spectrum $\Phi_L(q_x, q_y)$

$$\propto (q_x^2 + \eta^2 q_y^2)^{-s/2};$$

from this $\Phi_L$, $M_F(f)$ was derived by the second integral of equation (99); $M_F(f)$ was then Fourier transformed to give $A(\omega)$; and finally, $M_B(f)$ was obtained from $A(\omega)$ using equation (103). The chain of calculations was tested satisfactorily on functions with known transforms and of the form of the observed functions. Figure (7) shows results of the calculation for $\eta = 2$ and $\eta = 1/2$, which corresponds to irregularities elongated perpendicular to the solar wind velocity and parallel, respectively. Unfortunately, the calculation was done only for
FIGURE 6: Geometrical Smearing.
s = 3, which does not correspond exactly to the observations discussed in (§ 3). However, qualitatively, it shows that nulls of the Bessel spectrum $M_B(f)$ are enhanced (or disappear) for irregularities elongated perpendicular (or parallel) to the wind velocity. It may be noted that angular scattering observations (for example, Hewish and Wyndham, 1963) indicate that the irregularities important in scattering typically have axial ratios of two to one, longer in the radial direction out from the sun, for elongations $\varepsilon \geq 16^\circ$.

The fact has not gone unnoticed that for isotropic $M(q_x, q_y)$, there is a direct integral transform giving $M_B(f)$ in terms of $M_F(f)$:

$$M_B(f) = -\frac{1}{\pi} \left( \frac{u}{2\pi} \right) \int \frac{df'}{\sqrt{f^2 - f'^2}} \frac{df'}{df} M_F(f') \quad (107)$$

This may be derived from (99) and (103). The relation is potentially very useful for fast numerical calculation of Bessel transforms. However, for reasons not understood, not in any of several attempts on different computers has the author succeeded in obtaining correct $M_B(f)$ numerically from (107), even for noise free functions with known transforms.
FIGURE 7: Anisotropic Irregularities.
§2. **Data and Analysis:**

On eight different occasions during April, May, and June 1967 scintillation records were obtained at the Arecibo Observatory on the strong small-diameter source CTA-21. Signals received at 430 MHz with the line feed with predetection bandwidths of 3 and 8 MHz were detected, smoothed with low pass filters (flat response out to a cut off frequency of 31 Hz), and recorded digitally at 50 samples per second for about three minutes. The procedure was repeated with the antenna pointing at a nearby cold region of the sky, as discussed by Cohen et al. (1967). A running average over 15 seconds was subtracted from the digital data to suppress the very low frequencies.

Correlation functions were calculated from this data and from data smoothed to give equivalent sampling rates of 25 s/s and 12.5 s/s. Hence for each observation three correlation functions \( A(\gamma) \) were tabulated on the source and three off the source corresponding to Nyquist limits of \( f_N = 25, 12.5, \) and 6.25 Hz. Initially for each Nyquist limit, \( A(\gamma) \) was evaluated at \( N^0 = 200 \) lags for a maximum time lag of \( \gamma_m = N^0(2f_N)^{-1} \).

In some subsequent calculations, only the first half or first quarter of the available lags were used.

Bessel transforms of the 'on' and 'off' correlation functions were calculated using Simpson's rule with maximum time lags of 4 and 8 seconds and Nyquist limits of 6.25 and 12.5 Hz. A Gaussian lag window, \( \exp(-\gamma(\gamma/\gamma_m)^2) \) with \( \gamma' = 4 \) or 5, was introduced first, in order to suppress the
side-lobes of the Bessel transform frequency response. The lag window at the same time spoils the resolution giving a smearing in the frequency domain of width of the order of \( \approx \frac{1}{\sqrt{2\lambda \lambda_m}} \) (full width to e^{-1}). Bessel transforms obtained in this way on the source often showed abrupt fall-offs at 1-2 Hz. The fall-offs were found to be spurious, caused by the very strong low frequency components present in \( A(\tau) \).

An analogous problem arises in the calculation of Fourier transforms and may be avoided by 'prewhitening' (Blackman and Tukey, 1958). An alternative, prewhitened Bessel transform,

\[
\mathcal{M}_B(f) = -\frac{1}{(2\pi)^2} \int_0^\infty \tau \, d\tau \int_1 (2\pi f \tau) \frac{d}{d\tau} A(\tau)
\]

was used throughout and did not show abrupt fall-offs. The Bessel transform (108), which is equivalent to (103), was tested on functions having known transforms and also of the form of the observed \( A(\tau) \).

Fourier transforms were calculated both for high and low resolution, maximum time lags of one and eight seconds, and Nyquist limits of 6.25 and 12.5 Hz, using a Hanning lag window. The Fourier transform programs used were also tested on functions having known transforms and of the form of the observed \( A(\tau) \).

At high resolution, reliable Fourier spectra could be obtained only for frequencies \( f \ll 4 \) Hz, where the spectrum on the source was typically an order of magnitude larger.
than the 'off' or noise spectrum. For frequencies $f > 4$ Hz the 'on' spectrum approaches that of the 'off' so that high stability of both the 'on' and 'off' spectra is required to obtain a reliable difference spectrum, which is the spectrum of the source intensity. The r.m.s. statistical fluctuation expected in an 'on' (or 'off') spectrum, $\delta M_F(f)$, for a signal with Gaussian statistics (and a Hanning lag window) is:

$$\delta M_F(f)/M_F(f) \simeq 0.8 \left( \frac{\tau_m}{T} \right)^{1/2}$$  \hspace{1cm} (109)$$

where $T$ is the observation time. For our data $T \sim 3$ minutes. Thus the fractional fluctuation for $\tau_m = 1$ second is quite small, $\sim 0.06$.

In contrast, for the Bessel spectra obtained with the Gaussian lag-window mentioned above, the r.m.s. statistical fluctuation, $\delta M_B(f)$, is:

$$\delta M_B(f)/M_B(f) \simeq \left( \frac{\tau_m}{T} \right)^{1/2} \left( \frac{f \tau_m}{\gamma} \right)^{1/2}$$  \hspace{1cm} (110)$$

(Lovelace, 1969). Hence for $\tau_m = 8$ seconds and $\gamma = 4$, the fractional fluctuation is roughly 0.3 at $f = 1$ Hz, and twice as large at $f = 4$ Hz. The lack of stability in observed Bessel transforms ($\delta$3) for $f > 4$ Hz therefore could be due to our short observation time $T$ and not due to receiver noise (the 'off').
§ 3. Discussion of Observations:

A summary of the results of the analysis of the CTA-21 data is given in Table I. Figure (8), curve (b), shows one of the high resolution Fourier transforms $M_F(f)$, which is qualitatively typical of all occasions. For curve (b), the maximum time lag $= 8$ sec., and the Nyquist limit $= 12.5$ Hz, and the elongation $= 22.5^\circ$. Of the Bessel transforms obtained on the eight occasions, two gave pronounced interpretable minima, and Figure (8), curve (a), shows $f \cdot M_B(f)$ for one of these, which is the same occasion as for curve (b). For curve (a), $\tau_m = 8$ sec., and $f_N = 12.5$ Hz. The arrows indicate the successive zeros of the sine-squared term in equation (104) with an assumed speed of $u = 390$ km/sec. Figure (8), curve (c), also with arrows for $u = 390$ km/sec, is typical of two further occasions which gave less certain fits to equation (104). The 'error' bars in Figure (8) are from equation (109) or (110).

While the Bessel transforms on the remaining occasions could not be reliably fit to the zeros of the sine-squared function, it is gratifying to note that on the former four occasions values of $u$ between 350 and 500 km/sec were obtained, which is of the correct order of magnitude for solar wind speeds. Further observations over time periods longer than three minutes will be required to test whether one can reliably obtain solar wind speeds with a single antenna.
<table>
<thead>
<tr>
<th>elongation (ε)</th>
<th>1967 date</th>
<th>modulation index (m)</th>
<th>Bessel spectrum</th>
<th>Observed power spectrum $M_F(f) \propto f^{-s-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12^\circ$ (east)</td>
<td>30 April</td>
<td>0.43</td>
<td>interpretable sequence of minima. $u=460 \text{km/sec}$</td>
<td>$f^{-3.1}$ (1-6 Hz)</td>
</tr>
<tr>
<td>$11^\circ$ (west)</td>
<td>23 May</td>
<td>0.41</td>
<td>no appreciable modulation</td>
<td>$f^{-3}$ (1-3 Hz) $f^{-4}$ (3-6 Hz)</td>
</tr>
<tr>
<td>$13^\circ$</td>
<td>25 May</td>
<td>0.40</td>
<td>no appreciable modulation</td>
<td>$f^{-3.7}$ (1-3 Hz) $f^{-5.4}$ (3-4.5 Hz)</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>27 May</td>
<td>0.33</td>
<td>Figure 8c.</td>
<td>$f^{-3}$ (1-8 Hz)</td>
</tr>
<tr>
<td>$22.5^\circ$</td>
<td>3 June</td>
<td>0.24</td>
<td>interpretable sequence of minima. Figure 8a. $u=390 \text{ km/sec}$</td>
<td>$f^{-3}$ (1-10 Hz) Figure 9b.</td>
</tr>
<tr>
<td>$23.5^\circ$</td>
<td>4 June</td>
<td>0.27</td>
<td>questionable minima similar to $\epsilon = 15^\circ$.</td>
<td>$f^{-3}$ (1-12 Hz)</td>
</tr>
<tr>
<td>$33^\circ$</td>
<td>13 June</td>
<td>0.16</td>
<td>no appreciable modulation</td>
<td>$f^{-3.6}$ (1-8 Hz)</td>
</tr>
<tr>
<td>$36^\circ$</td>
<td>16 June</td>
<td>0.10</td>
<td>no significant modulation</td>
<td>$f^{-4}$ (1-4 Hz) Figure 9a.</td>
</tr>
</tbody>
</table>
FIGURE 8: Observed Scintillation Spectra.

\[ f = \text{Frequency (Hz)} \]
The nature of the Fourier power spectra is indicated in Table I. Figure (9) shows two spectra which illustrate the range of behaviour observed. The spectra are low resolution, maximum time lags = 1 second, and the Nyquist limits = 12.5 Hz. Curve (a) was at $\xi = 36^\circ$, and (b) at $22.5^\circ$. The asterisks mark the frequencies at which the 'on' spectrum is twice the 'off' spectrum, and the 'error' bars are from equation (109) including the effect statistical noise has on both the 'on' and 'off'.

On most occasions the power spectra could be represented by a power law, $M_F(f) \propto f^{-(s-1)}$, to within the expected statistical error over a range of frequencies, indicated in Table I. The indices $s$ listed in Table I were obtained by drawing by eye-ball straight lines through observed power spectra plotted on log-log paper as illustrated in Figure (9). The errors in $s$ by this method are probably small, $\pm 0.1$.

Broten et al. (1967) measured a fringe visibility

$$|V| \sim 0.5$$

for CTA-21 at a baseline $b = 4.6 \times 10^6 \lambda$ at 448 MHz. Thus the finite size of CTA-21 would decrease the observed $M_F(f)$ by a factor

$$|V|^2 \sim 0.25$$

at a frequency of $f = (u/D)(b/\lambda) \sim 12.0$ Hz, assuming $u = 390$ km/sec (see equation (43) of Salpeter, 1967). To correct the indices $s$ for this effect, we assumed a Gaussian visibility function $V(b)$ for the intrinsic shape of CTA-21: $V(b=0) = 1$ and $V(b) = 0.5$ for Broten et al.'s baseline. Then the observed $M_F(f)$ was divided by

$$|V(\frac{\lambda D}{u} \cdot f)|^2$$

This corrected $M_F$ was plotted on
FIGURE 9: Observed Irregularity Spectra.
log-log paper and new indices $s$ derived as above. The corrected index found in this way for a 'raw' index $s = (4.0 \pm 0.1)$ is: $s = (3.6 \pm 0.3)$. For numerical purposes in (44) we adopt the estimate $s = 3.6$ as typical of data given in Table I. It is recommended for future analysis that $s$ be derived by an appropriate weighted least-squares fit on a log-log plot.

Note that if the phase spectrum $g$ of equation (100) has a power law form,

$$g(Q) \propto Q^{-s}; \quad Q = \left\{ q_x^2 + \eta^2 q_y^2 \right\}^{1/2} \quad (111)$$

and $2\pi f = u q_x$, then equation (101) gives $M_F(f) \propto f^{-(s-1)}$ for $f > u (\lambda D)^{-1/2}$, and for $\eta$ not greatly different from unity. Assuming $\eta = 1$ we have verified this by evaluating $M_F(f)$ numerically from equations (96) and (99) when the frequency of the first zero of the sine-squared function, $f_F = u (\lambda D)^{-1/2} = 1$ Hz. The derived $M_F(f)$ is of the form of $f^{-(s-1)}$ for frequencies $f \gg 1$ Hz.

An alternative interpretation of a power law spectrum $M_F(f)$ is possible. A wavenumber spectrum $\Phi_L(q)$ which decreases sharply (i.e., as a Gaussian) with increasing $q$ could be significantly broadened by wide spread of the velocities along the line of sight. However, the observation of sharp cut-offs in the power spectra of some sources (for example, 3C-208) due to their finite diameter suggests that the velocity spread is small compared with $u$. This point was
made by Dr. E.J.G. Hardebeck (1968).

On the two occasions (ξ = 12° and 22.5°) when the Bessel transform could be fitted to the form of equation (104), we obtained absolute values of the function \( g_I(2\pi f/u) \) by dividing the assumed sine-squared function out from the observed Bessel transform for frequencies in the range 0.5 to 4 Hz. On both occasions the derived values of \( g_I(2\pi f/u) \) were proportional to \( f^{-s-1} \) with \( s \neq 4 \), compatible in slope and absolute value with those inferred from the Fourier transform and equation (99). For \( \xi = 22.5° \) the analysis indicates that the power law spectrum \( M_F(f) \propto f^{-s-1} \) covers a frequency range of 0.5 to 10.0 Hz.
§4. Weak Scintillation Parameters:

Admittedly the evidence adduced so far in support of a power law wavenumber spectrum of interplanetary irregularities is not by itself overwhelming. Even for the rather small range of distance from the sun covered for observations during 1967, the log slope $s$ varied from occasion to occasion and there was not always a unique (or single) value for it. Other data not included in Table I has corroborated the power law dependence of $M_F(f)$; and no data analyzed by the author thus far has contradicted it. The first indication of power law $M_F(f)$ was obtained in 1967 in analysis of the enhanced scintillations of CTA-21 (on March 27, 1967) on 611 MHz at $\zeta = 41^0$, which occurred during a period of solar activity (Sharp and Harris, 1967). The derived power spectrum was $M_F(f) \propto f^{-2.3}$ for frequencies $1 < f < 5$ Hz.

Data complementing that of Table I, for similar elongations, but taken during 1968 (by Zeissig, 1968), showed $M_F(f) \propto f^{-(3.0 \pm 0.3)}$ for $1 < f < 12$ Hz, for observations at 430 MHz of CTA-21 on 28 April, 26, and 30 May, and the same frequency range and dependence for observations of 3C-237 and 4C-14.40 both on September one. This data was Fourier analyzed by a different method and thus with a completely different computer program. But more importantly, the data suggests that the log slope $s$ does not undergo large yearly variations, and that it is at least roughly the same for all small sources at elongations of $10^0$ to $30^0$. 
Scintillation data on the source 3C-279 taken during 1966 at much smaller elongations ( \( \xi \approx 3^\circ \) to \( 7.5^\circ \) ) than in Table I were obtained by Cohen and Gundermann (1969). As yet only a preliminary analysis of the 3C-279 data for the log-slope \( s \) has been possible: for the observations of 3C-279 on October 9, 1966 at \( \lambda = 11 \) cm and \( \xi = 3^\circ \) (weak scintillations), the power spectrum could be represented as \( M_F(f) \propto f^{-2.9} \) for \( 1.5 < f < 7.0 \) Hz. This estimate is tentative, but still it suggests that the log-slope obtained in (\( \xi \approx 3^\circ \)) could also describe the interplanetary (or coronal) density fluctuations all the way into twelve solar radii (\( \xi = 3^\circ \)) from the sun. A thorough analysis of the 3C-279 spectra on log-log plots is of interest for establishing the value of \( s \) close to the sun.

In the following we investigate consequences observable for weak scintillations, and the corresponding scattering, of an isotropic power-law spectrum of irregularities \( F(q) = K_N q^{-s} \) with \( q = (q_x^2 + q_y^2 + q_z^2)^{1/2} \). Observations by Hewish and Wyndham (1963) indicate that irregularities important in scattering are elongated about two to one for \( \xi \approx 16^\circ \); however, inclusion of this effect would cause only minor changes in the formulae given below. In addition we assume that the screen is physically thin, \( L \ll D \), that there is no geometrical smearing, and that frozen flow holds. Inclusion of geometrical smearing and non-frozen flow may alter the shape of different spectra, but not their integrals or zeroth moments.
As a consequence of equation (43), \( \overline{Q}_L(q_x, q_y) \propto q^{-s} \) with \( q = (q_x^2 + q_y^2)^{1/2} \), for \( q_1 < q < q_2 \), where \( q_1 \) and \( q_2 \) are the limits of validity of the power law. For this kind of spectrum there is no special irregularity size \( \alpha \). Instead, mathematically because of the \( \sin^2(\frac{\lambda D}{4\pi} q^2) \) factor in equation (96) for \( N(q_x, q_y) \), the weak scintillations are determined by irregularities with sizes \( \sim (\lambda D)^{1/2} \). \( (\lambda D)^{1/2} \) is the Fresnel radius for the interplanetary screen. At \( \lambda = 70 \) cm (430 MHz) and \( D = 1 \) a.u., \( (\lambda D)^{1/2} = 320 \) km. We assume \( q_1 \ll (\lambda D)^{1/2} \ll q_2 \).

The importance of Fresnel size irregularities in weak scintillations may be understood as follows: consider phase fluctuations of size \( r \gg (\lambda D)^{1/2} \), which are caused by irregularities of the same size. Wave energy arriving at the earth from points separated by distances \( \sim r \) come from widely different Fresnel zones, and thus have large phase differences due to the different geometrical path lengths. For this reason, irregularities of size \( r \gg (\lambda D)^{1/2} \) do not contribute to intensity variations in spite of their relatively large phase shifts \( \sim (r^{-2} \overline{Q}_L(r^{-1}))^{1/2} \gg \{ (\lambda D)^{-1} \overline{Q}_L((\lambda D)^{-1/2}) \}^{1/2} \). Irregularities of size \( r \ll (\lambda D)^{1/2} \) cause only small phase shifts, which are much less than the phase shifts across a Fresnel radius, \( \sim \{\frac{1}{\lambda D} \overline{Q}_L(\frac{1}{\lambda D})\}^{1/2} \).

The modulation index, denoted by \( \kappa \) and defined as the r.m.s. intensity variations divided by the mean intensity, is a convenient parameter easily derived from observations. For weak scintillations \( \kappa^2 \ll 1 \). Because the mean intensity is
taken to be unity we have \( m^2 = M(r_x=0, r_y=0) \). Thus from equation (95),

\[
m^2 = 2\pi \int_{-\infty}^{\infty} df M_f(f) = (2\pi)^3 \int_0^{\infty} df M_B(f) = \int_{-\infty}^{\infty} d^2 \mathbf{q} M(q_x, q_y) \quad (112)
\]

Introducing \( M(q) = 4 \Phi_L(q) \sin^2(\frac{\lambda D}{4\pi} q^2) \) and \( \Phi_L(q) \propto F(q) \propto q^{-s} \) in the third integral of (112) one may obtain,

\[
m^2 = 16\pi K_N (\lambda r_e)^2 L \left( \frac{\lambda D}{2\pi} \right)^{s/2} C_2^{-2} \quad \begin{aligned}
\delta &= (s-2)/2 ; \quad 0 < \delta < 2
\end{aligned}
\]

where \( C_2 \) is the same constant given in equation (52).

Notice, by an amazing coincidence, that if the ubiquitous arbitrary length scale \( r_o \) of Part I is chosen as \( r_o = (\lambda D)^{1/2} \), then \( m = B \) of equation (52). Let us take this choice. Then \( B^2 \ll 1 \) is weak scintillations.

The parameter dependence of the modulation index in weak scintillations is that of \( B \):

\[
m \sim r_e K_N^{1/2} \lambda^{1+s/2} L^{1/2} D^{\delta/2} \quad (114)
\]

From the data of (§3), \( s \approx 3.6 \) or \( s \approx 0.8 \) and this would indicate that \( m \propto \lambda^{1.4} \). Also for \( s \approx 0.8, C_2^{-2} \approx 0.72 \), and therefore, for the observations discussed in (§3) at an elongation \( \xi = 22.5^\circ, K_N = 1.45 \times 10^{-6} \text{ (el./cm}^3 \text{)}^2 \text{ cm}^{-0.6} \), where
for the screen thickness we have used \( L = (2/\sqrt{3}) \sin(\xi) \) a.u., and \( D = \cos(\xi) \) a.u., where \( \xi \) is the solar elongation. The numerical value of \( K_N \) is extremely sensitive to \( \xi \) because of the \((\lambda D)^\delta\) factor in formula (113). Fortunately, however, observable parameters are not particularly sensitive to \( K_N \) directly.

The modulation index (squared) is the integral of the scintillation spectrum. Various ways are available for describing the width of the spectrum, but one practical measure is the first moment of the Fourier spectrum \( M_F(f) \), as suggested by Cohen and Gundermann (1969),

\[
\int_1 \equiv \int_0^\infty df f M_F(f) \int_0^\infty df M_F(f)
\]  

(115)

This is advantageous compared with the second moment used commonly (Cohen et al., 1967), because it is not sensitive to a high frequency tail on the spectrum which may be largely noise. For the power law \( \Phi_L(q) \) we find,

\[
f_1 = C_q \quad \mathcal{U} \quad (\lambda D)^{-1/2}
\]

\( 3 < s < 6 \quad \text{or} \quad 1/2 < \delta < 2 \)

\[
C_q = \frac{1}{\pi} \left( \frac{2}{\pi} \right)^{1/2} \frac{\Gamma(3-s/2)}{\Gamma(2-s/2)} \cos\left(\frac{3-s}{4} \pi\right) \cos\left(\frac{2-s}{4} \pi\right)
\]

(116)
From the data of (8.3), \( s = 3.6 \) and \( C_9 = 0.55 \). Thus for a solar wind speed of \( u = 390 \text{ km/sec} \) and \( D = \cos(22.5^\circ) \text{ a.u.} \), \( f_1 = 0.57 \text{ Hz} \). Or if we assume that \( s = 3.6 \) for Cohen and Gundermann's (1969) 11 cm data on 3C-279 at an elongation such that \( p = 0.1 \text{ a.u.} \) (21.5 solar radii), then we can use their observed first moment to calculate the wind speed. At 11 cm the Fresnel radius is about 128 km (for \( D = 1 \text{ a.u.} \)) and thus \( u = 233 f_1(\text{Hz}) \text{ km/sec} \). For the observed \( f_1 = 1.3 \text{ Hz} \), \( u = 300 \text{ km/sec} \). This is in quite good agreement with the wind speed 270 km/sec predicted by Parker's (1965) million degree corona (where we have used the wind speed formula given by Cohen and Gundermann, 1969).

The intensity correlation in weak scintillations is given by equations (95) and (96). Because of the assumption of isotropic turbulence \( M(r_x, r_y) = M(r) \) where \( r = (r_x^2 + r_y^2)^{1/2} \). Thus we may write

\[
M(r) = \gamma^2 G_M\left(\frac{r}{\sqrt{\lambda D}}\right)
\]

\[
G_M(\eta) = \pi C_2^2 \int_0^\infty d\eta \eta^{-2(\delta + 1)} \int (2\pi \eta \eta) \sin^2\left(\frac{1}{\alpha \eta^2}\right)
\]

\[
|G_M(\eta)| \leq 1
\]

where \( G_M(\eta) \) is a dimensionless function of a dimensionless argument. The form of \( G_M(\eta) \) is of interest because it is
directly measurable in bistatic experiments such as described by Hewish, Dennison, and Pilkington (1966). Asymptotic expansions for $G_M$ for $\zeta \ll 1$ are:

\[
\begin{align*}
G_M(\zeta) &\approx 1 - \zeta^{2\delta} \left(\frac{\pi}{2}\right)^{\delta} \frac{1}{\Gamma(1+\delta) \cos\left(\frac{\pi}{2} \delta\right)} ; \quad \text{Type A} \\
G_M(\zeta) &\sim \zeta^{2\delta} \ln \zeta \left(\frac{1}{\zeta}\right) ; \quad \delta = 1, \text{ Type C} \\
G_M(\zeta) &\approx 1 - \zeta^{2\delta} \left(\frac{\pi}{2}\right) \cot\left(\frac{\pi}{2} (\delta-1)\right) ; \quad \text{Type B}
\end{align*}
\]

whereas for $\zeta \gg 1$,

\[
\begin{align*}
G_M(\zeta) &\approx \frac{1}{2} C_2^2 \zeta^{-(4-2\delta)} \sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\frac{7}{2} - 2\delta\right)}{(2\pi)^{3/2}} \frac{\sin\left(\pi(2-\delta)\right)}{\left(2\pi\right)^{3/2}} ; \quad \delta \neq \frac{7}{4} \\
G_M(\zeta) &\sim \text{const.} \zeta^{-\frac{7}{2}} ; \quad \delta = \frac{7}{4}
\end{align*}
\]

(The table of Mellin transforms by Ditkin and Prudnikov (1965) was valuable in deriving these expansions.) Distinct behaviour of different types of irregularities is evident in the expansions. The large $\zeta$ limit of (119) is negative for type A irregularities, but positive for at least some kinds of type B irregularities, $1 < \delta < 7/4$. However, the singular nature of (119) at $\delta = 7/4$ is not understood.

From the data of (3.3), $s \approx 3.6$, and thus from equation (119) one finds $G_M(\zeta) \approx -0.028 \zeta^{-2.4}$ for $\zeta > 1$. This gradual fall-off of the intensity correlation is in con-
tract with the Gaussian fall-off for the case of Gaussian irregularities. For an illustration, consider observations at 430 MHz at an elongation $\xi = 22.5^\circ$. The Fresnel radius is $(\lambda D)^{1/2} \approx 311$ km. Thus the cross intensity correlation between two sites separated by a transverse distance $\sim 311$ km is $G_M \sim -0.028$. For the same separation, the correlation is much stronger at lower frequencies because the Fresnel radius is larger.

For the ideal case of frozen flow and no geometrical smearing, we have $A(\mathcal{V}) = M(r_x = u\mathcal{V}, 0) = m^2 G_M(u\mathcal{V})$ exactly. Figure (10) shows $G_M(u\mathcal{V})$ calculated numerically for $s = 3.6$ and $f_F = 1.5$ Hz ($= u(\lambda D)^{-1/2}$). Note that $G_M$ falls to $e^{-1}$ at a time lag $\tau_e \approx 0.32$ seconds; therefore, $\tau_e = 0.48 u^{-1}(\lambda D)^{1/2}$. For the case of Gaussian irregularities, $\tau_e \approx 2^{1/2} a/u$ (page 5). This gives the relation $a_{\text{eff}} \approx 0.34 (\lambda D)^{1/2}$ for the correspondence of the conventionally tabulated scale size and the Fresnel radius assuming $s \approx 3.6$. Observe that with $a_{\text{eff}} = 0.34 (\lambda D)^{1/2}$, then the scintillation parameter $\lambda D/\pi a_{\text{eff}}^2 \approx 1.38$. This is in reasonable agreement with the location of different observations in Figure (1) (e.g., Cohen et al., 1967). Also note that $G_M(u\mathcal{V})$ is qualitatively of the form of observed time auto-correlation functions (e.g., Dennison and Hewish, 1967).

* * *
FIGURE 10: Intensity Correlation.
Angle scattering or diffraction under conditions of weak scintillations: Part I (Section D) discusses angle scattering for type A, B, and C irregularities for both weak ($\beta^2 \ll 1$) and strong ($\beta^2 \gg 1$) scintillations. With $\beta^2 \ll 1$, or $m^2 \ll 1$, the characteristic instantaneous source size is: $\Theta_s \sim \frac{\lambda}{2\pi \delta x}$ with $\delta x \sim r_0 \beta^{-\frac{1}{2}} \sim (\lambda D)^{\frac{1}{2}} m^{-\frac{1}{2}}$, where these relations hold for all three kinds of irregularities. In all cases the scattering is strong, even though scintillations are weak, if $\delta x < q_1^{-1}$. We assume that $\delta x < q_1^{-1}$. As mentioned in (I; D; 7) for type B and C irregularities the source position varies with time, but not for type A. However, for the moment do not worry about position variations.

Observe that in weak scintillations $\delta x > (\lambda D)^{\frac{1}{2}}$, the turbulences which determine the typical angle scattering are larger than those causing the scintillations ($\sim (\lambda D)^{\frac{1}{2}}$). Corresponding to this $\Theta_s \sim \frac{\lambda}{2\pi \sqrt{\lambda D}}$. Thus spatial variations of the intensity in the observer's (x-y) plane arise from deflection angles in the tail of the probability density $W(\Theta)$ (equation (71) for type A; or (82) for type B; no density has been obtained for type C). This is because a deflection angle $\sim |\Theta|$ produces intensity variations with a scale $\sim \lambda/2\pi |\Theta|$(or $q = k |\Theta|$). This is evidenced by the identical analytical form of the large argument limits of $M(q_x, q_y)$ and $W(\Theta)$. 
For weak scintillations, the angle scattering source size $\theta_S$ subtends an area on the screen of size $\theta_S D$ less than the size $\delta x$ of a typical irregularity causing the scattering. Because of this a time $\tau \sim \delta x / u$ is required to obtain a stable average of the visibility (see I; D; §2). A problem in this connection arises for type B and C spectra, because during a time interval $\sim \delta x / u$ the source position changes by $< \Delta \theta (|u| \tau) >^{1/2} \sim \theta_S$, and by larger angles over longer time intervals. The only way for extracting $\theta_S$ from an observed visibility function would appear to be a minimization procedure requiring successive guesses of the source position as a function of time. However, the observations discussed in (§3) suggest that the interplanetary spectrum is type A, at least for small elongations.

Finally in this subsection let us consider an alternative to the proposal by Hewish and Symonds (1969) and others that interplanetary scintillations may be described by the far field (Fraunhofer limit) of Gaussian irregularities. In this limit the modulation index is proportional to radio wavelength, $m = 2^{1/2} \varphi_L \propto \lambda$ where $\varphi_L$ is the r.m.s. phase shift. From this assumption Hewish and Symonds derive a dependence on elongation of $m(p) \propto p^{-1.6}$; $p = \sin(\xi)$. Angle scattering, which gives $\theta_S \sim \frac{\lambda}{2 \pi a} \varphi_L$, is used to derive the scale size $a$. The scale size obtained from the scattering is found to be consistent with that gotten from scintillation data.
Below we investigate the possibility that a power law spectrum of interplanetary irregularities may also be consistent with observations for a range of elongations: $2.6^\circ < \xi < 33^\circ$. For $\xi < 33^\circ$ the interplanetary medium is quite thin and thus prior formulae apply. This upper limit also takes into account the possible effect discussed in \S 5.

Figure (11) shows $m(p)$ for most of the data given in Table I. The finite size of CTA-21 should not affect this data too strongly because $m^2 \ll 1$. Shown also in the Figure are modulation indices obtained by Cohen and Gundermann (1969) on 3C-279 at 11 cm during 1966 (denoted by C's) and those obtained by Bourgois (1969) on 3C-279 at 11 cm during 1968 (denoted by B's). If we assume that there are not large year to year variations of scintillation parameters including $s \approx 3.6$, and that $s$ is independent of elongation, then $m \propto \lambda^{1.4}$. Therefore, the 11 cm indices are also plotted in Figure (11) scaled by a factor $(70/11.1)^{1.4} \approx 13.2$. The vertical bars indicate the day to day variability of $m$, presumably not errors. The fact that for the elongations of overlap the 11 cm indices lie slightly above those at 70 cm could be due to departure from the weak scintillation approximation at the smaller elongations at 70 cm (Salpeter, 1970). The straight line in the Figure was eye-balled through both 70 cm and the scaled 11 cm indices. It gives,
\[ \phi = \sin(\xi) \rightarrow (\xi = \text{Solar elongation}) \]

FIGURE 11: Modulation Indices.
\[ m = 0.05 \left( \frac{\lambda}{70 \text{ cm}} \right)^{1.4} p^{-1.6} \] (120)

\[ 0.05 < p < 0.5 \ ; \ p = \sin(\varepsilon) \text{ a.u.} \]

Formula (120) is the observable modulation index for weak scintillations. The index happens to have the same dependence on \( p \) found by Hewish and Symonds (1969). At 70 cm, \( m(p) \) extrapolates to unity for \( p = 0.15 \) or \( \varepsilon = 8.8^\circ \), which agrees with Cohen et al. (1967); at 11 cm, \( m(p) \) extrapolates to unity for \( p = 0.03 \) or \( \varepsilon = 1.75^\circ \) \( (= 7 \text{ solar radii}) \) which agrees with Bourgeois (1969) and Cohen and Gundermann (1969) approximately.

From formula (120) it is now possible to deduce parameters for the angle scattering. The fringe visibility, time averaged over intervals \( \sim \delta x / u \), at a baseline \( b = (b_x^2 + b_y^2)^{1/2} \) for type A spectra \( (0 < \delta < 1) \) is given generally by

\[ V(b) = \exp \left\{ -\frac{1}{2} \right\} \frac{C_1^2}{\beta^2} \left( \frac{b}{\sqrt{\lambda D}} \right)^{2\delta} \] (121)

where \( C_1 \) is a constant defined in equation (52). The choice \( r_o = (\lambda D)^{1/2} \) is assumed, which implies \( \beta = m \) for \( m^2 \ll 1 \). For \( m^2 \) or \( \beta^2 \) much less than unity, \( V(b) \) is the visibility time averaged over intervals, \( \delta x / u \approx r_o \beta^{-1/6} \), much longer than the scintillation time scale. And in
marked contrast with the case for Gaussian spectra, the average visibility may be much less than unity for \( b \gg (\lambda D)^{\frac{1}{2}} \). However, most coronal scattering observations probably correspond to \( \beta^2 \gg 1 \), that is, baselines shorter than a Fresnel radius, where equation (121) is the visibility averaged over a scintillation time scale.

The scattering parameter is derived by the identification \( \beta = \frac{m}{\sqrt{2}} \) (of equation 120) even for \( \beta > 1 \), within the elongation limits of (120). From (121) the visibility e-folding baseline is \( b_e = \left( 2/c_1^2 \right)^{\frac{1}{2}} \xi (\lambda D)^{\frac{1}{2}} \beta^{-\frac{1}{6}} \). For \( \xi \approx 0.8 \), \( c_1^2 \approx 4.96 \), and thus \( b_e \approx 0.57 (\lambda D)^{\frac{1}{2}} \beta^{-1.25} \). From the prior discussion of (I; D; 2) the conventionally tabulated angle is: \( \theta_e = \frac{\lambda}{\pi b_e} \). Therefore,

\[
\theta_e \approx 0.56 \sqrt{\frac{\lambda}{D}} \beta^{1.25} \tag{122}
\]

From equation (120) and (122) with \( \beta = \frac{m}{\sqrt{2}} \) we find,

\[
\theta_e^{(\text{min. of arc})} \approx 0.4 \frac{\lambda}{\sqrt{\cos \xi}} \left( \frac{\lambda}{790 \text{ cm}} \right)^{2.25} \left( \frac{0.23}{p} \right)^{2.0} \tag{123}
\]

for \( 0.05 \leq p \leq 0.5 \). The wavelength normalization corresponds to 38 MHz; and an elongation \( p = 0.23 \) a.u. is 50 solar radii.

The observed dependence of \( \theta_e \) on elongation under the assumption of Gaussian irregularities is usually weaker than \( p^{-2} \) of formula (123), more like \( p^{-1.4} \) (Hewish and Wyndham, 1963). However, note that if the data were analyzed
using formula (123), then the slightly stronger wavelength
dependence would lead to a stronger variation of $\theta_e$ with $p$.

Figure (12) shows the scattering angle $\theta_e$ observed by
Hewish and Wyndham (1963) on the Crab Nebula in 1961 at 38
and 26.3 MHz. The 26.3 MHz data in the Figure is scaled by
a factor $(790/1140)^2$ to make it correspond to 38 MHz under
the Gaussian assumption. Thus in order for the 26.3 MHz
solid points of the Figure to correspond to formula (123) at
38 MHz, they should be multiplied by $(790/1140)^{0.25} \approx 0.91$, 
but this small correction factor is neglected. The solid
line in Figure (12) is equation (123), multiplied by $2^{1/2}$ to
account crudely for the fact that observations were made in
the wide scattering direction of irregularities typically
elongated two to one. The agreement of the power law theory
(fitted to 1966-1968 data, with $\delta \approx 0.8$ from 1967 data only)
and scattering observations (in 1961) is rather good in that
there are no free parameters in the comparison. The theoretical
$\theta_e$ is about a factor of two smaller than that which
would give excellent accord. A possible implication of this
agreement is that the solar wind turbulence extends from scales
$\sim 130$ km (the Fresnel radius at 11 cm) down to $\sim 1$ km
(the diffraction length at 790 cm) at distances $\sim 15$ solar
radii from the sun.
FIGURE 12: Angle Scattering.
§5. Relation of Scintillation Spectra to Large Scale Solar Wind Turbulence:

Coleman (1966, 1968) has published detailed spectra of the interplanetary magnetic field (and radial velocity) obtained from recordings of a magnetometer (and plasma probe) onboard Mariner 2. The observations covered a period of August to October 1962 during which time the spacecraft went from 1.0 a.u. to 0.87 a.u. and its solar equatorial latitude ranged from 7.5° to 5.5°. The average magnetic field was approximately 6.1 x 10^{-5} Gauss at an angle roughly 45° to the radius from the sun, outward over a 160° segment of longitude and inward over the remainder; the proton number density was \langle N \rangle \approx 5.6/cm^3; the average wind velocity was u \approx 490 km/sec, nearly in the radial direction; and some mean proton temperature was T_p \approx 1.7 x 10^5 °K.

Fluctuations of the three vector components and magnitude of the magnetic field (H_x, H_y, H_z and H = |H| in heliocentric polar coordinates with the polar axis aligned with the sun's angular momentum) were Fourier analyzed to give power spectra covering the frequency range f = 1.35 \times 10^{-6} to 1.35 \times 10^{-2} Hz. Coleman (1968) has concluded that field fluctuations with low frequencies f < 10^{-5} Hz reflect mainly polarity changes of the 'average' field; whereas higher frequencies f > 10^{-5} Hz are thought due mainly to a superposition of a transverse fluctuating field \delta H, with \langle \delta H \rangle = 0, on the average field. The transversality is concluded from the observation that for f > 10^{-5} Hz the power spectrum of
the magnitude of $\mathbf{H}$ is roughly a factor of 3-5 smaller than the sum of the spectra of the three field components. Because $|\mathbf{H}| \simeq |\mathbf{H}_0| + \mathbf{H}_0 \cdot \mathbf{\delta H}$, where $\mathbf{H}_0$ is the average field, the variations of $\mathbf{H}$ parallel to $\mathbf{H}_0$ must be a bit smaller than those perpendicular.

The field fluctuations seen by a slowly moving ($\sim 30$ km/sec) satellite around one a.u. arise primarily from the bulk motion of the solar wind past the probe with velocity $\mathbf{u}$ ($\sim 500$ km/sec), in that this is much larger than either the sound speed ($\sim 40$ km/sec) or the Alfvén velocity ($\sim 50$ km/sec). This may not be true much closer to the sun, because the Alfvén velocity is expected to increase and become larger than the wind velocity. Therefore, in a reference frame fixed in the solar wind, moving away from the sun with velocity $\mathbf{u}$, denote the magnetic field $\mathbf{\delta H}(x,y,z,t)$ tensor wavenumber power spectrum by,

$$\mathbf{F}_{ij}(q,\Omega) \equiv \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} \int d\Omega \int d^3r \ e^{i(q \cdot r - \Omega \tau)} \left< \mathbf{\delta H}_i(x,t) \mathbf{\delta H}_j(x+r,\tau+\tau(t)) \right>$$

(124)

where $i,j = (x',y',z')$ is an arbitrary coordinate system not necessarily the same as the $(x,y,z)$. A stationary satellite measures $\mathbf{\delta H}(x = ut, t)$ as a function of time. The power spectrum of this may be gotten from the inverse of (124),
\[ P_{ij}^H(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\pi f t} \left( \delta H_i(x,t) \delta H_j(x+t+R) \right) \]

\[ = \left\{ \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d^3q \int_{-\infty}^{\infty} d^3q \delta^\prime(q_\perp, \Omega = 2\pi f + q_x u) \right\} \]

(125)

where in the last integral we have chosen the x-axis parallel to the solar wind velocity \( u \), and \( u = |u| \).

If the frequencies important in (125) are those of linear magnetohydrodynamics, then \( \Omega \ll |q| \max(v_{Al}, v_S) \), where \( v_{Al} \) is the Alfven velocity and \( v_S \) is the sound speed. It follows that in (125) \( q_x = -2\pi f/u + \mathcal{O}\left(\max(v_{Al}, v_S)\right)/u \).

Thus for wind speeds much larger than Alfven or sound speeds, \( q_x \approx -2\pi f/u \); this limit is termed frozen flow, and it amounts to setting \( F_{ij}(q, \Omega) = F_{ij}(q) \delta(\Omega) \) in equation (125). One finds,

\[ P_{ij}^H(f) = \frac{1}{u} \left\{ \int_{-\infty}^{\infty} dq_y dq_z \int_{-\infty}^{\infty} dq_x \right\} \]

(126)

where

\[ F_{ij}^H(q) = \frac{1}{(2\pi)^3} \left\{ \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} \left( \delta H_i(x) \delta H_j(x+R) \right) \right\} \]

The mean-square field fluctuations in a frequency interval \( \Delta f \) are \( 2\pi \int_{\Delta f} df P_{ij}(f) \) with the normalization in (126).
In Figure (13) we have plotted Coleman's (1966) \( |H| \)
power spectrum divided by \( H_0^2 \). In our notation this is
essentially \( 2\pi F_\parallel^H(f)/H_0^2 \), where \( \parallel \) here and later
denotes the field component parallel to the average field. Also
shown in the Figure is a scaled estimate for the fractional
electron density spectrum corresponding to the parallel field
spectrum, which was obtained from the scintillation data dis-
cussed in (53), and where the scaling factor is discussed
below. The fact that the spectrum from scintillation data
lies approximately on the extrapolation of Coleman's field
spectrum to higher frequencies was found by the author in
1967 from the power law spectrum exhibited by the CTA-21 data
of Sharp and Harris (1967). Unfortunately, this was not pub-
lished, and now Cronyn (1969) has pointed out the possible
relationship independently.

Before venturing into an investigation of the relation
which could exist between magnetic field and density spectra,
let us summarize Coleman's (1968) conclusions, which are rele-
vant to the models we assumed for the density fluctuation wave-
number spectra in Part I. Coleman develops a model for the
solar wind turbulence based on Kraichnan's (1965) theory of
incompressible (zero sound speed) hydromagnetic turbulence.
The work by Kraichnan predicts that non-linear interactions
between a wide spectrum of Alfvén waves may under certain con-
ditions give an inertial range spectrum \( \propto f^{-3/2} \), whereas
Coleman finds \( f^{-1.2} \) (slightly different) for \( 10^{-5} < f < 10^{-2} \) Hz. The corresponding range of scales is \( q^{-1} \approx 10^7 \) to
Mariner 2, 1962
Parallel Field
Fluctuation Spectrum

Interplanetary
Scintillations,
CTA-21, 1967
Scaled Density Spectrum

Frequency (Hz) →
$10^4$ km, where the small scale is the spatial Nyquist limit of the Mariner 2 magnetometer. Coleman suggests that the energy input to the turbulence at large scales (our $q_1^{-1}$) is from differential motion or shear between streams in the solar wind; the energy extracted from the shear then cascades down the inertial range in roughly the way suggested by Kolmogoroff for hydrodynamic turbulence. Thus strong turbulence is expected according to this suggestion to continue out to distances from the sun to where the shear is destroyed. The dissipation range or inner scale (our $q_2^{-1}$ or $a_*$ of Part I), Coleman suggests involves cyclotron damping resulting in the heating of protons. We return later to this damping process. Here we comment that description of the solar wind turbulence with Kraichnan's theory just summarized may be qualitatively correct, but it is not evident that the interplanetary medium is even roughly incompressible at one a.u., where the sound speed is not very much smaller than the Alven velocity.

Coleman (1967) compared statistical properties of variables observed by Mariner 2 with those of a linearized uniform magnetic field model. The model was found compatible with most of the observed behaviour. Below we consider a possible relation which could exist between magnetic field and density fluctuations in a uniform linearized model. For the basic equations choose a coordinate system moving with the average solar wind velocity $\mathbf{u}$. Denote the fluid velocity perturbation by $\delta \mathbf{v}$. Expand the mass density $\rho = \rho_0 + \delta \rho$, and
the magnetic field $\mathbf{H} = \mathbf{H}_0 + \delta \mathbf{H}$, where $\rho_0$ is the constant mean density and $\mathbf{H}_0$ is the constant uniform field and the delta quantities are small perturbations. Then, discarding terms quadratic in smallness, one finds

$$\begin{align*}
\frac{\partial}{\partial t} \delta \mathbf{g} + \rho_0 \nabla \cdot \delta \mathbf{v} &= 0 \\
\rho_0 \frac{\partial}{\partial t} \delta \mathbf{v} + \nu_S^2 \nabla \delta \mathbf{g} + \frac{1}{4\pi} \mathbf{H}_0 \times (\nabla \times \delta \mathbf{H}) &= 0 \\
\frac{\partial}{\partial t} \delta \mathbf{H} - \nabla \times (\delta \mathbf{v} \times \mathbf{H}_0) &= 0
\end{align*}$$

(127)

where $\nu_S$ denotes the adiabatic sound speed. If the fluid perturbations are expanded in plane waves,

$$
(\delta \rho, \delta \mathbf{v}, \delta \mathbf{H})_{(x,t)} = (\delta \rho, \delta \mathbf{v}, \delta \mathbf{H})_{(q, \Omega)} e^{i(q \cdot \mathbf{x} - \Omega t)}
$$

(128)

then the basic equations become

$$\begin{align*}
\delta \mathbf{g} &= \left(\frac{\mathbf{q} \cdot \delta \mathbf{v}}{\Omega}\right) \rho_0 \\
\Omega \rho_0 \delta \mathbf{v} &= \nu_S^2 \mathbf{q} \delta \mathbf{g} + \frac{1}{4\pi} \mathbf{H}_0 \times (\mathbf{q} \times \delta \mathbf{H}) \\
\Omega \delta \mathbf{H} &= -\mathbf{q} \times (\delta \mathbf{v} \times \mathbf{H}_0)
\end{align*}$$

(129)

where $\delta \mathbf{g} = \delta \mathbf{g}(q, \Omega) = \delta \mathbf{g}(q)$, with the $\Omega$ dependence implicit in the latter, to illustrate subsequent notation.
Solutions to (129) are discussed in detail by Alfven and Falthammar (1963), for example. There are three modes: one, termed the Alfven mode (subscripted A), has the dispersion relation,
\[
\frac{\Omega_A^2}{q^2} = V_A^2 = V_{Al}^2 \cos^2(\gamma')
\]
\[
V_{Al}^2 \equiv \frac{H_o}{4\pi \rho_o}
\]
where \( V_A \) is the Alfven velocity and \( \gamma' \) is the angle between \( \mathbf{q} \) and \( \mathbf{H}_o \). For the Alfven mode there are no density fluctuations (\( \delta \rho = 0 \)) and the field fluctuations are perpendicular to the mean field (\( \delta H \cdot H_o = 0 \)). The velocity and field perturbations are perpendicular to the plane of \( \mathbf{H}_o \) and \( \mathbf{q} \). The other two modes, termed the fast (+ sign subscript) and the slow (- sign subscript) have the dispersion relation,
\[
\frac{\Omega_{\pm}^2}{|q|^2} = V_{\pm}^2 = \frac{1}{2} \left\{ V_{Al}^2 + V_S^2 \pm \left[ (V_{Al}^2 + V_S^2)^2 - 4 V_{Al}^2 V_S^2 \cos^2(\gamma') \right]^{1/2} \right\}
\]
Both the fast and slow modes entail density fluctuations, and field fluctuations both parallel and perpendicular to \( \mathbf{H}_o \). For both, the velocity and field perturbations are in the plane of \( \mathbf{H}_o \) and \( \mathbf{q} \).

For a simplification of the algebra in examining the three modes we assume \( v_S^2 / v_A^2 \ll 1 \). This is expected to be a fully valid approximation for distances closer to the sun than say half an a.u. For the Alfven mode,
\[ \delta H = - \frac{\delta V}{\Omega} \left| H_0 \right| \cos(\gamma) \] 

\[ \perp H_0 ; \delta \rho = 0 \quad (132) \]

without assuming \( \frac{v_s^2}{v_{\text{Al}}^2} \leq 1 \). For the fast and slow modes,

\[ 
\begin{align*}
\nu_+^2 & = \nu_{\text{Al}}^2 \left[ 1 + \frac{\nu_s^2}{\nu_{\text{Al}}^2} \sin^2(\gamma) \right] \quad \left\{ \nu_s^2/\nu_{\text{Al}}^2 \ll 1 \right\} \\
\nu_-^2 & = \nu_s^2 \cos^2(\gamma) \\
\tan \psi_\pm &= \frac{\sin(\gamma) \cos(\gamma)}{\cos^2(\gamma) - \nu_\pm^2/\nu_{\text{Al}}^2} \quad \left\{ \nu_s^2/\nu_{\text{Al}}^2 \ll 1 \right\} \\
\psi_+ &= \gamma + \frac{\pi}{2} + \mathcal{O} \left( \frac{\nu_s^2}{\nu_{\text{Al}}^2} \right) \\
\psi_- &= \gamma + \mathcal{O} \left( \frac{\nu_s^2}{\nu_{\text{Al}}^2} \right)
\end{align*} \]

where \( \psi_\pm \) is the angle between \( \delta v_\pm \) and \( \gamma \), following the convention of Alfvén and Falthammar (1963). For the fast mode, \( \delta v_\pm \) is perpendicular to \( H_0 \), and for the slow mode it is parallel. For both modes \( \delta v_\pm \) is in the \( \gamma - H_0 \) plane as mentioned. For the fast and slow mode magnetic fields we find,

\[ 
\begin{align*}
\frac{\delta H}{H_0} (\pm) &= \frac{\delta \rho}{\rho_0} \left\{ 1 - \frac{\cos(\gamma - \psi_\pm) \cos(\gamma)}{\cos(\psi_\pm)} \right\} \\
&= \left\{ \frac{\sin(\gamma - \psi_\pm) \cos(\gamma)}{\cos(\psi_\pm)} \right\}
\end{align*} \]

\[ \quad (134) \]
for arbitrary $v^2_S / v^2_{A1}$. The $\perp$ and $\parallel$ subscripts denote components perpendicular and parallel to $H_0$, respectively, and $H_0$ with no underline denotes the magnitude of the mean field. For the fast mode for $v^2_S / v^2_{A1} \ll 1$, one finds,

$$
\begin{align*}
\frac{\delta H_{\parallel}(+)}{H_0} & \doteq \frac{\delta \rho^+}{\rho_0} \\
\frac{\delta H_{\perp}(+)}{H_0} & \doteq \frac{\delta \rho^-}{\rho_0} \cot(\gamma')
\end{align*}
$$

Whereas for the slow mode in the same limit,

$$
\begin{align*}
\frac{\delta H_{\parallel}(-)}{H_0} & \doteq -\frac{v^2_S}{v^2_{A1}} \frac{\delta \rho^-}{\rho_0} \sin^2(\gamma') \\
\frac{\delta H_{\perp}(-)}{H_0} & \doteq \frac{v^2_S}{v^2_{A1}} \frac{\delta \rho^-}{\rho_0} \sin(\gamma')\cos(\gamma')
\end{align*}
$$

The density perturbations in fast and slow modes are related to those in velocity by equation (129). For the fast mode one finds,

$$
\frac{\delta \rho^+}{\rho_0} \doteq \frac{\delta v^+}{v^+_S} \sin(\gamma')
$$

where $\delta v^+_S \equiv |\delta v^+_S|$. For the slow mode,

$$
\frac{\delta \rho^-}{\rho_0} \doteq \frac{\delta v^-}{v^-_S}
$$
Equations (132) - (138) give the relationships between the fluid variables for each mode, but without an additional assumption no definite conclusions can really be drawn for a connection between field perturbations and those in the density for application to the solar wind, because the occupation of different modes is unknown. For example, one mode could be predominantly excited. Nevertheless, we discuss a plausible assumption, which does not appear to contradict observational data available at the moment and which implies some of the properties of fluid variables reported by Coleman (1968), Schubert and Coleman (1968), and which are observed in scintillations. We propose equality of the average energy density in the different modes at any given wavenumber. This is a kind of equipartition assumption and its validity or not is probably best determined from observations. A small additional assumption is also made below that contributions to fluid variables by different modes are statistically independent.

Let \( U_\alpha(q) \) denote the energy density of the wave with wavevector \( q \) in the \( \alpha \)th mode (\( \alpha = A, +, - \)). Then,

\[
U_\alpha(q) = \frac{1}{2} \rho_o \left< \delta v^2_\alpha \right> + \frac{1}{2} \rho_o v_s^2 \left( \frac{\delta \rho_\alpha}{\rho_o} \right)^2 + \frac{1}{8\pi} \left( \delta H_\alpha \right)^2 \tag{139}
\]

For the Alfvén mode,

\[
U_A = \frac{1}{2} \rho_o \left< \delta v^2_A \right> + \frac{1}{2} \rho_o v_s^2 = \rho_o \left< \delta v^2_A \right> \tag{140}
\]
and the fast and slow modes,

\[
U_+ = \frac{1}{2} \rho_0 \langle \delta v_+^2 \rangle + \frac{1}{2} \rho_0 \langle \delta v_-^2 \rangle \left( 1 + \mathcal{O} \left( \frac{V_S^2}{V_{Al}^2} \right) \right) = \rho_0 \langle \delta v_+^2 \rangle \\
U_- = \frac{1}{2} \rho_0 \langle \delta v_-^2 \rangle + \frac{1}{2} \rho_0 \langle \delta v_-^2 \rangle \left( 1 + \mathcal{O} \left( \frac{V_S^2}{V_{Al}^2} \right) \right) = \rho_0 \langle \delta v_-^2 \rangle
\]  \hspace{1cm} (41)

For the Alfvén mode there is automatically equipartition between kinetic and magnetic field energy, for the fast mode there is the same equipartition if \( \frac{v_S^2}{v_{Al}^2} \ll 1 \); whereas for the slow mode there is equipartition between kinetic and elastic energy density with only a small fraction of the energy in the magnetic field for \( \frac{v_S^2}{v_{Al}^2} \ll 1 \). Our main assumption, \( U_A(q) = U_+(q) = U_-(q) \), implies in addition that \( \frac{\delta v_A^2}{v_A^2} = \frac{\delta v_+^2}{v_+^2} = \frac{\delta v_-^2}{v_-^2} \) on the average, and independent of the direction of \( q \).

Coleman (1968) reports approximate equipartition between the radial kinetic energy and the radial field energy. Our assumption above implies that \( 1/3 \) of the total wave energy (all modes) is magnetic, \( 1/2 \) kinetic, and \( 1/6 \) elastic. The departure from equipartition between kinetic and magnetic energy is due to the slow mode and the assumption \( \frac{v_S^2}{v_{Al}^2} \ll 1 \), but these ratios are not directly applicable to Coleman's data which is for a single component of the fluid variables.
The wavenumber-frequency spectrum of the fluid mass density is given by

\[ F_\varrho(q, \Omega) = \langle \delta \varrho_+^2(q) \rangle \delta(\Omega - \Omega_+) + \langle \delta \varrho_-^2(q) \rangle \delta(\Omega - \Omega_-) \quad (142) \]

because of the assumed statistical independence of different modes. As a direct consequence of our assumption and equations (137) and (138), we see that the predominant contribution to \( F(q, \Omega) \) is from the slow mode for \( v_S^2 / v_{Al}^2 \ll 1 \). Thus the wavenumber spectrum of the mass density is approximately,

\[ F_\varrho(q) \approx \langle \delta \varrho_-^2(q) \rangle \quad (143) \]

The slow mode has a relatively small wave velocity (\( \sim v_S \ll u \)) even close to the sun where the Alfven velocity is expected to be comparable or larger than the wind speed \( u \). This could explain why interplanetary scintillation observations indicate frozen flow of density irregularities even close to the sun. Furthermore, part of our main assumption, that \( \langle \delta v_-^2(q) \rangle \) is isotropic, implies that \( F(q) \) is isotropic. This also is in rough accord with coronal scattering observations.

In contrast with the density, fluctuations in the magnetic field are larger in the Alfven and fast modes than in the slow by a factor \( v_S / v_{Al} \). For the mean-square perpendicular field one finds,
\[
\left\langle \delta H^2_{\perp}(q) \right\rangle / H_o^2 = \frac{\left\langle \delta v^2(q) \right\rangle}{V_{Ae}^2} (1 + \cos^2(\gamma)) \quad (144)
\]

For the field fluctuations parallel to the average field one finds,
\[
\left\langle \delta H^2_{\parallel}(q) \right\rangle / H_o^2 = \frac{\left\langle \delta v^2(q) \right\rangle}{V_{Ae}^2} \sin^2(\gamma) \quad (145)
\]

where \(\left\langle \delta v^2(q) \right\rangle = \left\langle \delta v_A^2(q) \right\rangle = \left\langle \delta v^2_+(q) \right\rangle = \left\langle \delta v^2_-(q) \right\rangle\), independent of the direction of \(q\), by part of our main assumption.

The ratio of parallel to perpendicular field variations, \(\sin^2(\gamma) \cdot (1 + \cos^2(\gamma))^{-1}\), is in general less than unity; the ratio of the averages of the two over all directions of \(q\) is one-half, which corresponds to two degrees of freedom in the perpendicular plane and only one in the parallel direction. The ratio of the one-dimensional spectra observable with space probes may be obtained from equation (126); one finds that \(P_H^f / P_{\perp}^f = 1/3\), independent of frequency and of the angle between \(H_o\) and \(u\). The observed ratio of parallel field spectrum to that of the total perpendicular field is of the order of \(1/5\) to \(1/3\), very roughly independent of frequency (Coleman, 1966). A further byproduct of our assumptions is the prediction that both parallel and perpendicular field spectra are non-isotropic with departures therefrom given by trigonometric functions of (144) and (145).
A relation connecting density fluctuations and parallel field variations may be obtained from (135) and (136). The contribution of the slow mode to parallel field variations is a factor \( v_S/v_{A1} \) smaller than that of the fast mode, and the Alfven mode does not contribute at all. Therefore,

\[
\frac{\langle \delta H_{\|}^2(q) \rangle}{H_o^2} \approx \frac{\langle \delta \rho^2(q) \rangle}{\rho_o^2}
\]

(146)

From equations (137) and (138) and our main assumption,

\[
\langle \delta \rho^2(q) \rangle = \sin^2(y) \frac{V_S^2}{V_A^2} \langle \delta \rho^2_{-}(q) \rangle
\]

(147)

where as mentioned, \( \langle \delta \rho^2(q) \rangle \) is isotropic, i.e., independent of the direction of \( q \). From (146) and (147) the relation of parallel field variations to the total density is:

\[
\frac{\langle \delta H_{\|}^2(q) \rangle}{H_o^2} = \sin^2(y) \frac{V_S^2}{V_A^2} \frac{V_S^2}{V_A^2} \langle \delta \rho^2(q) \rangle\frac{\rho_o^2}{\rho_o^2}
\]

(148)

for \( v_S^2/v_{A1}^2 \ll 1 \).

Observations relevant to relation (148) have been published by Schubert and Coleman (1968); the ratio of the proton number density spectrum to that of the field magnitude is found to be about \( 2.0 \times 10^{-10} \) cm\(^{-6}\) Gauss\(^{-2}\), independent of frequency approximately for a range \( 2 \times 10^{-6} < f < 2 \times 10^{-4} \) Hz, which is the range of the observations of the density. As mentioned earlier, the spectrum of the field magnitude is closely
related to that of the parallel field. Thus Schubert and Coleman's ratio normalized by the average field magnitude and number density is about 2.4 in absolute units. This is to be compared, by evaluation of equation (126) for the one-dimensional spectra, with \( \frac{2 \cdot v_{\text{Al}}^2}{v_S^2} = \frac{2 \cdot v_{\text{Al}}^2}{v_S^2} = 3.1 \) for \( v_{\text{Al}} = 50 \text{ km/sec} \) and \( v_S = 40 \text{ km/sec} \). Equation (148), although not in exact numerical agreement with observations (probably because \( v_S^2 \) is not terribly much smaller than \( v_{\text{Al}}^2 \)), does offer an explanation for the constancy with frequency of this ratio of spectra.

Now let us try comparing the electron density spectrum obtained from interplanetary scintillations (§3) with \( 2\pi \int_{||}^{H}(f) \) observed by Mariner 2. The density spectrum from scintillations may be extrapolated to 1 a.u., and assuming (148) valid we can find what the corresponding parallel field spectrum would be if the extrapolated density spectrum held at one a.u. For this we need the variation of \( \langle \delta^2_{\parallel}(q) \rangle/\langle \sigma \rangle^2 \) with distance from the sun, \( R \). Let us assume this ratio independent of \( R \). Hence in equation (148) we may set \( \langle \delta^2_{\parallel}(q) \rangle/\langle \sigma \rangle^2 = F(q)/\langle N \rangle^2 \), where \( F(q) \) is the electron density spectrum, and \( \langle N \rangle \) the average electron (or proton) density at some reference elongation. We choose our observations at \( \xi = 22.5^0 \) for illustration. An electron number density \( \langle N \rangle = 5.6/\text{cm}^3 \) at \( R = 1 \text{ a.u.} \) extrapolates to 38 el/cm\(^3\) at \( R = 0.38 \) (\( \xi = 22.5^0 \)) assuming an \( R^{-2} \) dependence. Hence from (§4) we have \( F(q)/\langle N \rangle^2 \approx 10^{-9} q^{-3.6} \text{ cm}^{-0.6} \). Now it is necessary to evaluate (126) for \( \langle \delta^2_{\parallel}(q) \rangle \) given
by (148). The x-axis was assumed in the direction of \( \mathbf{u} \). Let \( \mathbf{H}_0 \), assumed in the equatorial plane, be expressed as \( \mathbf{H}_0 = (H_0 \cos(\theta), 0, H_0 \sin(\theta)) \) with \( \theta \) the angle between \( \mathbf{u} \) and \( \mathbf{H}_0 \). The integral in (126) comes out independent of the angle \( \theta \):

\[
2\pi \frac{P_{\parallel,\parallel}^H(f)}{H_0^2} = \frac{2\pi}{u} \frac{V_S^2}{V_{Ae}^2} \left( \frac{K_N}{\langle N \rangle^2} \right) \int_0^\infty \frac{q^2 dq}{\left( q^2 + \left( \frac{2\pi f}{u} \right)^2 \right)^{3/2}} \tag{149}
\]

The sine-squared factor of (148) contributes simply a factor of one-half to the right side of (149). Performing the integral in (149),

\[
2\pi \frac{P_{\parallel,\parallel}^H(f)}{H_0^2} = \frac{1}{2} \frac{(2\pi)^{4-s} V_S^2}{V_{Ae}^2} u^{-3} \left( \frac{K_N}{\langle N \rangle^2} \right) f^{-(s-2)} H_z^{-1} \tag{150}
\]

For a numerical evaluation of (150) take \( u = 490 \text{ km/sec} \) (at \( R = 1 \text{ a.u.} \)), \( V_S^2/V_{Ae}^2 \approx 0.64 \) (\( R = 1 \text{ a.u.} \)), \( s \approx 3.6 \) (\( R = 0.38 \text{ a.u.} \)), \( K_N/\langle N \rangle^2 \approx 10^{-9} \text{ cm}^{-0.6} \) (\( R = 0.38 \text{ a.u.} \)). Then,

\[
2\pi \frac{P_{\parallel,\parallel}^H(f)}{H_0^2} = 1.7 \times 10^{-5} f^{-1.6} \tag{151}
\]

\( 0.5 < f < 10 \text{ Hz} \)

Equation (151) is shown in Figure (13) and is labeled IPS observations. However, this line does not correspond to IPS observations at one a.u., nor is it the predicted parallel field spectrum which could be measured at one a.u. The line represents the parallel field spectrum which would exist at 1 a.u. if the fractional density fluctuation spectrum as a function
of \( q \) at \( R = 0.38 \) a.u. also existed at \( R = 1 \) a.u. Implicit in the comparison in Figure (13) of Mariner 2 and IPS spectra is the assumption that the fractional fluctuation wavenumber spectrum is fundamental and invariant in form as a function of \( R \) under the condition that the fluctuations be described by the equations of magnetohydrodynamics. The reason the IPS line is not expected to correspond to observations at 1 a.u. is that ion cyclotron damping could become important for scale sizes \( \lesssim 100 \) km at this distance, that is, magnetohydrodynamics breaks down.

The agreement of the IPS and Mariner 2 spectra in Figure (13) provides an argument for a close connection of the physical processes causing density irregularities of sizes \( q^{-1} \sim 10 - 100 \) km observed in scintillations at \( R \lesssim 0.4 \) a.u. and magnetic field variations of scale sizes \( 10^4 \) to \( 10^7 \) km detected by Mariner 2 at \( R \sim 1 \) a.u. Thus the 'inertial' range of solar wind turbulence could span more than six decades in wavenumber. However, proof of the connection and the establishment of actual relationships between all fluid variables must await future space probes. Of particular interest will be observations much closer to the sun than one a.u.

Finally, on the subject of the magnetohydrodynamic modes, it is worth noting that certain inequalities may easily be derived from the linearized equations without particular assumptions such as our proposal of equipartition. For example, from equations (135) and (136) for \( \frac{v_S^2}{v_{A1}^2} \ll 1 \), it follows that,
\[ \left\langle \delta H_{\parallel}^2(q) \right\rangle / H_o^2 \leq \left\langle \delta \mathcal{F}^2(q) \right\rangle / \mathcal{F}_o^2 \] (152)

The equality of (152) implies parallel field variations a factor \( 2 \cdot \frac{v_{A1}^2}{v_S^2} \approx 3 \) times larger than equation (148) for the same density spectrum at \( R = 1 \) a.u.

* * *

Scarf et al. (1967) pointed out that the ratio of parallel temperature to perpendicular (relative to the magnetic field) deduced from Pioneer 6 data, \( T_\parallel / T_\perp = 2 \) to 5, could lead to growing ion cyclotron waves for propagation parallel to the interplanetary magnetic field. The maximum growth rate occurs at frequencies near the ion cyclotron frequency \( \Omega \sim \Omega_i = \frac{eH_o}{M_p c} \), where \( M_p \) is the proton's mass. For a mean field of \( 6 \times 10^{-5} \) Gauss, appropriate around one a.u., \( \Omega_i = 0.6 \text{ sec}^{-1} \). This work prompted the author (in some unpublished work) to suggest the importance of the characteristic wavenumber of growing or damping proton cyclotron waves to interplanetary scintillations. Previously, the proton cyclotron radius had been mentioned as a relevant scale for IPS by Cohen et al. (1967), but no specific process was suggested.

Left handed ion cyclotron waves propagating parallel to a uniform magnetic field with frequencies near \( \Omega_i \), that is, \( (\Omega_i - \Omega) / \Omega_i \ll 1 \), have small phase velocities, \( \approx 2v_{A1} (\Omega_i - \Omega) / \Omega_i \), and even smaller group velocities, \( \approx 2v_{A1} (\Omega_i - \Omega) / \Omega_i \)^{3/2}. Thus
these waves move slowly in a reference frame moving with the solar wind velocity. For $T_{||}/T_{\perp} > 1$, the waves can grow in amplitude absorbing parallel kinetic energy of the protons. The resulting temperature variations would be expected to cause density fluctuations. Waves not propagating parallel to the field are Landau damped for $(\Omega_L - \Omega_L)/|q| \lesssim v_{th}$, where $v_{th}$ is the mean or typical thermal velocity. The damped waves may also be expected to lead to density fluctuations in that $\Omega_L(R)$ increases with decreasing distance from the sun $R$: a spectrum of Alfvén, fast, and slow waves with low frequencies $\Omega$ can propagate, undamped by the cyclotron process, out from the base of the solar wind to a distance $R$ where $\Omega = \Omega_L(R)$ and damping sets in (i.e., a magnetic beach). In either case, the important scale size, denoted $q^{-1}$, of spatial variations is that which makes the Doppler shifted frequency of the wave, $(\Omega + qv_{th}) = \Omega_L$; that is, the important waves in the process are those which resonate with particles with thermal speed $v_{th}$. This gives

$$\left(\frac{q_*}{q_o}\right)^{3/2} = 1 - \left(\frac{v_{th}}{V_{Al}}\right)^{3/2}\left(\frac{q_*}{q_o}\right)$$

(153)

$$q_o^{-1} = \left(\frac{V_{Al}^2 v_{th}}{\Omega_L}\right)^{1/3}/\Omega_L$$

From (153) we have $q_* \leq q_o$; for $v_{th}/V_{Al} \ll 1$, $q_* \approx q_o$; whereas $q_* = (V_{Al}/v_{th})^{2/3}$, $q_o = \Omega_L/v_{th}$ ( = reciprocal ion cyclotron radius) at the other extreme. With numerical values at $R = 1$ a.u. of $v_{th} = 30\text{ km/sec}$ and $V_{Al} = 50\text{ km/sec}$, then $q_o^{-1} = q_o^{-1} = 70\text{ km}$. 
Coleman (1968) has proposed that the magnetic field wavenumber spectrum continues roughly like $q^{-3.2}$ (corresponding to a one-dimensional spectrum $f^{-1.2}$) down to $q = q_\star$; and that for smaller scales the spectrum falls off roughly as $q^{-5.8}$ (one-dimensional spectrum $f^{-3.8}$ for $0.2 < f < 2.0$ Hz) as deduced from the Ogo 1 data of Holzer et al. (1966). It is naturally tempting to conjecture that the wavenumber spectrum of the density has a similar form with a 'knee' occurring at $q = q_\star$. According to this the density wavenumber spectrum would be type A for $q < q_\star$ ($2 < s < 4$; see Part I, C and D), whereas for $q > q_\star$ it would be type B ($4 < s < 6$).

Previously, we discussed qualitatively the angle scattering expected for a hybrid AB spectrum exact of this form (I; D; §8). The important thing to notice here is that the scale size at the knee, $q_\star^{-1} = q_o^{-1}$ is expected to decrease with decreasing distance from the sun. For an estimate of the dependence, assume $v_{th} \propto T^{1/2} \propto$ constant (Scarf et al., 1967), $H_o \propto R^{-2}$, and $\langle N \rangle \propto R^{-2}$. Then from (153),

$$q_\star^{-1}(R) = \left( \frac{R}{1 \text{ a.u.}} \right)^{4/3} \cdot q_\star^{-1}(1 \text{ a.u.}) \quad (154)$$

The knee in the wavenumber spectrum moves to higher wavenumbers with about the inverse four-thirds power of distance from the sun. Thus $q_\star^{-1} = 70$ km at $R = 1$ a.u. corresponds to $q_\star^{-1} = 19$ km at $R = 0.38$ a.u. or $\xi = 22.5^\circ$. However, a knee is not evident in our observations (§3) at $\xi = 22.5^\circ$ for $200 > q^{-1} > 10$ km; probably $q_\star^{-1} < 10$ km. But our observations at $\xi = 33^\circ$ and $36^\circ$ do suggest that the density spectrum is steeper
( \( s \neq 4.6 \) or \( 5 \)) at larger elongations (\( R \)). This lends support to the possibility there is a knee in the density wavenumber spectrum dependent on elongation. On this admittedly tenuous basis we show schematically in Figure (14) conjectured IPS Fourier power spectra at two different elongations. The frequency of the knee is given by \( f_\kappa = u q_\kappa /2\pi \) with \( q_\kappa \) from (153) and \( u = 400 \) km/sec.

\[ * \quad * \]

Finally we end this Section (A) with a summary in Table II of the various indices of different spectra which hold under conditions of frozen flow and 'nearly' isotropic turbulence. Cronyn (1969) has obtained some of these relations independently.
FIGURE 14: Conjectured Model IPS Spectra.
TABLE II: Dependences of Different Spectra.

Density wavenumber spectrum:
\[ F(q_x, q_y, q_z) \propto q^{-s} \]
\[ q = (q_x^2 + q_y^2 + q_z^2)^{\frac{1}{2}} \]

Phase wavenumber spectrum:
\[ \mathcal{D}(q_x, q_y) \propto q^{-s} \]
\[ q = (q_x^2 + q_y^2)^{\frac{1}{2}} \]

Fourier power spectrum of weak IPS:
\[ M_F(f) \propto f^{-(s-1)} \]

Bessel spectrum of weak IPS:
\[ M_B(f) \propto f^{-s} \]

Prewhitened Bessel spectrum of weak IPS:
\[ f \cdot M_B(f) \propto f^{-(s-1)} \]

Satellite power spectrum:
\[ P(f) \propto f^{-(s-2)} \]
B. STRONG SCINTILLATIONS:

Introduction:

In strong scintillations intensity variations in the observer's plane may be comparable with the mean intensity or even exceed the mean by factors much larger than unity. The problem of interest in this Section is that of finding statistical properties of the intensity $I(\mathbf{x}) = |A(\mathbf{x})|^2$, where the amplitude $A(\mathbf{x})$ is given by equation (94), under these conditions. We are especially interested in the effect different kinds of irregularities have on the scintillations. Time variations of the intensity are assumed due to the rigid or frozen motion of the screen past the line of sight, except in subsection (§7).

Subsection (§1) discusses first qualitatively and then quantitatively strong scintillations in the case of Gaussian irregularities. We consider the spatial amplitude correlation; the spatial intensity correlation; temporal intensity correlations for the case of rigid motion of the screen across the line of sight; correlation of amplitude between two nearby wavelengths; reduction in intensity fluctuations due to a finite bandwidth; the temporal broadening of narrow pulses; the intensity probability density; and the spectral broadening of a nearly monochromatic source.

Subsection (§2) by dimensional analysis establishes the single parameter $\beta$ which distinguishes weak and strong scintillations for cases of type A, B, and C irregularity spectra.
In (§3), (§5), and (§6) we treat strong scintillations for cases of type A, B, and C irregularity spectra. In these subsections we attempt to treat all the points discussed in (§1) for Gaussian irregularities. There is at least one new effect which does not occur for Gaussian irregularities: A sequence of equi-spaced pulses shows random arrival times. We discuss this effect in detail in subsections (§3) and (§5) for type A and B spectra.

Subsection (§4) applies the results of (§3), for type A spectra, to observations of strong interplanetary scintillations.

Subsection (§7) derives formulae for situations where the frozen-flow approximation is invalid (thawing).
§1. Gaussian Spectrum:

The conditions for strong scintillations due to Gaussian irregularities are lucidly explained by Scheuer (1968). We follow his treatment initially. Salpeter (1967) has given a thorough quantitative treatment of scintillations for the Gaussian case. The relevant parameters are:

\[ a = \text{typical size of irregularities (equation 17).} \]
\[ D = \text{distance of screen from observer's plane.} \]
\[ \lambda = \text{radio wavelength; } k = 2\pi/\lambda = \omega/c. \]
\[ \theta_s = \text{typical scattering angle (equation 64).} \]
\[ \phi_L = \text{r.m.s. phase shift of screen (equation 41).} \]

The scattering angle \( \theta_s \) and phase shift \( \phi_L \) depend on the screen thickness \( L \) and r.m.s. electron density fluctuation, \( \langle \delta N^2 \rangle^{1/2} \), through,

\[
\begin{align*}
\phi_L &= (2\pi)^{1/4} (\lambda r_e) \left( \langle \delta N^2 \rangle \right)^{1/2} (L a)^{1/2} \\
\theta_s &= \frac{\lambda}{2\pi a} \phi_L \quad \text{for } \phi > 1,
\end{align*}
\]

equations (41) and (64). Figure (15) shows the ray geometry for this scintillation problem.

Scheuer's condition (i) for strong scintillations is that the phase shift imparted by the screen \( \phi_L \) be larger than unity. Form Part I (\( D; \theta_s \)) it is known that \( \phi_L > 1 \) implies strong scattering, which is a necessary condition for strong scintillations.

Condition (ii) of Scheuer is that \( D \theta_s > a \). In Part I (\( D; \theta_s \)) this was termed Hewish and Wyndham's case (a). It
FIGURE 15: Ray Geometry for Gaussian Irregularities.
corresponds to simulataneous reception of beams from different irregularities on the screen. The Hewish and Wyndham case (b), \( D \Theta_s \ll a \) gives simply source position variations. In their case (a), or under Scheuer's conditions (ii) and (i) the different beams incident at one point on the observer's plane may interfere thereby giving strong intensity variations. Condition (ii) may be written as \( D > k a^2/\phi_L \); that is, the distance of observer from the screen must be larger than a typical focal length of the irregularities, \( \ell_L = ka^2/\phi_L \) (Salpeter, 1967). Salpeter shows that rather unusual effects occur for \( D \sim \ell_L \) when \( \phi_L^2 \gg 1 \). Strong spikes of intensity \( \sim (\ell_L/D)^{-1/3} \) (focal lines) to \( (\ell_L/D)^2 \phi_L^{2/3} \) (symmetric focal spots) times the average, and of narrow width \( \sim a(D/\ell_L)^{-2/3} \phi_L^{-2/3} \) (in one or two directions) form by focusing of rays from single irregularities at the observer's plane. In most of the following, however, we assume \( D \gg \ell_L \) so that focusing is improbable.

Observe that the phase delay along a ray path as shown in Figure (15) is constituted by the delay by the screen \( \phi_L \) > 1, and that due to the extra geometrical path length, \( \frac{1}{2} k D \Theta_s^2 \), for a tilted beam. The latter may be written as \( \frac{1}{2} (D/\ell_L) \phi_L \). Hence because \( D \gg \ell_L \) the path length phase is much larger than that due to the screen, and by condition (i) it is much larger than unity.

Scheuer's condition (iii) is an instrumental requirement that the radio bandwidth used in observations, \( \Delta \lambda_j \), be not so
large as to cause the phase of a typical beam to change by a radian across it. If the phase change were much larger than a radian, intensity variations would be 'washed out'. Since the path length phase of a beam is typically much larger than that from the screen, conditions (iii) is: $\Delta \left(\frac{1}{2} kD \theta_s^2\right) < 1$, or equivalently $\frac{\Delta \lambda}{\lambda} < \left(\frac{3}{2} kD \theta_s^2\right)^{-1}$ is the condition on the fractional bandwidth, where we have used the fact that $\theta_s \approx \frac{\lambda}{2}$. We term $\frac{\Delta \lambda_D}{\lambda} = \left(\frac{3}{2} kD \theta_s^2\right)^{-1}$ the fractional bandwidth of the scintillations; it is a characteristic of the screen through $\theta_s$, and the distance of observer from screen.

In summary, strong scintillations occur for

\[
\begin{align*}
(i) & \quad \phi_L > 1 \\
(ii) & \quad D > L_L = a/\theta_s \\
(iii) & \quad \frac{\Delta \lambda}{\lambda} < \frac{\Delta \lambda_D}{\lambda} = \left(\frac{3}{2} kD \theta_s^2\right)^{-1}
\end{align*}
\]

in the case of Gaussian irregularities.

Related to the characteristic bandwidth $\Delta \lambda_D/\lambda$ of the scintillations is the time delay between different instantaneous ray paths, $\Delta t_D = \frac{1}{2} (D/c) \theta_s^2$ (see discussion by Salpeter, 1967). For a consideration of this delay imagine a narrow plane wave radio-frequency pulse of temporal width $\delta t$ and wavelength $\lambda$ incident normally on the screen (Figure 15). Necessarily such a pulse has a fractional wavelength spread of $\delta \lambda/\lambda \approx \frac{1}{4\pi} (\lambda/c \cdot \delta t)$. For $D \theta_s \gg a$, the wave energy arrives from a distribution of ray paths, the shortest being that with the smallest scattering angle, and
the largest with the largest angles. The characteristic value of the time delays is simply the delay along a typical geometrical path, which is \( \Delta t_D = \frac{1}{2}(D/c) \theta_5^2 \). For the spread in delays, \( \Delta t_D \), to be detectable we need \( \Delta t_D \geq \delta t \), or equivalently, \( (\delta t)^{-1} > (\frac{1}{2cD} \theta_5^2)^{-1} \). The latter implies \( \delta \lambda / \lambda > \Delta \lambda_D / \lambda \). Therefore, bandwidths larger than that characteristic of the scintillations are needed to detect temporal smearing of pulses.

Spatial variations of the intensity in the observer's plane arise from the non-normal angle of arrival of different rays; that is, interference terms of the form \( \cos(k \theta_n \cdot x) \) in the intensity where \( \theta_n \) is a beam scattering angle. Hence the characteristic scale size for transverse intensity variations is \( \sim r_D = \lambda / 2\pi \theta_5 \).

Time variations in intensity may arise for example from rigid motion of the screen past the line of sight with velocity \( \vec{u} \). Then the diffraction pattern on the ground also moves rigidly with velocity \( \vec{u} \). Hence the characteristic time scale for intensity variations at one antenna is \( \sim \tau_D = r_D / |\vec{u}| \).

We observe that there is a lower limit on the shortness of the time scale \( \tau_D \) of detectable intensity variations: Clearly, it is impossible to observe time variations shorter than that which would require a bandwidth \( (\delta \lambda > \frac{1}{4\pi c \tau_D} \lambda^2) \) larger than that characteristic of the scintillations. This
implies,

\[ \chi_D > \left( \frac{3}{16 \pi^2} \right)^{1/3} \left( \frac{\lambda^2 D}{c u^2} \right)^{1/3} \]  \hspace{1cm} (157)

as a condition for observable intensity variations. We emphasize that (157) places no restriction on observations of effects of scattering as manifested in the increased apparent size of a source or in temporal broadening of pulses, for example.

Before deriving explicit formulae for some of the effects just discussed qualitatively, it is useful to summarize observable scintillation and angle scattering parameters, for the case of incoherent sources. References below are included where no discussion of methods has been given.

\[ \chi_D = \lambda / 2 \pi \theta_S | u | \]  \hspace{1cm} from intensity time variations at one antenna.

\[ \frac{\Delta \lambda}{\lambda} = \left( \frac{3}{2} \right) \kappa D \theta_S^2 \]  \hspace{1cm} by determining the characteristic bandwidth of the scintillations at one antenna (Rickett, 1969).

\[ \Delta t_D = \frac{1}{2} (\% \theta_S^2) \]  \hspace{1cm} by observation of the widening of pulses at one antenna.

\[ \theta_S \]  \hspace{1cm} by angle scattering observations at two antennas (I; D; §) or at one (IV; H).

\[ u \]  \hspace{1cm} by observation of intensity variations at two or more antennas (Salpeter, 1969).

One consistency check on observations assuming Gaussian irregularities and a thin screen is the ratio obtained from the first
and second items above:

\[
\left( \frac{\Delta \lambda_D}{\lambda} \right)^{\frac{1}{2}} \cdot \zeta_D^{-1} \approx \left( \frac{4\pi}{3} \right)^{\frac{1}{2}} \frac{|u|}{\sqrt{\lambda D}} \tag{159}
\]

The ratio is of interest because it involves only the screen velocity and distance not the scattering angle \( \Theta_s \), as pointed out by Salpeter (1969). If exact formulae derived below are used, one finds that the numerical coefficient on the right of (159) is \((8\pi^2)^{\frac{1}{2}}\), when \( \zeta_D \) (as given in 158) is the e-folding time lag of the intensity correlation, but where \( \Delta \lambda_D/\lambda \) is the fractional rectangular bandwidth for which the modulation index is one-half. In later subsections we compare the ratio (159) for cases of type A and B irregularities.

\* \* \*

Below we describe a heuristic but quantitative theory from which correlation functions may be gotten in the limit \( D/\ell_L \gg 1 \) and \( \rho_L^2 \gg 1 \), which we term the Rayleigh limit. The theory we discuss is based on that developed by Salpeter (1967) and the yet different picture used by Little (1968). Mercier (1962) has shown how to derive the functions rigorously, and where there is overlap his results corroborate those gotten by the present theory. The advantage in the non-rigorous method is that of not concealing what is involved in the wave self-interference. Furthermore, it is applicable to type A spectra, whereas according to Mercier (1962) his is not. Some of the results in the following, which are specified, hold
in general, in weak and strong scintillations, in the focal spot domain or Rayleigh limit.

For a treatment of the theory, begin by noting that we need the amplitude on the screen over that area, say a square of size $\mathcal{L}$ by $\mathcal{L}$, which contributes to the wave received at the point of observation. The mesh size for $x$ on the screen, $(\Delta x, \Delta y, = \Delta x)$ where $\Delta x = \mathcal{L}/N$ with $N \gg 1$, should be small enough so that the amplitude $\exp(i\phi(x))$ changes typically by a small amount in a distance $\sim \Delta x$. The amplitude may be expressed as,

$$\exp(i\phi(x)) = \sum_n a_n \exp(ik\theta_n \cdot x) \tag{160}$$

where $\theta_n = (n_x \Delta \theta, n_y \Delta \theta)$, $x = (j_x \Delta x, j_y \Delta x)$ and $n = (n_x, n_y, j) = (0,0), (0, \pm 1), (\pm 1,0), (\pm 1, \pm 1)$, etc. The sum in (160) has $N^2 = (\mathcal{L}/\Delta x)^2$ terms. The angle increment $\Delta \theta = (k\mathcal{L})^{-1}$ is the angular size of a single term. $	heta_n$ is the scattering angle and $a_n$ the scattering amplitude for the $n$th term. $a_n$ is a raw (unaveraged) probability for the $n$th term and $\sum_n |a_n|^2 = 1$, because the modulus of (160) is unity.

Assume that angle scattering may be typified by a single angle $\Theta_\delta$ much less than a radian. $\Theta_\delta$ is determined by the distance over which the amplitude usually changes by itself. Denote this distance $r_D$. ($r_D$ is the same as the diffraction length of Part I (D) used in the discussion of angle scattering.) Then $r_D = \lambda/2\pi \Theta_\delta = a/\sigma_L$. The linear dimension of the area contributing radiation to the point of observation
is simply \( \mathcal{L} \simeq D \theta_5 \). Thus because \( \Delta x < r_D, N = D \theta_5 / \Delta x < k D \theta_5^2 \). The term probabilities are of order \( |a_n|^2 \sim N^{-2} \), for \( |\theta_n| \leq \theta_5 \), but \( |a_n|^2 \ll N^{-2} \), and decreasing to zero, for \( |\theta_n| \gg \theta_5 \).

The wave amplitude in the observer's plane may be gotten from (160) by including for each term the phase delay arising from the geometrical path length, \( \frac{1}{2} kD \theta_n^2 \), as shown in Figure (15). Therefore the rigorous expression,

\[
A(x) = \sum_n a_n \exp\left(ik \theta_n \cdot x - \frac{i}{2} kD \theta_n^2 \right)
\]  

(161)

Note that specifications on the expansion (160) are such to make the difference in path length phase between 'adjacent' terms in (161) of order unity: \( \frac{1}{2} kD \left( \theta_n^2 - \theta_{n+1}^2 \right) \simeq kD \theta_5 \Delta \theta \simeq 1 \). This does not however imply that different terms in (161) are independent, because the \( a_n \)'s have large phases of order \( \theta_L \).

To express \( A(x) \) as a sum of independent terms one needs to sum first over blocks of terms in (161) in order that the difference in path length phase between blocks be greater or of order \( \theta_L \); if \( \delta \theta_x \) and \( \delta \theta_x \) denote the angle area of a block, then \( kD \theta_5 \delta \theta_x \simeq \theta_L \) or \( \delta \theta_x \simeq \theta_L \simeq aD \). The number of terms in a block is \( \left( \frac{\delta \theta_x}{\Delta \theta} \right)^2 \simeq \theta_L^2 \gg 1 \). Therefore, the number of independent terms in (161) is of order \( \mathcal{N}^2 \equiv (D/\ell_L)^2 \gg 1 \). Observe that \( \mathcal{N}^2 \) is roughly the number of independent cells, each of area \( a^2 \), over which the phase \( \theta(x) \) may be freely specified in the area \( \mathcal{L} \) by \( \mathcal{L} \) which contributes to the received wave. The estimate here for
the number of independent terms in (161), $\mathcal{N}^2$, agrees with that by Salpeter (1967). However, in the present theory a block of terms (which is not a plane wave) corresponds to the plane wave beams of Salpeter.

The intensity $I(x) = |A(x)|^2$ may be written as

$$I(x) = 1 + \sum_{\eta, \eta' \neq n} a_{\eta} a_{\eta'}^* e^{i[k(\Theta_{\eta} - \Theta_{\eta'}) \cdot x - \frac{i}{2} kD(\Theta_{\eta}^2 - \Theta_{\eta'}^2)]} \quad (62)$$

where the sum isolates the interference terms. The typical value of $|\Theta_{\eta} - \Theta_{\eta'}|$ in the sum is $\Theta_5$; that of the path length phase $\frac{1}{2} kD |(\Theta_{\eta}^2 - \Theta_{\eta'}^2)| \approx \frac{1}{2} kD \Theta_5^2$, both in accord with prior discussion. Observe that (62) would be close to unity if the path length phases were much less than unity.

In the Rayleigh limit, however, these phases are large, of order $(D/\ell_L)\theta_L \gg 1$.

From (161) the amplitude correlation or visibility may easily be obtained. A spatial average of $A(x)A^*(x+r)$ over distances $\sim r_D = \sqrt[2]{\pi \theta_5}$ picks out just the diagonal elements of the beam product; that is, $\Theta_{\eta} = \Theta_{\eta'}$ implies $n = n'$. The necessity of averages over distances $\sim r_D$ corresponds to time averages over intervals $\tau \sim r_D/|u|$, required for stable estimates of the visibility as mentioned in Part I (D; 62). One finds,
\[ V(\mathbf{r}) = \sum_n |a_n|^2 e^{-ik\mathbf{\theta_n}\cdot\mathbf{r}} \]  \hspace{1cm} (162)

In (163) we may average the raw probabilities \( |a_n|^2 \) over a range of \( \delta n_x \) and \( \delta n_y \) or \( \delta \theta_x \) and \( \delta \theta_y \) in order to obtain stable probabilities denoted \( \langle |a_n|^2 \rangle_n \). The angle interval of this average \( \delta \theta_x = \delta \theta_y \) need not be specified other than by the restriction \( \Delta \theta \ll \delta \theta_x = \delta \theta_y \). The averaging of the \( |a_n|^2 \)'s gives the correspondence,

\[ \sum_n \langle |a_n|^2 \rangle_n e^{-ik\mathbf{\theta_n}\cdot\mathbf{r}} \rightarrow \int_{-\infty}^{\infty} d^2 \theta W(\mathbf{\theta}) e^{-ik\mathbf{\theta}\cdot\mathbf{r}} \]  \hspace{1cm} (164)

which indicates relation (162) is equivalent to (58) for the angle density \( W(\mathbf{\theta}) \). The Fourier transform connection of the visibility \( V(\mathbf{r}) \) and angle density \( W(\mathbf{\theta}) \) is fairly general; it holds in weak, strong scintillations, in the focal spot domain or Rayleigh limit for a thin screen of Gaussian (or type A) irregularities.

The intensity correlation, \( M(\mathbf{r}) = \langle I(x)I(x+r) \rangle^{-1} \), by an average over a distance \( \sim r_D \), may be written as

\[ M(\mathbf{r}) = \sum_{n,n' \neq n, m,m' = m, n + m = n' + m} a_n a_n^* a_m a_m^* \exp(i k (\mathbf{\theta_n} - \mathbf{\theta_{n'}}) \cdot \mathbf{x}) * \exp \left( \frac{-ik}{2D} \left[ \frac{\theta_n^2}{2} + \frac{\theta_{n'}^2}{2} + \frac{\theta_m^2}{2} + \frac{\theta_{m'}^2}{2} \right] \right) \]  \hspace{1cm} (166)
Observe that the last oscillatory factor in the summand in (165) is unity for terms with \( n = m' \) and thus also \( m = n' \). These terms correspond to the beating of two interference patterns; one is from \( I(\varphi) \) and the other from \( I(\varphi + \varphi') \); the first has spatial variation of the form \( \exp(ik(\varphi_n - \varphi_{n'}):x) \) and the second is just such that it 'cancels' the first, \( \exp(-ik(\varphi_n - \varphi_{n'}):x) \). Separating off these terms for which there is cancellation gives the identity,

\[
M(r) = \left| V(r) \right|^2 + \sum_{n \neq n' \neq m \neq m'} a_n a_n^* a_m a_m^* \exp(ik(\varphi_n - \varphi_{n'}):r) + \sum_{n + m = n' + m'}\exp\left(-\frac{i k}{2D}[\varphi_n^2 - \varphi_{n'}^2 + \varphi_m^2 - \varphi_{m'}^2]\right)
\]

(166)

The sum in (166) is small in the Rayleigh limit. The summation is over three independent indices, \( N^6 \) terms, each term of order \( N^{-4} \). Alternatively, the sum is over \( N^6 \) independent blocks of terms each of magnitude \( N^{-4} \). The large phase differences between different blocks means that the contributions usually add up in a random walk to a 'typical' value of \( (N^6)^{1/2} N^{-4} = N^{-1} \approx \frac{D}{D_L} \). Hence the sum may be neglected because \( D/D_L \gg 1 \) in the Rayleigh limit.

\[
M(r) \approx \left| V(r) \right|^2
\]

(167)

Equation (167) agrees with Mercier's rigorous result for the \( D/D_L \gg 1 \) limit, with \( \rho_L^2 \gg 1 \). The wavenumber spectrum of
the intensity is evidently just the Fourier transform of the square of the visibility function.

For an exact Gaussian irregularity spectrum \( F(q) \), \( M(\tilde{x}) = \exp(-\varphi_L^2(\tilde{x})) \), where \( \varphi_L(\tilde{x}) \) is given by equation (57). Approximately, \( M(\tilde{x}) = \exp(-\varphi_L^2 \tilde{x}^2 / a^2) \) for \( \varphi_L^2 \gg 1 \). And with the same approximation \( M(q) = (1/4\pi) \cdot (a^2/\varphi_L^2) \cdot \exp(-q^2 a^2/\varphi_L^2) \). We define \( r_D \) as the e-folding lag of the spatial intensity correlation; thus \( r_D = a / \varphi_L \). The corresponding wavenumber \( q_D = \varphi_L / a = r_D^{-1} \).

For the case where the screen moves rigidly across the line of sight with velocity \( \vec{u} \), the intensity time autocorrelation functions is \( A(\tau) = M(\tilde{x} = \vec{u} \tau) \). Hence the e-folding lag of the temporal correlation is: \( \tau_D = r_D / |\vec{u}| \).

Consider next the correlation of the wave amplitudes at two slightly different wavelengths, \( \lambda \) and \( \lambda' \) with \( (\lambda' - \lambda) / \lambda = \Delta \lambda / \lambda \ll 1 \). This correlation would be a measurable function if we had a wide band coherent source, a satellite for example. The wave fields at the two wavelengths differ because of the \( \lambda \)-dependence of the scattering amplitudes \( a_n \) and the path length dependence phases. Let \( a_n (\theta_n) \) denote the amplitudes (angles) at wavelength \( \lambda \), and \( \tilde{a}_n (\tilde{\theta}_n) \) those at \( \lambda' \). We may choose \( \tilde{\theta}_n = \lambda' / \lambda \theta_n \) for convenience. Then the cross wavelength amplitude correlation, defined as \( \gamma(\Delta \lambda) = \langle A_\lambda(\tilde{x}) A^*_\lambda'(\tilde{x}) \rangle \), is:
\[ \eta(\Delta \lambda) = \left( \sum_{n, n'} a_n \tilde{a}_n^* \right) e^{\frac{2\pi i}{\lambda} \left( \frac{1}{\lambda} \theta_n - \frac{1}{\lambda} \theta_n' \right)} e^{-i\pi D \left( \frac{1}{\lambda} \theta_n^2 - \frac{1}{\lambda} \theta_n'^2 \right)} \]

\[ = \sum_n a_n \tilde{a}_n^* e^{i \frac{\Delta \lambda}{\lambda} \frac{kD}{a} \theta_n^2} \quad (168) \]

Expression (168) may be simplified by observing that approximately \( \tilde{a}_n = a_n \) in the Rayleigh limit. This is because for a wavelength difference \( \Delta \lambda \) such that \( \frac{\Delta \lambda}{\lambda} \frac{kD}{a} \theta_s^2 \approx 1 \), there is a corresponding change in the phase \( \Theta(x) \), which determines the \( a_n \) (equation 160), of only \( \frac{\Delta \lambda}{\lambda} \Theta(x) \approx \frac{\ell_s}{D} \ll 1 \), a negligible amount. Hence,

\[ \eta(\Delta \lambda) = \sum_n |a_n|^2 e^{i \frac{\Delta \lambda}{\lambda} \frac{kD}{a} \theta_n^2} \quad (169) \]

We may average the raw probabilities \( |a_n|^2 \) over a range \( \delta n_x \) and \( \delta n_y \) (or \( \delta \theta_x \) and \( \delta \theta_y \)) such that \( \Delta \theta \ll \delta \theta_x \ll \theta_s \), etc. Then by the correspondence (164) and equation (64) for \( W(\triangle \theta) \) for Gaussian irregularities one finds,

\[ \eta(\Delta \lambda) = \left\{ 1 - i \frac{\Delta \lambda}{\lambda} (kD \theta_s^2) \right\}^{-1} \quad (170) \]

Previously, Uscinski (1965) and Budden (1965b) derived expressions similar to (170).

Correlation between intensities at slightly different wavelengths, but at the same point in the observer's plane, defined as \( \xi_2(\Delta \lambda) = \left\langle I_{\lambda}'(x) I_{\lambda}'(x) \right\rangle - 1 \), may be
derived in exactly the way used to obtain (167) in the Rayleigh limit. One finds,

$$\xi(\Delta \lambda) \equiv \left| \eta(\Delta \lambda) \right|^2$$  \hspace{1cm} (171)

analogous to relation (167). From equation (170) for Gaussian irregularities,

$$\xi(\Delta \lambda) = \left\{ 1 + \left[ \frac{4\lambda}{\lambda} kD \theta_0^2 \right]^2 \right\}^{-1}$$  \hspace{1cm} (172)

The cross wavelength intensity correlation is thus a Lorentzian function.

From the cross wavelength intensity correlation, we may easily calculate the mean-square intensity fluctuations or modulation index squared for observations with a finite, but narrow, radio wavelength bandpass. Let $B(\lambda)$ denote the normalized intensity response of the bandpass ($\int_{-\infty}^{\infty} d\lambda B(\lambda) = 1$). Then the mean-square intensity fluctuations through the bandpass are given by,

$$m_B^2 = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \xi(\lambda' - \lambda) B(\lambda') B(\lambda)$$  \hspace{1cm} (173)

where $m_B$ denotes the modulation index.

As an application of (173) consider a rectangular bandpass of wavelength width $\Delta \lambda_0$: $B(\lambda) = (\Delta \lambda_0)^{-1}$ for $\lambda + \frac{\Delta \lambda_0}{2} \gg \lambda \gg \lambda_0 - \frac{\Delta \lambda_0}{2}$, and zero otherwise, where $\lambda_0$ is the center wavelength. Then by integrating (173) by parts and using the fact that $\Delta \lambda_0 \ll \lambda_0$ one gets the relation,
\[ m_B^2 = 2 \int_0^{\Delta \lambda_o} \frac{d(\Delta \lambda)}{\Delta \lambda_o} \frac{\Delta \lambda - \Delta \lambda_o}{\Delta \lambda_o} \xi_3(\Delta \lambda) \]  

(174)

For the case where (172) applies,

\[
\begin{align*}
    m_B^2 &= \frac{2}{\Lambda} \tan^{-1}(\Lambda) - \frac{1}{\Lambda^2} \ln e (1 + \Lambda^2) \\
    \Lambda &\equiv \left( \frac{\Delta \lambda_o}{\lambda_o} \right) k D \theta_s^2 
\end{align*}
\]  

(175)

The modulation index \( m_B \) is unity for small bandwidths and falls to one-half for \( \Lambda \approx 4\pi \). This 'half-way' bandwidth, denoted \( B_h \) in frequency units, bears the relation,

\[ B_h = \frac{2 c}{D \theta_s^2} \]  

(176)

to the scattering angle. Little's (1968) original calculation of the bandwidth for which \( m_B = \frac{1}{2} \) gives \( B_h = c/D \theta_s^2 \) in our notation, where \( B_h \) is the e-folding half-width of a Gaussian bandpass intensity response. The agreement with (176) is quite good. The asymptotic dependence of (175) is:

\[ m_B = \left( \pi / \Lambda \right)^{\frac{1}{2}} \text{ for } \Lambda \gg 4\pi \]

and by inspection of (174), the fall-off with the inverse square root of \( \Lambda \) or \( \Delta \lambda_o \) is seen to be general for all rapidly diminishing functions \( \xi_3(\Delta \lambda) \).

The average temporal shape of a very narrow pulse \( \delta t \ll \Delta t_D \), and thus \( \delta \lambda \gg \Delta \lambda_D \) received through a thin screen in the Rayleigh limit has a simple form in the case of Gaussian irregularities. The fleetest ray follows the line of sight;
in our notation this ray corresponds to a delay \( \Delta t = 0 \). However, the ray suffers random delays from the screen of order \( \frac{1}{\omega} \theta_L \), but these are small compared with \( \Delta t_D \approx \frac{1}{\omega} (\partial \theta_L) \theta_L \) in the Rayleigh limit. Larger time delays occur for rays received from an annulus, \( r \) to \( r + dr \) on the screen with \( r = D \theta^D \) and \( \Delta t = \frac{1}{2} (D/c) \theta^2 \). The average intensity shape is evidently proportional to \( W(\theta) \cdot \theta d\theta \) with \( W(\theta) \) the angle probability density, given here by equation (64). Changing to the variable \( \Delta t \) one finds for the average pulse shape \( \mathcal{J}(\Delta t) \):

\[
\begin{align*}
\mathcal{J}(\Delta t < 0) &= 0 \\
\mathcal{J}(\Delta t > 0) &= (\Delta t_D)^{-1} \exp \left( -\frac{\Delta t}{\Delta t_D} \right) \\
\Delta t_D &\equiv \frac{1}{c} D \theta^2 \\
\int_0^\infty d(\Delta t) \mathcal{J}(\Delta t) &= 1
\end{align*}
\]

(177)

This definition of \( \Delta t_D \) replaces the prior one. A notable feature of \( \mathcal{J}(\Delta t) \) is its strong asymmetry.

It is of interest to note that a simple and fairly general relation exists between the cross wavelength amplitude correlation \( \eta(\Delta \lambda) \) and the average pulse shape \( \mathcal{J}(\Delta t) \):

\[
\mathcal{J}(\Delta t) = \frac{c}{\lambda \pi} \int_0^\infty d(\Delta k) e^{i\Delta k c \Delta t} \eta(\Delta k) \quad (177a)
\]

where \( \Delta k/k = -\Delta \lambda/\lambda \). Relation (177a) holds in strong and weak scintillations, in the Rayleigh limit or focal spot domain, for Gaussian (or type A) irregularities. For Gaussian irregularities in the Rayleigh limit \( \eta \) is given by (170), and in this case the integral in (177a) is trivial: for \( \Delta t > 0 \), one may
complete the $\Delta k$ integration in the upper half complex plane; the contour then encloses the simple pole of $\eta(\Delta k)$ at 

$$\Delta k = i k (\kappa D \frac{\theta_s^2}{\gamma})^{-1}$$

on the imaginary axis and this gives (177) for $\Delta t > 0$. For $\Delta t < 0$, the contour must be closed in the lower half $\Delta k$ plane where there are no poles so that the integral, $\mathcal{L}(\Delta t)$, is zero.

Statistics of the wave amplitude $A(\kappa)$ in the Rayleigh limit have been shown rigorously by Mercier (1962) to be Rayleigh. Equivalently, the intensity has an exponential probability density:

$$W(I) = \exp(-I); \langle I^n \rangle = m!; \int_0^\infty dI W(I) = 1 \quad (178)$$

The density (178) for $I(\kappa)$ arises whenever a signal, $A(\kappa)$, is constituted of a large number of randomly phased harmonic components (Rayleigh, 1964). The amplitude in equation (161) is exactly of this nature in the Rayleigh limit. The sum in (161) is over $\kappa^2 \simeq (D/\ell_L)^2 \gg 1$ independent terms, each of order of magnitude $\kappa^{-1}$. Typically the terms add up in a random walk giving an intensity 

$$I_{\text{typical}} \sim \left\{ \langle \eta^2 \rangle \right\}^{1/2} \left\{ \bar{n} \right\} = 1,$$

where by typical we mean the average of a large number of samples. However, occasionally, for a particular sample, the phase vectors of the different terms in (161) may 'line up' to give a spike of intensity of order 

$$I_{\text{max}} \sim \left\{ \langle \eta^2 \bar{n} \rangle \right\}^{1/2} = \kappa^2 \simeq (D/\ell_L)^2 \gg 1.$$ 

Intensity spikes in the Rayleigh limit may be much stronger than those found in the focal spot domain ($D \sim \ell_L$) for the same mean square
phase shift $\varphi_0^2 \gg 1$. However, spikes in the Rayleigh limit do not possess the distinguishing property of focal domain spikes of having spatial width inversely related to peak intensity (see Salpeter, 1967, for a treatment of focal domain spikes). Figure (16) illustrates a one dimensional sample of a Rayleigh signal.

\*

The previous discussion of the strong scintillations for Gaussian irregularities has emphasized the theory for effects observable in the scintillations of incoherent or broad-band sources. Of course, a coherent, very narrow bandwidth, source is a powerful tool for studying an irregular medium. As an illustration of the potentialities consider a monochromatic (defined shortly) plane wave, $\exp(ikz - i\omega t)$, incident on the thin screen of Figure (15). The wave amplitude on the ground $A(\tilde{x})$ is then given in general by equation (161). If the screen moves without distortion across the line of sight, then the amplitude (161) observed at a fixed site is a function of time, simply $A(\tilde{x} = ut)$. That is, there is an 'additional' time dependence of the wave, which multiplies the rapid time dependence of the incident wave, $\exp(-i\omega t)$. Therefore, the observed wave is not monochromatic. Let $\zeta(\tau) = \langle A(ut) A^*(u(t+\tau)) \rangle$ be the amplitude time auto-correlation function. Then exactly as in the derivation of (163), by a temporal average over a time interval of $\sim \lambda/2\pi q_0 |u|$, one finds $\zeta(\tau) = V(u, \tau)$, where
FIGURE 16: Rayleigh Signal.
\( V(\mathbf{r}) \) is the amplitude correlation or visibility for a transverse baseline of \( \mathbf{r} \). The average power density of the received wave, \( \mathcal{B}(f) \), is given simply by the Fourier transform of \( \xi(\mathbf{r}) \):

\[
\begin{align*}
\mathcal{B}(f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{r} \ e^{2\pi i f \cdot \mathbf{r}} \xi(\mathbf{r}) \\
2\pi \int_{-\infty}^{\infty} df \ \mathcal{B}(f) &= 1
\end{align*}
\]

(178a)

where \( f \) is the very low frequency deviation relative to the incident wave frequency. Because \( \xi(\mathbf{r}) = V(\mathbf{u} \cdot \mathbf{r}) \) and \( V(\mathbf{r}) \) is related to the angle density \( W(\theta) \) by a two dimensional Fourier transform, one finds,

\[
\mathcal{B}(f) = \frac{1}{k u} \int_{-\infty}^{\infty} d\theta_y \ W(\theta_x = \frac{2\pi f}{k u}, \theta_y) \quad (178b)
\]

where we have assumed \( \mathbf{u} = (u, 0) \). Equation (178b) is a general relation in the sense that it holds in weak or strong scintillations, in the focal spot domain or Rayleigh limit, of Gaussian or Type A irregularities. Notice that for isotropic irregularities a Bessel transform, analogous to (103), of the correlation \( \xi(\mathbf{r}) \) gives the angle density \( W(\theta) \).

For a concrete illustration of formula (178b), assume that the angle density is given by equation (64) for Gaussian irregularities, \( W(\theta) = \frac{2\pi}{\theta_0^2} \exp(-\theta^2/2\theta_0^2) \). Then one finds, \( 2\pi \mathcal{B}(f) = (\frac{1}{k u} f_0)^{-1} \exp(-f^2/2f_0^2) \), where \( f_0 = k u \theta_0^2/2\pi \). Thus observations of the frequency broadening \( f_0 \) could be compared with strong scintillations in the Ray-
leigh limit, where the scintillation time scale is

$$\frac{\lambda}{2\pi \theta_s |u|} = a/L \cdot (2\pi f_o)^{-1}.$$
§2. Power Law Spectra:

This subsection treats the transverse spatial spectrum of the instantaneous (monochromatic) intensity \( I( \vec{x} ) = |A( \vec{x} )|^2 \) in the observer's plane for cases of power law irregularity spectra, \( \Phi_L(q_x, q_y) \propto q^{-s} \) with \( 2 < s < 6 \) and \( q = (q_x^2 + q_y^2)^{\frac{1}{2}} \). We start from the intensity auto-correlation,

\[
M(r) + 1 = \langle I(\vec{x}) I(\vec{x} + \vec{r}) \rangle
\]

\[
= \left( \frac{k}{aD} \right)^4 \int_{-\infty}^{\infty} d^2k_1 \cdots d^2k_4 \ e^{-i \frac{k}{aD} \left[ (\vec{x} - \vec{x}_1)^2 + (\vec{x} + \vec{r} - \vec{x}_3)^2 - (\vec{x} + \vec{r} - \vec{x}_4)^2 \right]} \ast \left. \left\langle \mathcal{E}^i \left[ \phi(\vec{x}_1) - \phi(\vec{x}_2) + \phi(\vec{x}_3) - \phi(\vec{x}_4) \right] \right\rangle \right) \tag{179}
\]

obtained directly from Fresnel's formula (94). The angle brackets in (179) denote a spatial average over a suitably large, but for the moment unspecified distance.

As a temporary expedient for manipulating (179) introduce a regularized spectral function, that is, make the replacement \( \Phi_L(q_x, q_y) \rightarrow g(q_x, q_y) \Phi_L(q_x, q_y) \), where \( g(q_x, q_y) \) is unity for the range of the spectrum \( \Phi_L \); \( q_1 < q < q_2 \); but for \( q < q_1 \) or \( q > q_2 \) it is such that the new phase spectrum is well behaved in the sense that a phase correlation function (equation 35) is defined:

\[
\langle \mathcal{E}(\vec{x}) \mathcal{E}(\vec{x} + \vec{r}) \rangle = \Phi_L^2 \mathcal{S}_L(\vec{x}).
\]

We may then perform the average in (179) under the assumption that \( \mathcal{E}(\vec{x}) \) is a Gaussian random variable for which relation (45) holds. This will be the case for \( q_1 L \gg 1 \), but later it is seen that irregularities important in (179) are
much smaller than \( q^{-1} \) and thus the condition for Gaussian statistics for (179) is much weaker than \( q_1 \gg 1 \). For the average of (179) one finds,

\[
M(\chi) + 1 = \left( \frac{\kappa}{2\pi D} \right)^4 \left\langle \int_{-\infty}^{\infty} d\chi_1 \ldots d\chi_4 e^{-\frac{i}{\kappa} \int_{-\infty}^{\infty} [\chi_1^2 - \chi_2^2 + (\chi_1 - \chi_3)^2 (\chi_1 - \chi_4)^2] + \frac{1}{\kappa} \sum_{\chi} \chi_1^{\chi_1} + \chi_2^{\chi_2} + \chi_3^{\chi_3} + \chi_4^{\chi_4}} \right\rangle
\]

or

\[
H(\chi_1, \chi_2, \chi_3, \chi_4) = \left\langle \left\{ \phi(\chi_1) - \phi(\chi_2) + \phi(\chi_3) - \phi(\chi_4) \right\}^2 \right\rangle
\]

where we have set \( \chi = 0 \), because of the assumed statistical homogeneity (I; A; G2). The inequality on \( H \) is gotten by use of the fact \(-1 \leq \phi_L(\chi) \leq 1\). Expression (180) was obtained previously by Pisareva (1959), Mercier (1962), and Salpeter (1967).

A useful reduction of (180) to a tractable form was obtained by Pisareva (1959). She has shown that because of the translational invariance of \( H(\chi_1, \chi_2, \chi_3, \chi_4) \) (e.g., dependence only on differences between the four coordinates) half of the integrals in (180) are trivial. This fact has not been utilized in other works on scintillation theory. Pisareva's transformation is:
\[
\begin{align*}
  x'_1 &= x_1 \\
  x'_2 &= x_2 - x_1 \\
  x'_3 &= x_3 - x_1 \\
  x'_4 &= x_4 - x_1 \\

  H(x'_1, x'_2, x'_3, x'_4) &= H(x'_2, x'_3, x'_4) \\
  H(x'_2, x'_3, x'_4) &= 2 \phi^2_L \left[ 2 - \varphi^2_L(x'_2) + \varphi^2_L(x'_3) - \varphi^2_L(x'_4) \\
  &- \varphi^2_L(x'_2 - x'_4) + \varphi^2_L(x'_2 - x'_4) - \varphi^2_L(x'_3 - x'_4) \right]
\end{align*}
\]

Inserting the transformed \( H(x'_1, x'_2, x'_3, x'_4) \) in (180) one finds,

\[
M(\varpi) + 1 = \left( \frac{k}{2\pi D} \right)^4 \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2x_2 d^2x_3 d^2x_4 \ e^{-ikx'_2} \left[ x'_3 - x'_2 - x'_4 \right] \right\} \times
\left[ e^{-ikx'_2} \left[ x'_3 - x'_2 - x'_4 \right] - \varphi^2_L(x'_3 - x'_4) \right] \times
\left[ e^{-ikx'_2} \left[ x'_3 - x'_2 - x'_4 \right] - \varphi^2_L(x'_3 - x'_4) \right] \times
\left[ e^{-ikx'_2} \left[ x'_3 - x'_2 - x'_4 \right] - \varphi^2_L(x'_3 - x'_4) \right] \times
\left[ e^{-ikx'_2} \left[ x'_3 - x'_2 - x'_4 \right] - \varphi^2_L(x'_3 - x'_4) \right]
\]

\[
M(\varpi) + 1 = \left( \frac{k}{2\pi D} \right)^2 \int_{-\infty}^{\infty} d^2x_2 d^2x_3 d^2x_4 \ e^{-ikx'_2} \left[ x'_4 - x'_2 \right] \times
\left[ e^{-ikx'_2} \left[ x'_4 - x'_2 \right] - \varphi^2_L(x'_4 - x'_2) \right] \times
\left[ e^{-ikx'_2} \left[ x'_4 - x'_2 \right] - \varphi^2_L(x'_4 - x'_2) \right] \times
\left[ e^{-ikx'_2} \left[ x'_4 - x'_2 \right] - \varphi^2_L(x'_4 - x'_2) \right]
\]

Transforming \( M(\varpi) \) into an intensity wavenumber spectrum by use of the definition (95) one finds,
\[ M(q) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2 x' \ e^{-i q \cdot x'} \ e^{-H(D_{k}^{q}_{q}, D_{k}^{q}_{q}+x'_4, x'_4)} \] (183)

where the delta function contribution to \( M(q) \) at \( q = 0 \) is ignored. Finally, rewriting \( H \) in terms of the phase wavenumber spectrum one gets,

\[ M(q) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2 x \ e^{-i q \cdot x} \ e^{-E(x, \frac{\lambda D}{2\pi} q)} \]

\[ E(x, \frac{\lambda D}{2\pi} q) = 8 \int_{-\infty}^{\infty} d^2 q' \ \overline{\Phi}(q') \sin^2 \left( \frac{q \cdot x}{2} \right) \sin^2 \left( \frac{\lambda D}{4\pi} q \cdot q' \right) \] (184)

Equation (184) is a general relation between an arbitrary wavenumber phase spectrum and the intensity wavenumber spectrum.

For power law \( \overline{\Phi}(q) \propto |q|^s \) observe that the sine squared factors in the integrand of \( E \) in (184) give convergence at the lower limit (small \( |q| \)) for \( s < 6 \), whereas at the upper limit there is convergence for \( s > 2 \). This range of \( s \) includes type A (\( 2 < s < 4 \)), type B (\( 4 < s < 6 \)), and type C (\( s = 4 \)) spectra discussed in Part I, Section C. For these kinds of spectra the regularizing function \( g(q_x, q_y) \) introduced earlier is superfluous. (It is already omitted from (184).) With \( \overline{\Phi}(q) = 2\pi \ (\lambda r_e)^2 \ L \ F(q_x, q_y, q_z = 0) \); \( F(q_x, q_y, 0) = K_{N} \cdot (q_x^2 + q_y^2)^{-s/2} \), expression (184) may be non-
dimensionalized by the scalings,

\[ \tilde{q} = \frac{\lambda D}{\alpha \pi} q, \quad \tilde{x} = \frac{2\pi}{\lambda D} x, \quad \tilde{q}' = \frac{\lambda D}{\alpha \pi} q' \]  \hspace{1cm} (185)

which make the variables \( \tilde{x}, \tilde{q}, \tilde{q}' \) non-dimensional and gives,

\[ M(q) \, d^2 q = \tilde{M}(\tilde{q}) \, d^2 \tilde{q} \]

\[ \tilde{M}(\tilde{q}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^2 \tilde{x} \, e^{-i \tilde{q} \cdot \tilde{x}} \, e^{-C_2 B^2 \tilde{E}(\tilde{x}, \tilde{q})} \]  \hspace{1cm} (186)

\[ \tilde{E}(\tilde{x}, \tilde{q}) = \int_{-\infty}^{\infty} d^2 \tilde{q}' \, |\tilde{q}'|^s \, \sin^2\left(\frac{1}{2} \tilde{q}' \cdot \tilde{x}\right) \sin^2\left(\frac{1}{2} \tilde{q}' \cdot \tilde{q}\right) \]

\[ \delta \equiv (s-2)/2 \quad ; \quad 0 < \delta < 2 \]

\[ B^2 \equiv 16\pi K_N (\lambda r_e)^2 L \left( \frac{\lambda D}{2\pi} \right)^5 C_2^{-2} \]

\[ C_2^2 = 4 \delta \left[ \pi \Gamma(1-\delta) \cos\left(\frac{\pi \delta}{2}\right) \right]^{-1} \]

\( \tilde{E} \) is a dimensionless function of two dimensionless variables.

Observe that \( B^2 \) given in (186) is identical to that introduced first in Part I (Section C), if the ubiquitous arbitrary length scale \( r_o \) of Part I is chosen equal to the Fresnel radius \( r_o = (\lambda D)^{\frac{1}{2}} \). This definition of \( r_o \) and that of the constants \( C_1 \) and \( C_2 \) make \( B \) the modulation index in weak scintillations, \( B^2 \ll 1 \). Therefore, the parameter
dependence of $\beta$ is given by equation (114). The dimensionless spectrum $\tilde{M}(\tilde{q})$ of dimensionless wavenumber depends on only a single parameter, $\beta^2$. It follows that the modulation index given by $m^2 = \sum_0^\infty d^2 \tilde{q} \tilde{M}(\tilde{q})$ depends on only $\beta^2$ for $\beta \ll 1$.

The weak scintillation approximation discussed previously (Part I; A, §4) is obtained as an asymptotic expansion of (186) for $\beta^2 \ll 1$. A Taylor expansion of $\exp(-C_2^2 \beta^2 \tilde{E}) \approx 1 - C_2^2 \beta^2 \tilde{E}$ in (186) gives, to first order in $\beta^2$,

$$\tilde{M}(\tilde{q}) = \frac{1}{2} C_2^2 \beta^2 \left| \tilde{q} \right|^2 \sin^2 \left( \frac{1}{\beta} \tilde{q} \right)$$

or

$$M(q) = 4 \Phi_L(q) \sin^2 \left( \frac{2q}{4\pi} q^2 \right)$$

$$\sum_{-\infty}^{\infty} d^2 q \tilde{M}(q) = \beta^2 \ll 1$$

(187)

where the delta function contributions at $q = 0$ in $M$ are ignored.

That equation (187) is generally only an asymptotic limit may be seen from the second order contribution in to the expansion of $\exp(-C_2^2 \beta^2 \tilde{E})$ in (186). The second order spectrum is:

$$\left\{ \tilde{M}(\tilde{q}) \right\}_{2nd} = \frac{1}{8} C_2^2 \beta^4 \sum_{-\infty}^{\infty} d^2 q \left\{ \sin^2 \left( \frac{1}{2} \tilde{q} \tilde{q}' \right) \sin^2 \left( \frac{1}{2} \tilde{q} \cdot \tilde{q}' \right) \right. -$$

$$\left. 2 \frac{\sin^2 \left( \frac{1}{2} \tilde{q} \tilde{q}' \right) \sin^2 \left( \frac{1}{2} \tilde{q} \cdot \tilde{q}' \right) \left( \tilde{q} \cdot \tilde{q}' \right)}{\left| \tilde{q} \right|^2 \left| \tilde{q}' \right|^2} \right\}$$

(188)
Inspection of the integrand of (188) shows that without some renormalization procedure there is not convergence at small $|q|$ for any log-slope in the range $4 < s < 6$ corresponding to type B, and for $s = 4$, or type C. The mathematical reason for this divergence of $(\mathcal{M})_{2\text{nd}}$ is due to an essential singularity of the expansion in $\beta^2$ at $\beta^2 = 0$ of the full wavenumber spectrum (186). However, the singularity does not invalidate the asymptotic weak scintillation spectrum (187).
§3. Type A Spectrum:  

This subsection shows that strong scintillations caused by type A irregularities (2 < s < 4 or 0 < \delta < 1) may be described by a Rayleigh limit alone. A new finding also discussed below is a way for observing the very large turbules by measurement of arrival time fluctuations of narrow pulses.

From Part I (D; §2) we have the estimate \( \Theta_\delta \approx \frac{\lambda}{2\pi \delta x} \) for the typical scattering angle where \( \delta x = r_0 \beta^{-\gamma_\delta} \). With the choice \( r_0 = (\lambda D)^{1/2} \), assumed hereon, \( \Theta_\delta \approx \frac{1}{2\pi \sqrt{D \beta^{-\gamma_\delta}}} \) and \( \delta x \approx (\lambda D)^{1/2} \beta^{-\gamma_\delta} \). In strong scintillations, for \( \beta^2 \) increasing through unity to \( \beta^2 \gg 1 \), one sees that irregularities important in the scattering (of size \( \delta x \)) diminish in size to dimensions smaller than a Fresnel radius \( (\lambda D)^{1/2} \). Around \( \beta \approx 1 \) the received radiation comes from an area of Fresnel radius size on the screen, which at the same time is the size of turbules important in the scattering. For \( \beta^2 \gg 1 \), the received radiation comes from an area, say a square of size \( \mathcal{L} \) by \( \mathcal{L} \), with \( \mathcal{L} \approx \Theta_\delta D \), which includes many turbules of size \( \delta x \). Hence we may expand the wave amplitude on the screen in plane waves exactly as in the case of Gaussian irregularities (equation 160). The number of terms in the expansion is \( N^2 \approx \left( \frac{\Theta_\delta D}{\delta x} \right)^2 \approx \beta^{4/3} \gg 1 \). This is also the number of independent terms. The angular size of a term in the expansion is \( \Delta \theta \approx \frac{\delta x}{D} \ll \Theta_\delta \). For the wave amplitude at the earth, by inclusion of the path length phases, expansion (161) holds. The typical path length phase difference be-
tween 'adjacent' terms is $\sim k D \theta_s \Delta \theta \sim 1$. Evidently, the Rayleigh limit discussed in (§1) applies to type A irregularities for $\beta^2 \gg 1$. Note that only one parameter, $\beta$, is required to specify this limit whereas for the Rayleigh limit of Gaussian irregularities two parameters $D/\ell_L$ and $\theta_L$ were needed.

From formula (167) the intensity auto-correlation is given by the square of the amplitude correlation or visibility in the Rayleigh limit, $\beta^2 \gg 1$. Thus from equation (121) for the visibility,

$$M(x) = \exp \left\{ - C_1^2 \beta^2 \left( \frac{|r|}{\sqrt{\lambda D}} \right)^2 \delta \right\} \tag{189}$$

The e-folding lag of the intensity correlation is $r_D = (\lambda D)^{\frac{1}{2}} C_1^{-\frac{\beta}{2}} \beta^{-\frac{1}{2}} \delta$. From equation (67) for $\theta_s$, one finds the relation $r_D = \left( \frac{\lambda}{2\pi \theta_s} \right)^{\frac{1}{2}} \Delta^\frac{1}{2} \delta$.

The wavenumber spectrum of the intensity is given by a two dimensional Fourier transform (equation 95) of (189) and this may be written as,

$$M(q) = (2\pi q_D^2)^{-1} G_A \left( \frac{|q|}{q_D} \right) \tag{190}$$

$$q_D \equiv \frac{1}{\sqrt{\lambda D}} C_1^{\frac{1}{2}} \beta^{\frac{1}{2}} \delta = r_D^{-1} = k \theta_s \Delta \frac{1}{2} \delta \right\} \tag{190}$$

where $G_A(\zeta)$ is the dimensionless function of dimensionless argument defined in equation (67). A notable contrast between the type A and Gaussian intensity wavenumber spectra is that
that for type A declines very gradually, only as $|q^\prime|^{-s}$ for $|q^\prime| \gg q_D$, whereas the fall-off in the Gaussian case is Gaussian. This tail on $M(q)$ arises from the long tail on the angle density discussed in Part I (D; §2) in that spatial variations of $I(x)$ with wavevector $q$ correspond to deflection angles $\theta$ roughly by $q \rightarrow k \theta$. Observe also that the weak scintillation intensity wavenumber spectrum (A; §4) declines with the same power of wavenumber as (190) for large values thereof. From equation (114) for the parameter dependence of $\beta$, $q_D \propto \lambda^{\delta}$; the width of the spectrum depends on wavelength more strongly than the first power of $\lambda$.

For the case of rigid motion of the screen across the line of sight, the intensity time auto-correlation function is simply $A(\tau) = M(u \tau)$. Thus the e-folding time lag of $A(\tau)$ is $\tau_D = r_D / |u|$. The Fourier power spectrum of $A(\tau)$ is $M_F(f)$, as given by equation (99); for large $f \gg \tau_D^{-1}$, $M_F$ has the form $M_F \propto f^{-(s-1)}$, which is also the form of $M_F$ in weak scintillations. From the definition of the first moment $f_1$ of $M_F$ in equation (115), one finds the relation, $f_1 = \tau_D^{-1} (1/\pi^2) \Gamma(1-\frac{1}{2\delta})$ for $\frac{1}{2} < \delta < 1$.

From equation (169) one may obtain a general relation between the cross wavelength amplitude correlation $\eta$ and the amplitude correlation $V$ in the Rayleigh limit:

$$\eta(\Delta \lambda) = i \frac{\lambda}{\Delta \lambda} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda D} e^{-i \pi \frac{\lambda}{\Delta \lambda} \frac{r^2}{D^2}} V(x)$$ (191)

And again using the type A visibility of equation (121), one finds,
\[ \eta(\Delta \lambda) = G_{\eta} \left( \frac{\Delta \lambda}{\lambda} / \frac{\Delta \lambda_0}{\lambda} \right) \]

\[ \frac{\Delta \lambda_0}{\lambda} = 2^{\frac{5}{6}} \pi C_1^{-2/5} \beta^{-2/5} \]

\[ G_{\eta}(\nu) \equiv i \int_0^\infty d\xi \ e^{-\nu \xi} \xi^\delta e^{-i \xi} \]

\[ |G_{\eta}(\nu)| \leq 1 ; \quad G_{\eta}(0) = 1 \]

\[ C_1^2 = \text{defined in equation (52).} \]

\[ G_{\eta}(\nu) \approx 1 - \nu^\delta \Gamma(1+\delta) e^{-\nu \frac{i \pi \delta}{2}} \]

\[ G_{\eta}(\nu) \approx \frac{1}{\nu} \frac{i}{\delta} \Gamma\left(\frac{1}{\delta}\right) \]

\[ (192) \]

\[ (193) \]

\( G_{\eta}(\nu) \) is another dimensionless function of dimensionless argument. Some asymptotic limits of it are:

For large \( \nu \) the fall-off of the amplitude correlation \( \eta \) has the same dependence as that for Gaussian irregularities, \( \sim (\Delta \lambda)^{-1} \). The cross wavelength intensity correlation is given by \( \xi(\Delta \lambda) = \left| G_{\eta}\left(\frac{\Delta \lambda}{\lambda} / \frac{\Delta \lambda_0}{\lambda} \right) \right|^2 \) and from equation (173) one may obtain the modulation index \( m_B \) for observations with a finite bandwidth. For 'large' bandwidths \( \Delta \lambda_0 \), \( m_B \) falls-off as \( m_B \propto (\Delta \lambda_0)^{-1/2} \), the same as for Gaussian irregular-
ities. The parameter dependence of the fractional scintillation bandwidth of equation (192) is:

\[
\frac{\lambda}{\Delta \lambda_D} \sim K_N \frac{\rho}{r_e}^{2/3} \lambda^{1+2/3} L^{1/3} D
\]  

(194)

The relation between \( \tau_D \) and \( \Delta \lambda_D/\lambda \), analogous to equation (159) for Gaussian irregularities, is:

\[
\left( \frac{\Delta \lambda_D}{\lambda} \right)^{1/2} \tau_D = \sqrt{\pi} 2^{1/2} 6^{1/2} \frac{|u|}{\sqrt{\lambda D}}
\]

(195)

Formula (195) differs from (159) by only a rather small numerical factor for \( \xi \) not too close to zero.

The average temporal shape of narrow pulses received through a type A thin screen may be obtained following the discussion prior to equation (177). As there, \( \Delta t = 0 \) corresponds to the time delay along the line of sight, including the delay from the screen.

\[
\mathcal{E}(\Delta t < 0) = 0
\]

\[
\mathcal{E}(\Delta t > 0) = \frac{1}{2} \left( \Delta t_D \right)^{-1} G_A \left( \sqrt{\frac{\Delta t}{\Delta t_D}} \right)
\]

\[
\Delta t_D = \frac{1}{2c} D \Theta_5^2 ; \quad \int_0^\infty \mathcal{E}(\Delta t) \mathcal{E}(\Delta t) \, d\sigma = 1
\]

(196)

For large \( \Delta t \gg \Delta t_D \), and from the asymptotic fall-off of \( G_A(\xi) \) given in equation (71), one sees that \( \mathcal{E}(\Delta t) \) dimin-
ishes quite gradually, as \( (\Delta t)^{-s/2} \), faster than the first, but slower than the second inverse power of \( \Delta t \). This is markedly different from the exponential fall-off (equation 177) for Gaussian irregularities. However, as in the Gaussian case, the pulse shape (196) is highly asymmetrical. The averaging time needed to obtain a stable pulse shape is of the order of \( \delta x / |\mathbf{w}| \approx \mathcal{C} \delta \).

The average pulse shape (196) is not the full story. The delay along the line of sight (\( \Delta t = 0 \)) contains a large contribution by the big turbules in the screen. For rigid motion of the screen across the line of sight, the difference in arrival times of narrow pulses over a long time interval \( \mathcal{C} \), \( \Delta t_{Ar}(\mathcal{C}) = t_{Ar}(t + \frac{\mathcal{C}}{\Delta}) - t_{Ar}(t - \frac{\mathcal{C}}{\Delta}) \), is:

\[
\Delta t_{Ar}(\mathcal{C}) = \frac{1}{\omega} \left\{ \phi\left(\frac{x + \frac{\mathcal{C}}{2}}{\Delta}\right) - \phi\left(\frac{x - \frac{\mathcal{C}}{2}}{\Delta}\right) \right\} \tag{197}
\]

with \( \mathcal{C} = u \mathcal{C} \). The average \( \langle \Delta t_{Ar}(u \mathcal{C}) \rangle = 0 \), and the r.m.s. arrival time difference is:

\[
\langle \Delta t_{Ar}^2(u \mathcal{C}) \rangle^{1/2} = \frac{C_1}{\omega} \beta \left(\frac{|u \mathcal{C}|}{\sqrt{\lambda D}}\right)^s \tag{198}
\]

from equation (52) with \( r_o = (\lambda D)^{1/2} \) and subject to the constraint \( q_2^{-1} < |u \mathcal{C}| < q_1^{-1} \). Formula (198) applies for \( \beta^2 \) larger or smaller than unity, with different conditions on the interval \( \mathcal{C} \). Independent of \( \beta^2 \leq 1 \), we need \( |u \mathcal{C}| > \delta x \) (\( \delta x \) assumed \( > q_2^{-1} \)) for validity of geometrical optics on which (198) is based. For \( \beta^2 \gg 1 \),
we need \( |u \tau| > (\lambda D)^{1/2} \beta \frac{2-\delta}{\delta^2} \) for (198) to hold because the arrival time fluctuations must be larger than the smearing time, \( \Delta t_D \), to be defined. For the range of intervals, \( \delta x < |u \tau| < (\lambda D)^{1/2} \beta \frac{2-\delta}{\delta^2} \), arrival time fluctuations are relatively small and cannot be separated from the pulse smearing; this range of \( \tau \) ( \( > \) scintillation time scale \( \tau_D \)) permits one to do the average over \( \tau_D \) required to get a stable mean pulse shape. For \( \beta^2 < 1 \), one must wait for an interval \( \tau \) long enough in order for \( \langle \Delta t_{AR}^2 \rangle^{1/2} \) to be larger than the intrinsic pulse width or structure.

Observe that \( \langle \Delta t_{AR}^2 \rangle^{1/2} \propto \lambda^2 \) has the same wavelength dependence as the ordinary dispersion caused by a uniform medium of free electrons, and that this is much weaker than the pulse smearing time \( \Delta t_D \propto \lambda^2 + 2/\delta \). Furthermore, it is evident that the scaled instantaneous arrival time fluctuation, \( \Delta t_{AR} (u \tau )/\lambda^2 \), is independent of wavelength, because it depends only on electron fluctuations along the line of sight. For \( |u \tau| \ll L \), following the arguments in Part I (C), \( \Delta t_{AR} (u \tau) \) has a Gaussian probability density.

For \( \beta^2 \gg 1 \), it is convenient for comparison with observations to cast formula (198) into a non-dimensional form, by measuring \( \langle \Delta t_{AR}^2 \rangle^{1/2} \) in units of the pulse smearing time scale \( \Delta t_D = \frac{1}{\omega} \frac{\lambda^2}{D} \theta_5^2 \), and also the interval \( \tau \) in multiples of \( \Delta t_D \), that is, \( \tau = M \Delta t_D \). Also \( \beta \) may be expressed as a power of \( \omega \Delta t_D \). One finds,
\[
\langle \Delta t_{Ar}^2 \rangle^{1/2} / \Delta t_D = \frac{2}{\pi^{1-\delta}} \left[ \frac{|kw|}{c} K D M \right]^{\delta} \left( \frac{\omega \Delta t_D}{2\pi} \right)^{3\delta-2} (198a)
\]

For observation of a finite \( \langle \Delta t_{Ar}^2 \rangle^{1/2} \), the ratio on the right of (198a) should be larger than unity. (Later, in subsection 5.5, it is shown that type B spectra also give arrival time fluctuations. These are described by a formula similar to (198a), but with \( 1 < \delta < 2 \).)

For type A spectra, arrival time fluctuations provide the only way we have yet found for measuring the very large turbules.

For type A irregularities in the Rayleigh limit, statistics of the wave amplitude are Rayleigh, or equivalently the intensity has an exponential probability density, as defined by equation (178). The amplitude is a sum of \( N^2 \sim \beta^{4/5} \gg 1 \) independent randomly phased harmonic components. The maximum intensity, which occurs when all of the phases 'line up', is \( \sim I_{max} \sim N^2 \sim \beta^{4/5} \gg 1 \).

* * *

Finally to end this subsection we mention one effect which is observable in the wave received from a coherent source. The spectral broadening of a 'nearly' monochromatic wave received through a type A screen may be obtained from
equation (178a):

\[ 2\pi B(f) = \left( 2\pi f_0 \right)^{-1} G_B \left( f/f_0 \right) \]

\[ f_0 = \frac{1}{2\pi} \left( \frac{C_\lambda^2}{2} \right)^{1/2} \frac{\left| u \right|}{\sqrt{\lambda D}} \beta^{1/8} = k \left| u \right| \theta_3 / 2\pi \] (198b)

\[ G_B(\nu) \equiv 2\sum_0^\infty d_\xi \cos(\nu \xi) e^{-\xi^2/2} \]

\[ \int_0^\infty d\nu \ G_B(\nu) \equiv 2\pi \]

where \( G_B(\nu) \) is yet another dimensionless function of dimensionless argument. For \( \nu \gg 1 \), one sees that \( G_B(\nu) \propto \nu^{-(s-1)} \) declines quite slowly in contrast with the Gaussian fall-off in the case of Gaussian irregularities. Equation (198b) holds for \( \beta^2 \leq 1 \). However, in the Rayleigh limit, \( \beta^2 \gg 1 \), the spectral broadening, as characterized by \( f_0 \), may be compared for example with the e-folding lag of the intensity time auto-correlation, for which there is the relation:

\[ f_0 = \left( \frac{1}{2\pi \zeta_D} \right) 2^{-1/2} \delta \]
§4. **Strong Interplanetary Scintillations:**

Recent interpretations of strong interplanetary scintillations, in terms of the theory for Gaussian irregularities, is in a muddled state. This is partly due to the apparent coincidence that the angular sizes intrinsic to most strong but small sources at meter wavelengths is just such as to prevent observation of their very strong scintillations, because their finite size smears out the short scale variations in the received diffraction pattern (as discussed by Salpeter, 1967). (In Part IV a way for circumventing this limitation imposed by continuous sources, using pulsars, is investigated.) Another cause of the muddle may be the use of the scintillation theory for Gaussian irregularities. The IPS data discussed in Section A (§3 and §4) suggests the assumption of type A irregularities, at least for a range of distance from the sun, $0.05 < p < 0.5$ a.u.

To illustrate the current interpretation of IPS observations let us summarize some recent representative estimates for the r.m.s. phase shift:

\[
\phi_L = 0.027 \left( \frac{\lambda}{70 \text{m}} \right) p^{-1.6} ; \quad 0.1 \leq p \leq 0.8 \text{a.u.} \quad (199)
\]

\[
\phi_L = 0.015 \left( \frac{\lambda}{70 \text{m}} \right) p^{-1.05} ; \quad 0.028 \leq p \leq 0.3
\]

\[
\phi_L = 0.062 \left( \frac{\lambda}{70 \text{m}} \right) p^{-1.43} ; \quad 0.185 \leq p \leq 0.93
\]
Expression (199) is from Hewish and Symonds (1969), and (200) from Cohen and Gundermann (1969). The small elongation formula in (200) joins the large at about 0.19 a.u. Hewish and Symonds' estimate for $\theta_L$ is about a factor of two greater than (200) at $p = 0.8$ a.u., whereas at $p = 0.2$ it is only a factor 1.1 larger. The phase shift determination by Bourgeois (1969) agrees with (199) for $0.1 \leq p \leq 0.15$. Along with the estimates (199) and (200) it is customary under the Gaussian assumption to derive the radial variation of the scale size, $a(p)$. For example, Cohen and Gundermann (1969) found $a(p) \propto p^{1.17}$ for $0.038 \leq p \leq 0.14$, but $a(p) \propto p^{0.55}$ for $0.14 \leq p \leq 0.6$.

Under the assumption of isotropic type A irregularities with $s = 3.6$ independent of elongation, and frozen flow let us compare values of $\beta$ obtained from equation (120) with those of formulae (199) and (200). It is convenient to do this at four selected elongations corresponding to four wavelengths. At each wavelength Cohen and Gundermann (1969) have reported a distinct change in slope of the frequency width measure of the scintillation spectrum as a function of elongation at the elongations in question. On the Gaussian assumption in the Fraunhofer limit ($D \gg 2\pi a_0^2/\lambda$) the change in slope is expected, according to Mercier's (1962) theory, to occur at $\theta_L = 0.8$ (Cohen and Gundermann, 1969). The advantage in using these only semi-strong scintillation points is that the concomitant parameters are probably not
corrupted to any large extent by finite source size effects. For the type A spectrum the single parameter of scintillations is $\beta$ (§2); therefore, a change in slope of the intensity wavenumber spectrum with elongation must occur at $\beta = C_{10}$, where $C_{10}$ is an unknown numerical constant (derivable in principle through evaluation of equation (186)) which depends only on the log-slope $s$. Values of $\beta$ are gotten from equation (120), which was obtained from weak scintillation modulation indices. We have the relation $\beta = m$ even for $\beta$ comparable or larger than unity. Below is a summary of the relevant data at the four elongations:

\[
\begin{array}{|c|c|c|c|c|}
\hline
p & \sin(\varepsilon') a.u. & \lambda & \Phi_L (eqn. 200) & \Phi_L (eqn. 199) & \beta (eqn. 120) \\
0.027 a.u. & 11 cm & 0.81 & 0.60 & 1.1 \\
0.043 & 21 & 0.98 & 0.55 & 1.4 \\
0.19 & 70 & 0.79 & 0.40 & 0.7 \\
0.30 & 154 & 0.76 & 0.37 & 1.0 \\
\hline
\end{array}
\]

$\Phi_L \approx 0.8$ for most of the Cohen and Gundermann numbers because this is the way they derived the first of equations (200). Their phase shift is reasonable in describing the change in slope of the intensity spectrum width, but it requires a rather ungainly combination of power laws as a function of $p$. Hewish and Symond's phase shift (199) is systematically small for interpretation with Mercier's theory; and, furthermore, the numbers obtained from it in (201) show a suspicious trend towards smaller values at longer wavelengths, or larger elon-
gations. On the other hand, the values of $\beta$, although having a fair bit of scatter, do not have an evident trend.

From the average of the numbers in (201) for $\beta$, we obtain the estimate $C_{10} \approx 1.0$ for the value of $\beta$ at which the wavenumber intensity spectrum shows a change in its width as a function of $\beta$.

For really strong scintillations, $\beta^2 \gg 1$, from equations (190) and (192) and formula (120) for $\beta = m$ and with $\delta = 0.8$, we find,

$$\tau_D(\text{sec.}) = 0.15 \sqrt{\cos \theta} \left( \frac{400 \text{~km}}{u} \right) \left( \frac{70 \text{~cm}}{\lambda} \right)^{1.25} \left( \frac{p}{0.1 \text{~a.u.}} \right)^{2.0}$$

$$\Delta \lambda_D/\lambda = 0.024 \left( \frac{70 \text{~cm}}{\lambda} \right)^{3.5} \left( \frac{p}{0.1 \text{~a.u.}} \right)^{4.0}$$

where $\tau_D$ is the e-folding lag of the intensity time autocorrelation and $\Delta \lambda_D/\lambda$ is the characteristic fractional bandwidth of the scintillations. For a numerical example assume $p = 0.0316$ a.u. (6.8 solar radii), $\lambda = 21$ cm, and $u = 154$ km/sec (from Parker's (1965) million degree corona as given by Cohen and Gundermann (1969)). One finds $\tau_D = 0.17$ sec and $\beta = 2.3$. From (93), the relation between the first moment of the scintillation power spectrum and $\tau_D$ is $f_1 = 0.24/\tau_D$ for $\delta = 0.8$; thus for this example $f_1 = 1.4$ Hz. Considering that the $p$ chosen is outside the suggested range of (202), this is in good agreement with observations by Cohen and Gundermann (1969), who found a first moment of $f_1 \approx 4.7$ Hz in measurements at $p = 0.0316$ at 21 cm of 3C-279. At the same wavelength and elongation one finds $\Delta \lambda_D/\lambda \approx (63)^{-1}$. 
Further observable parameters in strong scintillations are the pulse smearing time $\Delta t_D$ and the r.m.s. arrival time fluctuation $\langle \Delta t^2_{Ar} \rangle^{1/2}$ discussed in (§3). For $p = m$ in formula (120) (with $\delta = 0.8$ for $0.05 \leq p \leq 0.5$ a.u.) one may insert numbers (ugh) in equations (196) and (198) and thereby finds,

$$\Delta t_D (\mu \text{sec}) = 5.1 \times 10^{-4} \left( \frac{\lambda}{70 \text{ cm}} \right)^{3.5} \left( \frac{0.1 \text{ a.u.}}{p} \right)^{4.0}$$

$$\langle \Delta t^2_{Ar} (\mu \text{sec}) \rangle^{1/2} = 1.4 (\cos \xi)^{-0.4} \times\left( \frac{u}{400 \text{ km sec}^{-1}} \right)^{0.8} \left( \frac{\lambda}{70 \text{ cm}} \right)^{2} \left( \frac{0.1 \text{ a.u.}}{p} \right)^{1.6}$$

$$0.05 \leq p \leq 0.5 \text{ a.u.}$$

It must be emphasized that formula (202a) for the pulse smearing time depends on very small turbulence of size $\delta x \simeq (\lambda_D)^{1/2} \beta^{-1.25}$, whereas (202b) depends on the very large turbulence of size $q^{-1} \simeq |u| \geq (\lambda_D)^{1/2} \beta^{1.87}$. Thus the correctness or not of (202a) and (202b) depends on the (unknown) range of validity of the power law irregularity spectrum $\Phi_L(q) \propto |q|^{-3.6}$, which we have assumed.

As a numerical illustration for (202a) and (202b) consider observations at $\xi = 5.7^\circ$ or $p = 0.1$ a.u. (21.5 solar radii) and at a wavelength of $\lambda = 750$ cm ($= 40$ MHz).
Then from equation (120) one finds $\beta = 55.1$ and $\Delta t_D = 2.1 \mu$ sec. For evaluation of $\langle \Delta t_{Ar}^2 \rangle^{\frac{1}{2}}$, assume $u = 270 \text{ km/sec}$ which is appropriate to Parker's (1965) million degree corona. Then for a time interval $\tau$ of one hour one finds $\langle \Delta t_{Ar}^2 \rangle^{\frac{1}{2}} = 116 \text{ sec}$. A positive measurement of $\langle \Delta t_{Ar}^2 \rangle^{\frac{1}{2}}$ over intervals of the order of hours would be valuable in providing a measurement of electron density fluctuations with sizes $q^{-1} \lesssim \mu \tau \lesssim 10^6 \text{ km}$, which have no affect on scintillations. However, according to the estimate (202b) the possibility of deriving $\langle \Delta t_{Ar}^2 \rangle^{\frac{1}{2}}$ from delay fluctuations of radar pulses bounced off of Venus appears at this time to be marginal for parameters of the Arecibo radar at 430 MHz (D. Cambell, 1970).

The spectral broadening of a monochromatic plane wave received through the interplanetary medium may be obtained from equation (198b). We have the relation $f_o = \left( \frac{1}{2 \pi \tau_D} \right)^{\frac{1}{2}}$ for the characteristic width of the broadening, for $\beta^2 \gg 1$.

$\tau_D$ is given numerically by (202) in this limit, and thus one finds,

$$f_o (\text{Hz}) = \frac{0.7}{\sqrt{\cos \xi}} \left( \frac{u}{400 \text{ km/sec}} \right) \left( \frac{\lambda}{70 \text{ cm}} \right)^{1.25} \left( \frac{0.1}{p} \right)^{2.0}$$

for $0.05 < p < 0.5 \text{ a.u.}$

which holds for $\beta^2 \geq 1$. The spectral width $f_o$ depends on the small irregularities of size $\delta x \sim (\kappa D)^{\frac{3}{2}} \beta^{-1.25}$. Thus the correctness of (202c) depends on the validity of the power law spectrum $\Phi_L(q) \sim |q|^{-3.6}$ for turbules of size $q \lesssim \delta x$. 

For a numerical illustration of (202c) assume \( p = 0.0316 \) a.u. \( (6.8 \) solar radii \( ) \), \( \lambda = 21 \) cm, and \( u = 154 \) km/sec, the latter appropriate to Parker's (1965) million degree corona, as given by Cohen and Gundermann (1969). Then one finds \( f_0 = 0.6 \) Hz.

\[ \star \quad \star \]

Finally to end this subsection let us discuss the apparently paradoxical behaviour of IPS data (Cohen and Gundermann, 1969) that with decreasing elongation modulation indices of the smallest strong sources \( ( \text{for example CTA-21 and 3C-279}) \) increase up to a certain point and then for yet smaller elongations diminish in the way expected for a source with a finite diameter, whereas at the same time the width of the frequency and therefore wavenumber spectrum increases monotonically, much more rapidly for elongations less than that of the index maximum, with decreasing elongation with no evident saturation even for the smallest elongations contrary to the thin screen theory for Gaussian irregularities including the finite source size. For a qualitative consideration of the paradox let us assume that the source visibility function has a rather sharp cut-off, as in a Gaussian, at a baseline of \( b_e = \lambda / n \theta^2 \), where \( \theta^2 \) is the intrinsic source size. Then from Salpeter (1967) we know that the effect of the source size on the scintillation spectrum is crudely that of filtering out wavenumber components higher than \( q^2 = 2^{-\frac{1}{2}} \left( \frac{\theta^2}{D} \right)^{-1} \).
Let us compare modulation indices and intensity spectrum widths for Gaussian and type A spectra, both assumed in the Rayleigh limit. In both cases the intensity spectrum is roughly constant out to $q = q_D$ and for larger $q$ it diminishes, in the Gaussian case very rapidly, whereas quite gradually for type A, for point sources. In both cases $q_D$ is expected to increase rapidly with decreasing elongation and the height of the spectrum at $q = 0$ decrease just so as to make the integral of the $q$-spectrum unity (modulation index of one) for point sources $q_I \to \infty$. For both Gaussian and type A irregularities, the width, measured say by the second moment, of the scintillation spectrum will continue to increase up to the point where $q_D \gg q_I$, and for yet smaller elongations the width will saturate at a constant value of the order $q_I$. Let us adjust the Gaussian and type A irregularities to give the same modulation index a fair bit less than unity, for elongations less than that of the index turnover, where a not insignificant part of the fluctuation power is filtered out. Then it is clear that because of the long tail on the type A spectrum we need $q_D(\text{type A}) \ll q_D(\text{Gaussian})$ to get the same index. This implies that for the type A case that there will be a larger range of elongations where $q_D(\text{type A}) \ll q_I$, that is, where the spectrum width increases but the index decreases, than for the Gaussian case. Thus a type A interplanetary irregularity spectrum could explain the observed paradox, but we cannot say so for sure without doing the prior discussion numerically.
§5. Type B Spectrum:

This subsection discusses first qualitatively and then semi-quantitatively the strong scintillations arising from type B irregularities ($4 < s < 6$ or $1 < \delta < 2$). For most observable quantities it is found that the wave propagation phenomena conspires to hide effects directly related to the very large irregularities. This fact was noted by Salpeter (1969) who previously treated some aspects of scintillations for type B spectra. As discussed below, however, a sequence of equi-space narrow pulses received through a type B screen show random arrival times. This effect also occurs for type A spectra, but not for a Gaussian spectrum. It provides another method, in addition to the source position variations treated in Part I (D; §4), for observing large intense type B irregularities.

From Part I (D; §3) we have an estimate for the typical diffraction angle $\theta_s \sim \frac{\lambda}{2\pi \delta x}$ (determined by turbules of size $\delta x$ = diffraction length), and for the geometrical optics angle $\Theta_s \sim \frac{r_s}{D}$ (determined by turbules of size $r_s$). With the choice $r_o = (\lambda D)^{1/2}$ (assumed hereon), $\theta_s \sim \frac{1}{2\pi \sqrt{\lambda D}} \beta^\delta$ and $\Theta_s \sim \frac{1}{2\pi \sqrt{\lambda D}} \beta^\delta$, from equations (82) and (84) respectively. Because $\delta^{-1} < (2-\delta)^{-1}$ for type B, it follows that $\Theta_s \gg \theta_s$ for $\beta^2 \gg 1$.

From ($\delta_2$), strong scintillations are possible for $\beta^2 \gg 1$. The scale size of the intensity pattern in the observer's plane at a distance D from the screen may be estimated
as \( \sim r_D \equiv \lambda / \sqrt{\alpha} \), because \( \theta_s \gg \theta_s \). \( \theta_s \) in \( r_D \)

is taken as the e-folding radius of the angle density (84) by analogy of the definition of \( r_D \) for Gaussian irregularities. Keeping all of the ugly numerical factors one finds,

\[
 r_D \simeq \sqrt{\lambda D} \beta^{\frac{1}{2-\delta}} \left\{ \sqrt{\alpha} \pi \left[ \frac{C_7^2 + C_8^2}{2 \pi \alpha} \right]^{\frac{1}{4-2\delta}} \right\}^{-1}
\]

(203)

where the numerical constants \( C_7 \) and \( C_8 \) are given in equation (83). From the parameter dependence of \( \theta_s \) given in equation (85), one finds \( r_D \sim \lambda^{-\frac{\delta}{2-\delta}} \).

For the case of rigid motion of the screen across the line of sight with velocity \( u \), the scintillation time scale is \( \sim \tau_D \equiv r_D / |u| \). This estimate has the same parameter dependence as that first obtained by Salpeter (1969).

The scintillation bandwidth \( \Delta \lambda_D \) is derived with the aide of Figure (17). In the Figure, \( \theta (\chi_o) \) is the reference angle of Part I (D; 33), and \( \chi_o \) is the coordinate on the screen of this received ray; \( \Delta \theta (\chi_o, r) \) is the change in ray direction from the point \( \chi_o \) to \( \chi_o + r \) as defined by equation (75); \( \phi (\chi_o) \) is the phase shift from the screen at the point \( \chi_o \); and in the neighborhood of \( \chi_o \), the phase of the screen is expanded as \( \phi(\chi) = \phi(\chi_o) + k_{\chi} \cdot \theta(\chi_o) + \phi(\chi, \chi_o) \), where \( k_{\chi} = \chi - \chi_o \), as in equation (80). To help in understanding the Figure we have associated the slowly varying screen phase shift \( \phi(\chi_o) + k_{\chi} \cdot \theta(\chi_o) \) with the horizontal line, and the rapidly varying component \( \phi(\chi, \chi_o) \) with the tilted line. From the Figure one sees that the difference in
\[ x_0 = \Theta(x_0)D \]

\[ \text{Delay} = \phi(x_0) - \frac{kD}{2} \Theta^2(x_0) + \frac{kD}{2} \Delta \Theta^2(x_0, r) \]

\[ \Delta \Theta(x_0, r) \]

FIGURE 17: Ray Geometry for Type B Irregularities.
phase along the two rays shown is \( \hat{\phi}(x, x_0) - \frac{1}{a} k D A(\hat{\phi}(x_0, r)) \). The typical value of \( A(\hat{\phi}(x_0, r)) \) for the rays received at any given instant is \( A_S \), and the corresponding range of \( |x| \) is \( \ll r_s \), at which separation \( \hat{\phi}(x_0 + r_s, x_0) \sim \frac{kD \theta^2}{2} \). Therefore, as discussed for Gaussian irregularities the scintillation bandwidth may be estimated as \( \frac{A\lambda_D}{\lambda} \sim \left( k D \theta_S^2 \right)^{-1} \). From equation (84) with \( r_o = (\lambda D)^{1/2} \), one obtains,

\[
\frac{A\lambda_D}{\lambda} \sim \beta \frac{-2}{2-\delta} \left\{ \frac{1}{2\pi \left[ \frac{2\pi^2}{C_T^2 + C_S^2} \right]^{1/2-\delta}} \right\} \quad (204)
\]

Formula (204) agrees in dependence with the estimate previously obtained by Salpeter (1969). The parameter dependence of \( \frac{A\lambda_D}{\lambda} \) is:

\[
\frac{\lambda}{A\lambda_D} \sim K \frac{1}{2-\delta} \frac{2}{r} \frac{2-\delta}{\lambda^{2-\delta}} \frac{2+\delta}{D \delta} \quad (205)
\]

The relation between \( A\lambda_D \) and \( \tau_D \) for type B irregularities is roughly the same as that for Gaussian,

\[
\left( \frac{A\lambda_D}{\lambda} \right)^{1/2} \tau_D^{-1} \sim \sqrt{\pi} \frac{|x|}{\sqrt{\lambda D}} \quad (206)
\]

The numerical constant on the right of (206) is rather uncertain.

The shape of a narrow pulse received through a type B screen depends on the distribution of delays,
\[ \Delta t(x_o, r) = \frac{1}{\omega} \phi(x_o + r, x_o) - \frac{1}{\Delta c} D \frac{\Delta \theta^2}{\Delta x^2}(x_o, r) \] (207)

for different values of \( r \) around the point of origin of the reference ray. As yet we have not obtained an analytic expression for the average pulse shape, and cannot say even if the shape is asymmetric or not as a function of time. However the spread in instantaneous arrival times of different rays may be estimated from (207) because \( |r| \lesssim r_s \) for most of the received rays. This gives the estimate

\[ \Delta t_D \sim \frac{1}{2c} D \Delta \theta_S^2 = \frac{1}{2\omega} \frac{\lambda}{\Delta \lambda_D} \] (208)

for the pulse smearing time scale. From equation (205), one finds \( \Delta t_D \propto \lambda^{4/(2-S)} \).

As emphasized in Part I, Section C, type B spectra have very strong large turbules. This immediately suggests that a sequence of equally spaced pulses received through a type B screen, which is moving across the line of sight, will show a random variation in arrival time with a long time scale, similar to the case for the type A screen (§3). If one considers the difference in phase delay between a ray received from the point \( x_o + \frac{r}{2} \) and that from \( x_o - \frac{r}{2} \) on the screen, one finds the difference in arrival times to be:
\[ \Delta t_{Ar}(x_0, r) = \frac{1}{\omega} \left\{ \phi(x_0 + \frac{r}{2}) - \phi(x_0 - \frac{r}{2}) - r \cdot \nabla \phi(x_0) \right\} \] (209)

First note that we have \( \langle \Delta t_{Ar} \rangle = 0 \). The root mean square arrival time difference is given by equations (81) and (82). Utilizing the fact that \( \frac{r}{\omega} \approx u \cdot \tau \) for observations over a long enough time interval \( \tau \), in the case of rigid motion of the screen with velocity \( u \), one obtains,

\[ \left\langle \frac{\Delta t_{Ar}^2}{(u \cdot \tau)} \right\rangle^{\frac{1}{2}} = \frac{C_6}{\omega} \beta \left( \frac{|u \cdot \tau|}{\sqrt{\lambda D}} \right)^{\frac{5}{2}} \] (210)

where we have put \( r_o = (\lambda D)^{\frac{1}{2}} \), where \( C_6 \) is the constant defined in equation (82), and where a necessary condition on \( \lambda \) is \( q_2^{-1} < |u \cdot \tau| < q_1^{-1} \); that is, turbules of size \( q^{-1} \approx |u \cdot \tau| \) cause the arrival time difference observed in an interval \( \tau \). Formula (210) applies for \( \beta^2 \) larger and smaller than unity with different conditions on the interval \( \tau \). Independent of \( \beta^2 \leq 1 \), we need \( |u \cdot \tau| > \delta x \) (\( \delta x \) assumed larger than \( q_2^{-1} \)) for validity of geometrical optics on which (210) is based. For \( \beta^2 \gg 1 \), we need \( |u \cdot \tau| > r_s \approx (\lambda D)^{\frac{1}{2}} \beta^{-\frac{1}{2}} \) in order for \( \left\langle \Delta t_{Ar}^2 \right\rangle^{\frac{1}{2}} \) to be detectible; that is, for \( \left\langle \Delta t_{Ar}^2 \right\rangle^{\frac{1}{2}} \approx \Delta t_D \). For the range of intervals defined by \( \delta x \approx (\lambda D)^{\frac{1}{2}} \beta^{-\frac{1}{2}} \), \( |u \cdot \tau| < r_s \approx (\lambda D)^{\frac{1}{2}} \beta^{-\frac{1}{2}} \delta x \) the arrival time fluctuations are much smaller than the pulse smearing time \( \Delta t_D \); the existence of this range of \( \tau \).
longer than the scintillation time scale  \( \tau_D \sim (\lambda D)^{1/2} \beta^{-1/2} \), permits one to obtain a stable pulse shape. For \( \beta^2 < 1 \), one must wait for an interval long enough in order for \( \langle \Delta t_{Ar}^2 \rangle^{1/2} \) to be larger than the intrinsic pulse width or structure.

Observe that \( \langle \Delta t_{Ar}^2 \rangle^{1/2} \propto \lambda^2 \) has the same wavelength dependence as the ordinary dispersion caused by a uniform medium of free electrons, and that this is much weaker than the pulse smearing time \( \Delta t_D \propto \lambda^{4/(2-\delta)} \). Also, as for type A instantaneous arrival time fluctuations, the type B, after scaling thusly \( \Delta t_{Ar} \left( \frac{u}{\omega} \right) / \lambda^2 \), are independent of wavelength. Furthermore, for \( |u/\omega| \ll 1 \), it is reasonable to expect \( \Delta t_{Ar} \left( \frac{u}{\omega} \right) \) to have a Gaussian probability density.

In the \( \beta^2 \gg 1 \) limit it is convenient for comparison with observations to non-dimensionalize formula (210) by measuring \( \langle \Delta t_{Ar}^2 \rangle^{1/2} \) in units of the pulse smearing time \( \Delta t_D = \frac{1}{2cD} \Theta^2 \), and also the time interval in multiples of \( \Delta t_D \), \( \tau = m \Delta t_D \). We may also express \( \beta \) as a power of \( \omega \Delta t_D \). One obtains,

\[
\langle \Delta t_{Ar}^2 \left( \frac{u}{\omega} \right) \rangle^{1/2} / \Delta t_D = C_{11} \left[ m \left( \frac{|u|}{c} \sqrt{\frac{\lambda}{D}} \sqrt{\frac{\omega \Delta t_D}{2\pi}} \right)^{\delta} \right] \]

\( C_{11} \equiv \frac{C_6}{2^{\frac{\delta-1}{2}} \sqrt{C_I^2 + C_8^2}} \) \( \quad \) \( \text{(210a)} \)

For there to be a possibility of observing \( \langle \Delta t_{Ar}^2 \rangle^{1/2} \), the ratio on the right of (210a) should be larger than unity.
For type B irregularities we may make a list of observable parameters of the scattering and strong scintillations corresponding to the list (158) for Gaussian irregularities.

\[ \tau_D = \lambda / \sqrt{2} \pi \sigma_0 \mu_0 \]
from intensity variations at one antenna.

\[ \Delta \lambda_D = (kD \sigma_s^2)^{-1} \]
by determining the characteristic scintillation bandwidth at one antenna.

\[ \Delta t_D = \frac{1}{2c} D \sigma_s^2 \]
by observation of the widening of pulses at one antenna.

\[ \sigma_s \]
by measurement of increase in source size due to scattering, with two antennas.

\[ \mu \]
by observation of intensity variations at two antennas.

\[ \left\langle \Delta \sigma^2 (\mu \tau) \right\rangle^{1/2} \sim \tau^{5/4} \]
by observation of position variations of a source over long time intervals, at two antennas.

\[ \left\langle \Delta t^2 (\mu \tau) \right\rangle^{1/2} \sim \tau^{5/8} \]
by observation of random variation in arrival times of pulses over long time intervals, at one antenna.

The first five items above are essentially of the form of those in (158) for Gaussian spectra. The last two do not occur for Gaussian spectra, and the second to last does not occur for a type A spectrum. Thus together the last two items provide a 'footprint' of a type B spectrum.

* * *
To end this subsection let us examine mathematically the wavenumber spectrum of the type B monochromatic intensity, $M(q)$. The integral over $M(q)$, the modulation index squared, provides a handle on statistics of the intensity. For this purpose it is convenient to simplify the problem by treating the analogous one-dimensional problem of finding the intensity spectrum $M(q_x)$ given that $\overline{\Phi}_{L}^2(q_x) = 2\pi (\pi r_e)^2 L \cdot F(q_x,0) \text{ and } F(q_x,0) = k_N q_x^{-\delta - 1}$. The one-dimensional phase spectrum $\overline{\Phi}_{L}$ has a log-slope smaller by unity than that in the two dimensional case in order to make the range $4 < s < 6$ in one-dimension correspond to type B in two. The one-dimensional analogue of equation (186) is:

$$
\tilde{M}(\tilde{q}_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x} \tilde{y} e^{-i \tilde{q}_x \tilde{x}} \tilde{y} e^{-C_{12}^2 \beta^2 \tilde{E}(\tilde{x},\tilde{y})} e^{\int_{-\infty}^{\infty} d\tilde{q}_x' q_x'^{-\delta - 1} \sin^2(\frac{1}{2} \tilde{q}_x' \tilde{x}) \sin^2(\frac{1}{2} \tilde{q}_x' \tilde{y})}
$$

$$
\tilde{E}(\tilde{x},\tilde{y}) = \int_{-\infty}^{\infty} \frac{d\tilde{q}_x'}{q_x'} \sin^2(\frac{1}{2} \tilde{q}_x' \tilde{x}) \sin^2(\frac{1}{2} \tilde{q}_x' \tilde{y})
$$

$$
2 < s < 6 ; \quad \delta = (s-2)/2
$$

$$
\beta^2 \equiv 16\pi K_N (\gamma_e \lambda)^2 L \left( \frac{\lambda D}{2\pi} \right)^\delta / C_{12}^2
$$

$$
C_{12}^2 \equiv 4 \delta \left[ \Gamma(1-\delta) \cos \left( \frac{\pi}{2} \delta \right) \right]^{-1}
$$

$$
\tilde{q}_x = \sqrt{\frac{\lambda D}{2\pi}} q_x ; \quad \tilde{q}_y = \sqrt{\frac{\lambda D}{2\pi}} q_y' ; \quad \tilde{x} = \sqrt{\frac{2\pi}{\lambda D}} x
$$
For $\beta^2 \ll 1$, expansion of (211) shows that $\beta = m$ is the modulation index for weak scintillations.

Progress in evaluating (211) is made by approximating the sine-squared factors in $\tilde{E}$, for example, thusly:

$$\sin^2\left(\frac{1}{2} \tilde{q}_x' \tilde{x}\right) = \frac{1}{4} (\tilde{q}_x' \tilde{x})^2, \quad \text{for } \tilde{q}_x' \tilde{x} \leq C_{13}; \quad = \frac{1}{2} \quad \text{for } \tilde{q}_x' \tilde{x} > C_{13},$$

where $C_{13}$ is an unknown numerical factor dependent on the log-slope, but of order unity. With this approximation one finds,

$$\tilde{E}(\tilde{x}, \tilde{q}_x) = \frac{1}{8} \frac{C_{13}^{6-S}}{6-S} \tilde{x}^2 |\tilde{q}_x|^{S-4} + \frac{1}{4} \frac{C_{13}^{4-S}}{S-4} \tilde{x}^2 \left|\tilde{q}_x\right|^2 |\tilde{x}|^{S-4} + \frac{1}{2} \frac{C_{13}^{2-S}}{S-2} |\tilde{x}|^{S-2}$$

$$\tilde{E}(\tilde{x}, \tilde{q}_x) = \text{above with } \tilde{q}_x \leftrightarrow \tilde{x}$$

$$|\tilde{x}| \geq |\tilde{q}_x|$$

The approximation in (212) could be refined by use of two constants for the partition of the range of integration of $\tilde{E}$ in (211) but this would not alter our later estimate for the modulation index.

The integral in (211) over $\tilde{x}$ may now be performed by separating the range into $|\tilde{x}| \leq |\tilde{q}_x|$ and $|\tilde{x}| > |\tilde{q}_x|$. One finds that for $\beta^2 \gg 1$, the main contribution to (211) is from $|\tilde{x}| \leq |\tilde{q}_x|$. Therefore, it is useful to introduce the
variable $\zeta$ defined thusly: $\tilde{x} = \zeta |\tilde{q}_x|$. Equation (211) then becomes,

$$
\tilde{M}(\tilde{q}_x) = \frac{1}{2\pi |\tilde{q}_x|} \int_{-\infty}^{\infty} dy \ e^{-i\tilde{\zeta} \tilde{q}_x} e^{\tilde{E}'(\zeta, \tilde{q}_x)}
$$

$$
\tilde{E}'(\zeta, \tilde{q}_x) = C_{12}^2 \beta^2 |\tilde{q}_x|^s - \left( C_{14}^2 \zeta^2 + C_{15}^2 |\zeta|^s \right)
$$

$$
C_{14}^2 = \frac{1}{8} \frac{C_{13}^{6-s}}{6-s} + \frac{1}{4} \frac{C_{13}^{4-s}}{s-4}
$$

$$
C_{15}^2 = \frac{1}{2} \frac{C_{13}^{2-s}}{s-2} - \frac{1}{4} \frac{C_{13}^{4-s}}{s-4}
$$

Note that for $4 < s < 6$, $C_{14}^2 > 0$ and $C_{14}^2 + C_{15}^2 > 0$, whereas for $2 < s < 4$, $C_{15}^2 > 0$ and $C_{14}^2 + C_{15}^2 > 0$, which follow from the fact that $\tilde{E}'$ is positive definite.

The integral in (213) may be performed by noting that for a typical wavenumber $\tilde{q}_x \sim (\lambda D)^{\frac{1}{2}} r^{-1} \gg 1$, the important range of $\zeta$ is: $0 \leq \zeta^2 \ll 1$. This permits neglect of the contribution to $E$ proportional to $|\zeta|^s \ll \zeta^2$ for $s > 4$ and $\zeta^2 \ll 1$. One then obtains,

$$
\tilde{M}(\tilde{q}_x) \approx \frac{1}{2\pi |\tilde{q}_x|} \int_{-\infty}^{\infty} dy \ e^{-i\tilde{\zeta} \tilde{q}_x} e^{\tilde{C}_{12}^2 \frac{C_{14}^2}{C_{15}^2} \beta^2 |\tilde{q}_x|^s \zeta^2}
$$

(214)
The integral in (214) is trivial, and reinstating the \((2^3\pi)^{1/2}\) factors, one gets

\[
M(q_x) \approx \frac{1}{\sqrt{\pi} q_D} \left| \frac{q_D}{q_x} \right|^{\delta - 1} \exp \left\{ - \left| \frac{q_x}{q_D} \right|^{4 - 2\delta} \right\}
\]

(215)

\[
q_D \equiv \frac{1}{\sqrt{\lambda D}} \beta \frac{1}{2 - \delta} \left\{ \frac{1}{2\pi} (2 C_{12} C_{14}) \left( \frac{1}{2 - \delta} \right) \right\}
\]

Notice that \(q_D\) of (215) is in agreement with \(r_D\) of (203), \(q_D \sim r_D^{-1}\). Furthermore, the analytic form of (215) is closely similar to the type B angle density (equation 84), as it should be because of the correspondence of wavevector components of the intensity \(q\) with deflection angles \(\Theta\), by \(q \rightarrow k \Theta\). The spectrum (215) is characterized by the single wavenumber \(q_D\). This conflicts with the conjecture by Salpeter (1969) that the spectrum has two characteristic scales.

The mean-square intensity fluctuations or modulation index squared may be obtained from the integral of \(M(q_x)\) in (215), \(m^2 = \int_0^\infty dq_x M(q_x)\). The integral comes out fortunately independent of the unknown constant \(C_{12}\)

\[
m = \left\{ \frac{1}{\sqrt{\pi}} \frac{1}{2 - \delta} \sqrt{\frac{\delta}{4 - 2\delta}} \right\}^{1/2}
\]

(216)

For \(\delta = 1, 5/4, 3/2, 7/4, 2\), one finds modulation indices \(m = 1.0, 0.92, 1.0, 2.74, \infty\). That is, for \(1 < \delta < 3/2\), the index is approximately unity, whereas for \(3/2 < \delta < 2\), the index is appreciably larger than one. The latter range
of $\delta$ corresponds to greater strength of the larger turbules relative to the former range. Thus the index being larger than unity in the range $3/2 < \delta < 2$ could be due to focusing of the large turbules of size $\sim r_s$, as suggested by Salpeter (1969). With this interpretation, the log-slope range $1 < \delta < 3/2$, with index of order unity, would correspond to the action of turbules smaller than $r_s$ in spoiling the focusing by the large turbules. The range $1 < \delta < 3/2$ would then be expected to have an approximate Rayleigh amplitude distribution or equivalently an intensity probability density $W(I) \sim \exp(-I)$.

As a check on our approximation in equation (212), and thereby also a check on formula (216), we have derived the spectrum $M(q_x)$ in the above way for type A spectra ($2 < s < 4$). The resulting $M(q_x)$ is of the form of (190) and the corresponding index is exactly unity, in good agreement with the results of (§3).
§6. Type C Spectrum:

This subsection discusses qualitatively the strong scintillations arising from a type C \((s = 4\ or\ s = 1)\) irregularity spectrum.

From Part I (Di §5) we have the estimate \(\theta_s \approx \frac{\lambda}{2\pi \delta x}\) for the typical diffraction angle, and \(\delta x \approx (\lambda D)^{1/2} \beta^{-1}\) for the diffraction length assuming \(r_o = (\lambda D)^{1/2}\). For \(\beta^2 \gg 1\), the geometrical optics angular size \(\Theta_S\) of the instantaneous received radiation is larger than \(\theta_s\), but only by a logarithmically large factor, \(\sim (\ln (\beta^2))^{1/2}\). For \(\beta^2 \gg 1\), the turbules of size \(\delta x\) which determine \(\theta_s\) are much smaller than the Fresnel radius, whereas the angle \(\Theta_S\) is determined by turbules in the range \(\delta x\) to \(\Theta_S D\), sizes ranging from much smaller to much larger than the Fresnel radius.

As for type B irregularities in the \(\beta^2 \gg 1\) limit, we may estimate the scale size of the intensity pattern in the observer’s plane as \(\sim r_D \approx \frac{\gamma}{2\pi \Theta_S}\), where \(\Theta_S\) is given by equation (89). Because \(\Theta_S\) depends on wavelength slightly more strongly than \(\gamma^2\), \(r_D\) depends on \(\gamma\) a bit stronger than the inverse first power. The time scale of the intensity scintillations is estimated as usual as \(\sim \tau_D \approx r_D / |\dot{u}|\) for rigid motion of the screen across the line of sight with velocity \(\dot{u}\).

The characteristic scintillation bandwidth may be estimated as \(\Delta \lambda_D \sim (k D \Theta_S^2)^{-1}\), and the pulse smearing time scale as \(\Delta t_D \sim \frac{1}{2c} D \Theta_S^2\).
In addition to the pulse smearing, there are random variations in the pulse arrival times larger than $\Delta t_D$ over long time intervals. A formula describing the arrival time fluctuations may easily be derived following the discussion prior to equation (198) or (210). The behaviour of the type C fluctuations are qualitatively similar to those for type A and B.

We may obtain a semi-quantitative expression for the one-dimensional type C monochromatic intensity wavenumber spectrum $\tilde{M}(q_x)$ for $\beta^2 \gg 1$, by following the calculation of $M(q_x)$ done in detail for type B in subsection (5). The integral of the resulting $\tilde{M}(q_x)$ provides some information on the type C intensity statistics. Taking the $s \to 4$ limit of equation (213) one finds,

\[
\tilde{M}(\tilde{q}_x) = \frac{1}{2\pi} \left| \tilde{q}_x \right| \int_{-1}^{1} d\xi e^{-i\xi \tilde{q}_x^2} e^{-\tilde{E}'(\xi, \tilde{q}_x)}
\]

\[
\tilde{E}'(\xi, \tilde{q}_x) = \frac{c_{12}^2 \beta^2 \xi^2 \tilde{q}_x^2}{16} \left\{ C_{16}^2 \left[ \frac{1}{4} \ln \left| \xi \right| \right] \right\}
\]

\[
\beta = \sqrt{\pi} K_N^{1/2} r_e^{3/2} \lambda^{1/2} L^{1/2} D^{1/2}
\]

\[
C_{16}^2 = \frac{1}{16} C_{13}^2 + \frac{1}{4} C_{13}^{-2} ; \quad C_{12}^2 = \frac{8}{\pi}
\]
An approximate, but accurate, evaluation of the integral in (217) may be obtained by transforming to the variable of integration \( y = \xi / \beta^2 \). This makes the logarithmic contribution to \( \tilde{E}' \) in (217) small in the important range of \( y \). After reinstating the \( (2\pi / \lambda D)^{\frac{1}{2}} \) factors one finds,

\[
M(q_x) = \frac{1}{\sqrt{\pi} q_D} \exp \left\{ - \frac{q_x^2}{q_D^2} \right\} \\
q_D \equiv \frac{1}{\sqrt{\lambda D}} \beta \left\{ 2 \sqrt{2\pi} C_{12} \lambda h \right\} \\
h^2 \equiv C_{16}^2 + \frac{i}{4} \ln(e(\beta^2))
\]

(218)

Thus the type C wavenumber spectrum is a Gaussian. Interestingly this is the form of the wavenumber spectrum for the case of Gaussian irregularities, as given below equation (167).

The width of the type C spectrum depends on \( \lambda \) slightly more strongly than the first power because of the \( \ln_e(\beta^2) \) term in \( h^2 \). This agrees with the prior conclusion that \( r_D \) (\( \sim q_D^{-1} \)) varies slightly more strongly than the inverse first power of wavelength.

The type C modulation index \( m \) is given by the integral of \( M(q_x) \) in (218), \( m^2 = \int_{-\infty}^{\infty} dq_x M(q_x) \). It is exactly unity. This suggests that the focusing action of turbules with sizes \( \sim D \delta_s \) is completely spoiled by the phase shifts introduced by small turbules of sizes \( \sim \delta_x \). If this interpretation is correct, we may expect the type C intensity to have an expo-
nential probability density \( W(I) \approx \exp(-I) \).
§7. Departures from Frozen Flow:

Here we consider situations where it is necessary to describe the density fluctuations by their full frequency-wavenumber spectrum \( F(q_x, q_y, q_z; \Omega) \), defined by equation (15), in the reference frame moving with the average velocity of the medium \( \bar{u} \) (transverse to the line of sight). In this fluid-fixed frame one may derive a relation analogous to formula (43) connecting phase and density spectra. Defined in this frame is the auto-correlation function of the phase,

\[
\phi_L^2 \rho_L(r_x, r_y; \tau) \equiv \left\langle \phi(x, y, L; t) \phi(x + r_x, y + r_y, t + \tau) \right\rangle
\]

and

\[
\overline{\Phi}_L(q_x, q_y; \Omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d \tau \int_{-\infty}^{\infty} e^{-i(q_x r_x + q_y r_y - \Omega \tau)} \phi_L^2 \rho_L(r_x, r_y; \tau)
\]

where \( \phi(x, y, L; t) \) is the thin screen phase. Under the plausible assumption that the only significant contributions to the spectrum \( F(q, \Omega) \) are from components with \( |\Omega/\bar{u}| \ll c \), one obtains,

\[
\overline{\Phi}_L(q_x, q_y; \Omega) = 2\pi (\lambda c)^2 L \int F(q_x, q_y, q_z = 0; \Omega)
\]

The z-axis is the wave propagation direction as usual.

The intensity received in a plane a distance \( D \) from the screen is denoted \( I_u(x, t) \) in the x-y coordinate system moving with the velocity \( \bar{u} \). For completely frozen flow \( I_u(x, t) \) would be independent of time. As before,
\( I(\vec{x}, t) \) denotes the intensity on the ground at the earth. Let \( M_u(\vec{x}, \tau) \), \( M_u(\vec{q}, \tau) \), and \( M_u(\vec{q}, \Omega) \) denote the space-time auto-correlation, wavenumber-time auto-correlation, and wavenumber-frequency spectrum of the intensity \( I_u(\vec{x}, t) \). Then there are the relations,

\[
M_u(\vec{r}, \tau) = \left< I_u(\vec{x}, t) I_u(\vec{x} + \vec{r}, t + \tau) \right> - 1
\]

\[
M_u(q, \tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\vec{r} e^{-i\vec{q} \cdot \vec{r}} M_u(\vec{r}, \tau)
\]

\[
M_u(q, \Omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\vec{r} d\tau e^{-i(\vec{q} \cdot \vec{r} - \Omega \tau)} M_u(\vec{r}, \tau)
\]

\[
M_u(q, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{i\Omega \tau} M_u(q, \tau)
\]

The relation between the intensity \( I(\vec{x}, t) \) at an antenna on the earth and \( I_u(\vec{x}, t) \) is simply \( I(\vec{x}, t) = I_u(\vec{x} - ut, t) \). Therefore, the time auto-correlation of intensity at the earth, \( A(\tau) = \left< I(\vec{x}, t) I(\vec{x}, t + \tau) \right> - 1 \), is given by

\[
A(\tau) = M_u(u \tau, \tau)
\]

From equation (221) there is the relation,

\[
A(\tau) = \int_{-\infty}^{\infty} dq e^{i\vec{q} \cdot \vec{u} \tau} M_u(q, \tau)
\]

which is useful later.
In the fluid-fixed frame the amplitude $A_u(\tilde{x}, t)$ in the observer's plane is given by

$$A_u(\tilde{x}, t) = \frac{k}{\alpha \pi D} \int_{-\infty}^{\infty} dx' e^{-\frac{i k}{\alpha D} (\tilde{x} - x')^2} e^{i \phi(x', t')},$$

$$t' = t - \frac{D}{c} - \frac{1}{2 D c} (\tilde{x} - x')^2; \quad \phi(x', t') = \phi(x', y, L; t')$$

where $t'$ is the retarded time. The intensity $I_u(\tilde{x}, t) = |A_u(\tilde{x}, t)|^2$. For practical use of equation (224), we want the conditions under which $t' \approx t - D/c$. An obvious necessary condition is that for frequencies $\Omega$ of components of significant strength in $F(q, \Omega)$, $\Omega/\omega \ll 1$.

The sufficient conditions are of greater interest, and even these impose only weak conditions on $F(q, \Omega)$ and the scintillation parameters in most (but not all) situations.

Observe that $|t' - t + D/c| \approx \Delta t_D$, where $\Delta t_D$ is the pulse smearing time. $\Delta t_D \approx \frac{1}{\alpha c} D \Theta^2_s$ or $\frac{1}{2c} D \Theta_s^2$ appropriate to the actual kind of irregularities. A sufficient condition for $t' \approx t - D/c$ to be a valid approximation is that the phase on the screen $\phi(\tilde{x}, t)$, over the area $\mathcal{L}$ by $\mathcal{L}$ which contributes to the received wave, not vary by more than say a fraction $\epsilon$ of a radian in a pulse smearing time $\Delta t_D$. This condition is applied to all irregularity sizes $r$. For $r$ large enough so that the phase difference of equation (56) $\rho_{L_2}(r) > 1$, the condition takes the form
\[ \epsilon \frac{1}{\psi_L^2}(r), \nu(r) \ll \Delta t_D, \] where \( \nu(r) \) is the turbulent velocity of irregularities of size \( r \). For small \( r \) such that \( \psi_L^2(r) = \epsilon \ll 1 \), the condition is \( r/\nu(r) \ll \Delta t_D \).

As an illustration of the sufficient condition for \( t' = t - D/c \), assume type A irregularities and scintillations in the Rayleigh limit, and a turbulent velocity \( \nu(r) \) independent of irregularity size. Then the strongest limitation is on irregularities of diffraction length size \( \delta x \), where \( \psi_L(\delta x) = 1 \).

The explicit condition one finds is \( \Theta_5 < \epsilon \psi_3(\psi_5 \lambda)^{1/3} \). This requirement on \( \Theta_5 \) is related to that on \( \zeta_D \) (equation 157), which is a limit on the rapidity of observable intensity variations.)

In what follows we assume \( t' = t - D/c \). This allows evaluation of the intensity auto-correlation \( M_u(\xi, \zeta) \) from (224) exactly as done in detail in subsection (\( \xi_2 \)) for \( \zeta = 0 \). The replacement of \( H(\xi_1, \xi_2, \xi_3, \xi_4) \) in equation (180) is:

\[
H(\xi_1, \xi_2, \xi_3, \xi_4; \zeta) = 2\psi_2^2 \left[ 2 - \psi_L(\xi_2 - \xi_1; 0) - \psi_L(\xi_4 - \xi_3; 0) + \psi_L(\xi_4 - \xi_2; \zeta) - \psi_L(\xi_4 - \xi_1; \zeta) + \psi_L(\xi_3 - \xi_1; \zeta) - \psi_L(\xi_3 - \xi_2; \zeta) \right]
\]

Utilizing Pisareva's transformation (\( \xi_2 \)), the eight dimensional integral for \( M_u(\xi, \zeta) \) may be reduced to four. The resulting expression corresponds to equation (182). Then by
using the second Fourier transform identity of (221), one finds,

\[
M_u(q, \tau) = \left( \frac{1}{2\pi} \right)^2 \left\{ \int_{-\infty}^{\infty} dx \ e^{-i q \cdot x} \ e^{-E(x, \frac{\lambda D}{2\pi} q; \tau)} \right\}
\]

\[
E(x, \frac{\lambda D}{2\pi} q; \tau) = 8 \int_{-\infty}^{\infty} dq' d\Omega \ \Phi_L(q', \Omega) \sin^2 \left( \frac{\lambda D}{2\pi} q' q \right) \]

* \frac{1}{\tau} \left[ q' \cdot x - \lambda \tau \right] \]

This corresponds to equation (184); in fact, for \( \tau = 0 \), equation (226) is identical to (184). Expression (226) along with (222) provides a mathematical basis for investigating effects of non-frozen flow on scintillations.

First observe the rather obvious fact that the modulation index \( m \) observable at an antenna on the earth is not affected at all by non-frozen flow. The index is given by

\[
m^2 = A(\tau = 0) = M_u(\tau = 0, \tau = 0) \]

This is the same as the instantaneous index \( m^2 = M(\tau = 0, \tau = 0) \).

Consider next weak scintillations where we may expand \( \exp(-E) \approx 1 - E \). Then performing the trivial integrations in (226) one gets,

\[
M_u(q, \tau) = 4 \sin^2 \left( \frac{\lambda D}{2\pi} q^2 \right) \int_{-\infty}^{\infty} d\Omega \ \Phi_L(q, \Omega) e^{-i \Omega \tau} \]

which is the analogue of formula (187). From equation (222), one may get the intensity time auto-correlation at an antenna fixed on the earth,
\[
A(\tau) = 4 \iint_{-\infty}^{\infty} dq \, d\Omega \, \bar{\Phi}_L(q, \Omega) \sin^2 \left( \frac{\lambda D q_2^2}{4 \pi} \right) e^{i(q \cdot u - \Omega \tau)} \tag{228}
\]

The intensity cross-correlation between two antennas separated by a transverse distance \( b \) on the ground is: \( B(\tau) = M_u(b + u \tau, \tau) \). Hence,

\[
B(\tau) = 4 \iint_{-\infty}^{\infty} dq_1 \, dq_2 \, d\Omega \, \bar{\Phi}_L(q_1, \Omega) \sin^2 \left( \frac{\lambda D q_2^2}{4 \pi} \right) *
\]

\[
* e^{i(b + u \tau) \cdot q} e^{-i\Omega \tau} \tag{229}
\]

Consider spectra \( \bar{\Phi}_L(q, \Omega) \) such that for fixed \( q \) there is a range of frequencies \( \Omega \leq V(q)|q| \), where \( V(q) \) is the phase velocity of wavevector component \( q \). Then for \( V(q) \ll u \), formulae (228) and (229) may be approximated by setting \( \exp(i(q \cdot u - \Omega \tau)) = \exp(i(q \cdot u - V(q)|q|)\tau) \)

\[
\approx \exp(iq \cdot u \tau) \), which significant \( \tau \) dependence comes from the spatial fluctuations. This is termed the frozen-flow approximation.

For an application of equation (228), assume an isotropic spectrum of non-dispersive waves; for example, sound waves. Then in the fluid-fixed frame \( \bar{\Phi}_L(q, \Omega) = \frac{1}{2} g(q) * 
\]

\[
* \left( \delta(\Omega - V q) + \delta(\Omega + V q) \right), \text{ where } q = (q_1^2 + q_2^2)^{1/2}, \text{ where } g(q) \]

is the wavenumber spectrum, and where \( V \) is the constant wave velocity. For this spectrum the integrations over \( \Omega \) and
the direction of \( q \) in (228) are trivial and one finds,

\[
A(\tau) = 2\pi \int_0^\infty q dq \, J_0(q u \tau) \cos(q v \tau) \sin^2\left(\frac{2D}{4\pi} q^2\right) g(q)
\] (230)

In (230) the departure from frozen flow is contained entirely in the \( \cos(q v \tau) \) kernel in the integrand. We may expand \( \cos(q v \tau) = J_0(q v \tau) - 2 J_2(q v \tau) + \ldots \), and need retain only the \( J_0 \) term in (230) for \( v^2/u^2 \ll 1 \). Then the Bessel transform of \( A(\tau) \) of Section A (31) gives,

\[
M_B(f) = \frac{1}{2\pi} \int_0^\infty \tau d\tau \int_0^\infty J_0(2\pi f \tau) A(\tau)
\]

\[
M_B(f) = 4 \int_0^\infty q dq \, K\left(\frac{2\pi f}{u}; q\right) g(q) \sin^2\left(\frac{2D}{4\pi} q^2\right)
\]

\[
K\left(\frac{2\pi f}{u}; q\right) = \int_0^\infty \tau d\tau \int_0^\infty J_0(2\pi f \tau) J_0(q u \tau) J_0(q v \tau)
\]

\[
K = \left\{ \pi q^2 u V \left[ 1 - \frac{1}{4q^2 u^2 V^2} (2\pi f)^2 q^2(u^2 + V^2)^2 \right]^2 \right\}^{-1}
\]

where \( K \), which may be termed the smearing function, is non-zero only in the range \((2\pi f/u)(1 + V/u)^{-1} < q < (2\pi f/u)^* \) \* \((1 - V/u)^{-1} \). The especially interesting feature of \( K \) is that it has sharp, but integrable, singularities at \( q = (2\pi f/u)(1 \pm V/u)^{-1} \), which correspond to two effective screen velocities, \( u \pm V \). This effect is explained by the greater likelihood of these velocities in \( |u + V| \) for an isotropic
distribution of $V$. Observe that the smearing by $K$ is symmetrical and of quite a different form than the geometrical smearing described by equation (106).

For small enough $V/u \ll 1$, $K(2\pi f/u; q) \simeq \frac{1}{u^2} \delta(q - 2\pi f/u)$, and then equation (231) is the same as (104). The necessary condition on $V/u$ for there to be even a possibility for observing nulls of the sine-squared function in the Bessel spectrum $M_B(f)$ is that the smearing by $K$ in (231), of frequency width $= (2V/u)f$, be less than the frequency difference between the second and first nulls, $(2^{1/2}-1)f_F$, where $f_F = u(\lambda D)^{-1/2}$. Hence the weakest condition is: $V/u \leq 0.2$. If our observations of nulls in the Bessel spectrum at $\theta = 22.5^0$ are valid as interpreted in Section A (23), then $V < 80$ km/sec, in that the nulls indicate $u \approx 390$ km/sec. The case for believing $V/u \ll 1$ for the interplanetary medium is strongly supported by Dennison and Hewish's (1967) measurements of $B(\gamma)$, which indicate that the lifetime of the weak scintillation diffraction pattern is longer than the e-folding lag of the intensity time auto-correlation ($= 0.6$ sec).

Investigation of the effects of non-frozen flow on strong scintillations requires consideration of the different kinds of irregularities. As yet we have not completed analysis of all cases, particularly that of type B irregularities. However, for either Gaussian or type A spectra in the Rayleigh limit, one may obtain from (226) the result,
\[
M_u(x, \gamma) = \exp \left[ - \varphi^2_L(x, \gamma) \right]
\]

where \( \varphi^2_L(x, \gamma) \) is the time lagged r.m.s. phase-difference.

The condition in order for \( A(\gamma) \) and \( B(\gamma) \) to satisfy the frozen flow approximation is not significantly stronger than that for the case of weak scintillations, \( \Omega/|q| \sim V \ll |u| \).

For example, consider the Rayleigh limit of type A irregularities for the spectrum \( \overline{\Phi}_L(q, \Omega) = \frac{1}{2} g(q) \times \delta(\Omega - Vq) + \delta(\Omega + Vq) \) with \( g(q) \approx q^{-s} \), \( q = (q_x^2 + q_y^2)^{1/2} \), and \( V = \text{constant} \). Then the integration over \( \Omega \) and the direction of \( q \) in the phase difference is trivial. The intensity time auto-correlation, \( A(\gamma) = M_u(x, \gamma, \gamma) \), is found to be given by,

\[
A(\gamma) = \exp \left[ - C_1 \frac{h^2 \beta^2}{(\frac{|u\gamma|}{\sqrt{\lambda D}})^2} \right]
\]

\[
h^2(v \gamma) = \frac{\int_0^\infty \xi^{-\delta-1} \left[ 1 - \cos \left( \frac{v \gamma}{u} \xi \right) J_0(\xi) \right] d\xi}{\int_0^\infty \xi^{-\delta-1} \left[ 1 - J_0(\xi) \right] d\xi}
\]

where \( \beta \) and \( C_1 \) are defined as before in equation (186).

For \( V/u = 0, \ h^2 = 1, \) and \( A(\gamma) \) is identical to equation (189) with \( |x| = |u\gamma| \). For finite \( V/u < 1, \ h^2 \) is larger than unity. Therefore, finite \( V/u \) will give an
e-folding lag of the intensity time auto-correlation $A(\tau)$ smaller than the e-folding lag of the instantaneous spatial auto-correlation $M(r, \tau = 0)$ with $|\mu| = |\mu_r|$, by a factor of $h^{-1/6}$. This corresponds to a widening of the spectrum of $A(\tau)$ by the non-frozen fluid motion. In the extreme limit $V^2/u^2 \gg 1$, one finds from (233),

$$
\begin{align*}
\lambda^2 &= C^{-2} C_{17} \left( \frac{V}{u} \right)^2 \delta^2 \\
A(\tau) &= \exp \left\{ -C^{-2} \beta^2 \left( \frac{|V\tau|}{\sqrt{\lambda D}} \right)^2 \delta \right\} \\
C_{17} &= C_1^{-2} \lambda^{2\delta-1} \frac{\Gamma(2\delta) \Gamma(1+\delta) \cos(\pi\delta)}{\Gamma(1-\delta) \Gamma(1-2\delta)} ; \ 0 < \delta < 1
\end{align*}
$$

In this limit no dependence of $A(\tau)$ on the average velocity $\mu$ remains. However, it is interesting that for the model spectrum assumed, the analytic form of $A(\tau)$ is independent of $V/u$.

We may also obtain the cross intensity correlation, $B(\tau) = M_\mu(\mu + \mu_r, \tau)$, between two antennas on the earth for the case of type A irregularities in the Rayleigh limit. From (232) and for the same spectrum assumed for (233), one finds,

$$
\begin{align*}
B(\tau) &= \exp \left\{ -C^{-2} \left( \frac{|V\tau|}{\sqrt{\lambda D}} \right)^2 \delta \right\} \\
\lambda^2 &= \lambda^2 \left( \frac{|V\tau|}{|b + \mu_r|} \right) ; \lambda^2 \text{ given in (233)}.
\end{align*}
$$
For \((V\gamma)^2 \ll (\bar{w} + u\gamma)^2\), \(h^2 \equiv 1\) and thus measurement of the displacement from zero time lag of the centroid of \(B(\gamma)\) in (235) allows derivation of \(u\). The first observations of \(B(\gamma)\) for interplanetary scintillations with derivation of \(u\) were described by Hewish, Dennison, and Pilkington (1966).

A number of cases of (235) may be distinguished for different limits on the maximum of the argument of \(h^2\), of value \((V/u)(\sin \gamma)^{-1}\), where \(\gamma\) is the angle between \(\bar{w}\) and \(u\). If \((V/u)^2 \ll (\sin \gamma)^2\), we have \(h^2 \equiv 1\) for all \(\gamma\) in formula (235) and the maximum correlation occurs at a lag \(\gamma_m = -(b/u)\cos(\gamma)\), where \(b = (b_x^2 + b_y^2)^{1/2}\). However, for \(1 > (V/u)^2 > (\sin \gamma)^2\), the \(h^2\) function may be appreciably larger than unity for a range of lags. The range of lags where the argument of \(h^2\) is larger than unity is bounded by \(\gamma_{\pm} = (b/u)(1 - V^2/u^2)^{-1}(\cos(\gamma) \pm (V^2/u^2 - \sin^2(\gamma))^{1/2})\).

Inside this range of lags, which includes \(\gamma_m = -(b/u)\cos(\gamma)\), \(B(\gamma)\) is corrupted by the non-frozen flow, and is smaller than it would be for \(V/u = 0\). The lag of the peak correlation requires evaluation of (235); however, the 'wings' or large \(\gamma\) part of \(B(\gamma)\) is unaffected by the finite \(V^2/u^2\) and may still be used for determining \(u\). For \(V^2/u^2 > 1\), the corruption by the dependence of \(h^2\) covers an even larger range of lags. For \(V^2/u^2 \gg 1\), \(B(\gamma)\) is independent of \(u\); for \(|\gamma| < b/V\), the correlation is roughly constant, \(B \sim \exp(-c_1^2 \beta^2 (b/\sqrt{AD})^{3\delta})\), whereas for \(|\gamma| > b/V\), \(B \sim \exp(-c_{17}^2 \beta^2 (|\gamma|/\sqrt{AD})^{3\delta})\).
PART III

Geometrical and Wave Optics
of the Thick Screen
Introduction to Part III:

Part III is in three sections. Section A is a general development of geometrical optics applied to ray motion in randomly irregular media. Section B, utilizing results of A, develops a heuristic theory for wave propagation in and through a thick screen. Section C describes the wave-optics of a multiple thin screen model for a thick screen. The development in C, although independent, is complementary to that in B.
A. Geometrical Optics within the Slab:

In this Section geometrical optics equations describing rays and wavefronts in random media are derived. The eventual purpose is to identify the main parameters for wave propagation in a thick medium, which are related to $\lambda$ and $\epsilon$ introduced earlier in Part I, Section B. Several statistical quantities are evaluated: the mean-square angle of a ray to the z-axis, the mean-square transverse displacement of a ray, and the mean-square phase delay accumulated along a ray. These are expressed in terms of the phase shift $\phi_z$, the radio wavelength $\lambda$, the scale size of the irregularities $a$, and the propagation distance $z$. The derivations below are initially given without criteria for the validity of geometrical optics. The criteria are discussed in (§4). For simplicity the statistical calculations are done assuming Gaussian irregularities or spectra.

In the eikonal equation (27) let $\psi(x,y,z) = \text{constant}$ represent a wave surface or front. Rays may be introduced as a vector field everywhere normal to the different wave surfaces; that is, the local ray direction is that of $\nabla \psi$. A 'single' ray is described by a three dimensional curve $X(s)$, where $s$ is a parameter, analogous to the arc-length of the curve, but reduced by the refractive index. $s$ is termed the inverse optical path length, and is defined by,
\[ ds^2 = \left( dX(s) \right)^2 \left/ \left( 1 + \mu(x) \right)^2 \right. \]  \hspace{1cm} (236)

The optical path length is simply \( \psi^2 \). Expressed in differential form it is given by,

\[ d\psi^2 = \left( dX(s) \right)^2 \left( 1 + \mu(x) \right)^2 \]  \hspace{1cm} (237)

The tangent to the ray-curve may be chosen to be

\[ \frac{d}{ds} X(s) = \frac{i}{k} \nabla \psi \]  \hspace{1cm} (238)

Then (238) satisfies the eikonal equation (27). By operating on equation (27) with \( \nabla \) one may obtain

\[ \frac{d^2}{ds^2} X(s) = \nabla \frac{1}{2} \left( 1 + \mu(x) \right)^2 \]  \hspace{1cm} (239)

which is the ray-equation. Equation (239) is a close relative of the particle equations of motion of classical mechanics. This analogy is of practical use later in Section B (§ 4). A Hamiltonian describing the ray-dynamics is:

\[ \mathcal{H} = \frac{1}{2} \mathcal{P}^2 - \frac{1}{2} \left( 1 + \mu \right)^2 ; \quad \mathcal{P} = \frac{d}{ds} X(s) \]  \hspace{1cm} (240)

where \( \mathcal{P} \) is the momentum canonically conjugate to \( X \). Parameterizing the ray-curve \( X \) with distance \( z \) rather than \( s \) gives,
\[ \mathcal{H} = \frac{1}{\hbar} p_x^2 p_y^2 - \mu - \frac{i}{\hbar} \mu^2 \]

\[ p_z = 1 + \int_0^z d\zeta' \sqrt{1 + p_z^2(z')} \frac{\partial}{\partial z'} \mu[x(z'), z'] \]

\{ (241) \}

where \( \mathbf{x} = (x_x, x_y, z) \), \( \mathbf{x} = (x_x, x_y) \), and \( \mathbf{p} = (p_x, p_y) \).

In actual wave propagation problems of interest, \( \mu \) is extremely small suggesting for a first approximation an iteration of the Hamiltonian (241) in powers of \( \mu \). Thus discard the term \( \frac{1}{2} \mu^2 \) and set \( p_z = 1 \). Then the ray equation is:

\[ \frac{d^2}{dz^2} x(z) = \nabla \mu [x(z), z] \]

\{ (242) \}

We assume the boundary conditions:

\( x(0) = 0 \)

\( \frac{d}{dz} x(0) = 0 \)
§1. Ray Angle:

The first integral of the ray equation (242) is the angle of the ray to the z-axis,

\[ \Theta(z) = \frac{d}{dz} \Xi(z) = \int_0^z d\xi' \nabla \mu \left[ \Xi(z'), z' \right] \] (243)

The second boundary condition assumed on (242) indicates that \( \Theta(z = 0) = 0 \). That is, the wave is incident normally on the slab. At any distance \( z > 0 \) the average \( \langle \Theta \rangle \) is zero by symmetry or because \( \langle \mu \rangle = 0 \). The mean-square angle is obtained from the expression,

\[ \langle \Theta^2(z) \rangle = \int_0^z dz' dz'' \langle \nabla \mu \left[ \Xi(z'), z' \right] \cdot \nabla \mu \left[ \Xi(z''), z'' \right] \rangle \] (244)

The average in (244) is carried out by considering first \( |z' - z''| > a \) where the values of \( \Xi(z') \) and \( \Xi(z'') \) are effectively independent of the refractive index \( \mu \) at the points \( \Xi(z'), z' \) and \( \Xi(z''), z'' \). For this limit the correlation is zero because the range over which \( \mu \) and its derivatives are correlated is \( \sim a \). For \( |z' - z''| < a \), it is permissible as verified later to assume \( |\Xi(z') - \Xi(z'')| \ll |z' - z''| \). Thus equation (244) may be expressed as,

\[ \langle \Theta^2(z) \rangle = -\int_0^z dz' dz'' \nabla^2 \mu \left[ \Xi, z' \right] \mu \left[ 0, z'' \right] \]

\[ = -\mu^2 \int_0^z dz' dz'' \left[ \nabla^2 \Theta \left[ \Xi, z'-z'' \right] \right]_{z=0} \] (245)
The transverse gradient operator acting on the second $\mu$ factor has been shifted onto the first, as in integration by parts, which is justified by the fact that statistical averages are assumed equivalent to spatial averages. A further simplification is obtained by assuming that $\zeta_N$ is isotropic in three-dimensions. Then,

$$\left\langle \hat{\omega}^2(z) \right\rangle = -4 \mu^2 \int_0^z dz' (z-z') \frac{1}{z'} \frac{2}{\partial z'} \zeta_N(z') \quad (246)$$

Evaluating (246) with the standard Gaussian $\zeta_N = \exp(-z^2/2a^2)$,

$$\left\langle \hat{\omega}^2(z) \right\rangle = 4 \mu^2 a^{-2} \int_0^z dz' (z-z') \zeta_N(z') \quad (247)$$

or

$$\left\langle \hat{\omega}^2(z) \right\rangle \frac{1}{2} = \frac{\lambda}{\sqrt{2} \pi a} \phi_z \quad (248)$$

where equation (39) for $\phi_z$ has been used. Note that the r.m.s. angle (248) is identical to the r.m.s. scattering angle of equation (65) for the thin screen in the $\phi_z \gg 1$ limit.

The probability density for $\hat{\omega}$ is obtained by considering higher order moments. First note that third order moments such as $\hat{\omega} \hat{\omega}^2$ involve an average of three $\mu$ factors. For a non-zero third moment, each of the three arguments of the $\mu$ factors must not be much more than a few correlation lengths $\alpha$ from another argument. This allows one to estimate the magnitude of third moments as

$$\left\langle \hat{\omega} \hat{\omega}^2 \right\rangle \frac{1}{3} \sim \left\langle \hat{\omega}^2 \right\rangle (a/z)^{1/6}.$$ 

Therefore, for $z \gg a$, third and
and other odd moments may be neglected. The fourth order moment of interest is:

\[
\langle \tilde{\theta}^4 \rangle = \int_0^Z \int_0^Z \langle \nabla_1 \mu(1) \cdot \nabla_2 \mu(2) \cdot \nabla_3 \mu(3) \cdot \nabla_4 \mu(4) \rangle \tag{249}
\]

where \( \mu(1) = \overline{\mu[\chi(z_1), z_1]} \), etc. Unlike the case for third moments, a large \( \sim \langle \tilde{\theta}^2 \rangle^2 \), contribution to \( \langle \tilde{\theta}^4 \rangle \) may be obtained in the \( z \gg \) limit, by approximating the average in (249) by the sum of the separate averages of all possible pairs of \( \mu \) factors:

\[
\langle \tilde{\theta}^4 \rangle = \int_0^Z \int_0^Z \langle \nabla_1 \mu(1) \cdot \nabla_2 \mu(2) \cdot \nabla_3 \mu(3) \cdot \nabla_4 \mu(4) \rangle \left\{ \langle \mu(1) \mu(2) \rangle \langle \mu(3) \mu(4) \rangle + \langle \mu(1) \mu(3) \rangle \langle \mu(2) \mu(4) \rangle + \langle \mu(1) \mu(4) \rangle \langle \mu(2) \mu(3) \rangle \right\}
\]

\[
= 4z^2 \overline{\mu^2}^2 \left\{ \frac{\nabla_1^2 \nabla_3^2 + 2(\nabla_1 \cdot \nabla_3)^2}{\nabla_1^2} \right\} \int_0^Z \int_0^Z \langle \mu(1) \rangle \langle \mu(3) \rangle \tag{250}
\]

It is not necessary to go to higher order moments to obtain the \( \tilde{\theta} \) probability density. If the previous assumptions on \( \mu \) averages is continued, \( \tilde{\theta} \) is automatically a Gaussian random variable, as described by the density,
\[ W(\Theta) = (2\pi \Theta_s^2)^{-1} \exp\left\{ -\frac{\Theta^2}{2 \Theta_s^2} \right\} \]
\[
\int_{-\infty}^{\infty} d\Theta_x \int_{-\infty}^{\infty} d\Theta_y \; W(\Theta_x, \Theta_y) = 1
\]

where \( \langle \Theta^2 \rangle_{\Theta_s} = \sqrt{2} \Theta_s \)
§2. Ray Displacement:

The second integral of the ray equation (242) is the ray coordinate,

\[
\tilde{X}(z) = \int_0^z d\xi (z-\xi) \nabla \mu \left[ \tilde{X}(\xi), \xi \right] \tag{252}
\]

Because of the first boundary condition on (242), \( \tilde{X}(z) \) has average value of zero. Following the methods used in evaluating \( \langle \tilde{\beta}^2 \rangle \), the mean-square of the transverse ray displacement is easily obtained as,

\[
\langle \tilde{X}^2(z) \rangle = \int_0^z d\xi (z-\xi) (z-\xi) \nabla \mu \left[ \tilde{X}(\xi), \xi \right] \cdot \nabla \mu \left[ \tilde{X}(\xi), \xi \right] = -2 \mu^2 \int_0^z d\xi (z-\xi)^2 \frac{1}{\xi} \frac{\partial}{\partial \xi} \mathcal{S}_N(\xi) \tag{253}
\]

Consider the limit of interest \( z \gg a \). Then except for a negligible contribution from two small regions \( 0 \leq \xi \ll a \) and \( z \gg z' \gg z' - a \), the limits of integration for the second integral in (253) are effectively \( \pm \infty \). Hence

\[
\langle \tilde{X}^2(z) \rangle = -2 \mu^2 \int_0^z d\xi (z-\xi)^2 \int_{-\infty}^{\infty} d\xi \frac{1}{\xi} \frac{\partial}{\partial \xi} \mathcal{S}_N(\xi) = -4 \mu^2 \int_0^z \frac{d\xi}{\xi} \frac{\partial}{\partial \xi} \mathcal{S}_N(\xi) \tag{254}
\]

Evaluating (254) for \( \mathcal{S}_N = \exp(-\xi^2/2a^2) \) gives
\[ \langle X^2(z) \rangle = \frac{2}{3} \left( \frac{\lambda Z}{2\pi a^2} \right)^2 \phi^2 \alpha^2 \] (255)

Because the angle \( \theta \) is assumed much smaller than a radian, there is a simple geometrical relation between the ray displacement \( X \) and \( \theta \), which is \( \langle X^2 \rangle^{1/2} \sim z \langle \theta^2 \rangle^{1/2} \). This is illustrated in Figure (18).

Following arguments used to obtain the \( \theta \) probability density, one finds for the \( X \) density,

\[
\mathcal{W}(X) = (2\pi X_0^2)^{-1} \exp \left\{ -\frac{X^2}{2X_0^2} \right\} \left\{ \int_{-\infty}^{\infty} dX_x dX_y \mathcal{W}(X_x, X_y) \equiv 1 \right\} \] (256)

where \( \langle X^2 \rangle^{1/2} = 2^{1/2} X_0 \).
FIGURE 18: Ray Meanderings.
§3. Ray Phase:

The optical path length \( \psi \) of equation (237) or phase of equation (20) may be integrated and thereby expressed as,

\[
\phi = \phi_1 + \phi_2
\]

\[
\phi_1 = k \int_0^z dz' \mu \left[ X(z'), z' \right]
\]

\[
\phi_2 = \frac{1}{2} k \int_0^z dz' \left( \mu X(z') \right)^2
\]

(257)

The ray phase differs from that for the thin screen (equation 21) first because the refractive index variations added into \( \phi_1 \) are those encountered along a ray path, instead of the fluctuations along a straight line through the medium, and secondly because of the additional term \( \phi_2 \), which is the phase delay arising from the geometrical path length.

The mean-square phase consists of three terms \( \langle \phi^2 \rangle = \langle \phi_1^2 \rangle + 2 \langle \phi_1 \phi_2 \rangle + \langle \phi_2^2 \rangle \). First note that \( \langle \phi_1 \rangle = 0 \).

The mean-square of \( \phi_1 \) is given by,

\[
\langle \phi_1^2 \rangle = k^2 \int_0^z dz' \int_0^z dz'' \mu \left[ X(z'), z' \right] \mu \left[ X(z''), z'' \right]
\]

\[
\langle \phi_1 \rangle = 2 k^2 \mu^2 \int_0^z dz' \phi (z')
\]

\[
\langle \phi_1 \rangle^{1/2} = \phi_z
\]

(258)

The phase \( \phi_1 \) in the respect of its r.m.s. and also higher moments is the same as the thin screen phase.
The cross term \( \langle \varphi_1 \varphi_2 \rangle \) is assumed zero because the average involves an odd number, three, \( \mu \) factors. Before examining the term \( \langle \varphi_2^2 \rangle \), note that the path length phase \( \varphi_2 \) is always positive. Its mean value is:

\[
\langle \varphi_2 \rangle = \frac{1}{\alpha} k \int_0^z dz' \langle \frac{\alpha^2}{\varphi_2} (z') \rangle \quad (259)
\]

which from equation (248) is,

\[
\langle \varphi_2 \rangle = \frac{1}{\alpha} k \left( \frac{\lambda}{\sqrt{\pi a}} \right)^2 \int_0^z dz' \varphi_2^2 \quad (260)
\]

In the limit \( z \gg a \), \( \varphi_z^2 = (z'/z)\varphi_z^2 \) and,

\[
\langle \varphi_2 \rangle = \frac{1}{4} k \langle \frac{\alpha^2}{\varphi_2} (z) \rangle z \quad (261)
\]

This is the typical average optical path length, which is additional to the vacuum path length \( k z \).

Of greater interest is the fluctuation in the path length phase,

\[
\langle \left[ \varphi_2 - \langle \varphi_2 \rangle \right]^2 \rangle = \frac{1}{4} k^2 \int_0^z dz' \int_0^z dz'' \int_0^{z'} dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_0^{z_3} dz_4 \times
\]

\[
* \left\{ \langle \prod_{\mu(1)} \prod_{\mu(2)} \prod_{\mu(3)} \prod_{\mu(4)} - \langle \prod_{\mu(1)} \prod_{\mu(2)} \prod_{\mu(3)} \prod_{\mu(4)} \rangle \langle \prod_{\mu(1)} \prod_{\mu(2)} \prod_{\mu(3)} \prod_{\mu(4)} \rangle \right\} \quad (262)
\]

where \( \mu(1) = \mu \left[ X(z_l), z_l \right] \), etc. Equation (262) is simplified by assuming in the limit \( z \gg a \) that in the average involving the four \( \mu \) factors that the largest contribution arises
from the different possible averages of pairs of $\mu$ factors separately. Thus equation (262) is approximately equal to,

$$\left\langle \left[ \phi_2 - \langle \phi_2 \rangle \right]^2 \right\rangle = \frac{1}{4} k^2 \left\langle \left( \sum_1^4 d z_1 \cdots d z_4 \sum_1^4 \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \right)^* \right\rangle \left\{ \langle \mu(1) \mu(3) \rangle \langle \mu(2) \mu(4) \rangle + \langle \mu(1) \mu(4) \rangle \langle \mu(2) \mu(3) \rangle \right\}$$

$$= \frac{1}{2} k^2 \mu^2 \sum_0^\infty \left\{ \left( \frac{z'}{z''} \right) \int_0^\infty \left( \frac{z'}{z''} \right) \right\} \left\{ \int_0^\infty \left( \frac{z'}{z''} \right) \right\}$$

where the brackets $\{z', z''\}$ denote the smaller of $z'$ or $z''$. Performing the trivial integrations in (263) gives,

$$\left\langle \left[ \phi_2 - \langle \phi_2 \rangle \right]^2 \right\rangle = \frac{1}{3} k^2 \mu^2 \sum_0^\infty \left\{ \left( \frac{z'}{z''} \right) \int_0^\infty \left( \frac{z'}{z''} \right) \right\}$$

and invoking the isotropy of $\rho_N$,

$$\left\langle \left[ \phi_2 - \langle \phi_2 \rangle \right]^2 \right\rangle = \frac{2}{3} k^2 \mu^2 \sum_0^\infty \left\{ \left( \frac{z'}{z''} \right) \int_0^\infty \left( \frac{z'}{z''} \right) \right\}$$

or finally,

$$\left\langle \left[ \phi_2 - \langle \phi_2 \rangle \right]^2 \right\rangle = \frac{1}{4} \sqrt{\frac{2}{3}} k \left\langle \Theta^2(z) \right\rangle$$  (264)

Thus the r.m.s. fluctuation in path length (264) is of the order of the mean path length (261).
§4. Dimensionless Parameters of the Wave Equation:

The dimensionless scaling parameters $\alpha$ and $\varepsilon$ are defined in equation (24). From equation (257) observe that $\vartheta_1 \propto k \mu$ so that from (24), $\vartheta_1 = O(\varepsilon/\alpha)$. And similarly $\vartheta_2 \propto k \mu^2$ so that $\vartheta_2 = O(\varepsilon^2/\alpha^2)$. We have $\vartheta_1 \sim \vartheta_2$, and the path length phase may be written as $\vartheta_2 \sim (\lambda z \varphi_z / 2 \pi a^2) \vartheta_2$. Therefore, ignoring numerical factors,

$$\varepsilon = \frac{\lambda z}{2 \pi a^2} \varphi_z ; \quad \alpha = \frac{\varepsilon \varphi_z}{\lambda z} = \frac{\lambda z}{2 \pi a^2}$$

Let $\alpha = z / z_o$ where $z_o = 2 \pi a^2 / \lambda$. Let $\alpha = z / \ell_z$ where $\ell_z = z_0 / \varphi_z$. $z_o$ and $\ell_z$ are fundamental lengths in the random wave propagation problem; $z_o$ is the Fresnel distance for turbules of size $a$. For reasons evident in the next section, $\ell_z$ is the focal length for turbules of size $a$. Our use of the lengths $z_o$ and $\ell_z$ follows the work by Salpeter (1967).

Observe that prior formulae are conveniently summarized in terms of $\ell_z$ and $\varphi_z$ below,

$$\Theta \sim a / \ell_z \quad ; \quad \chi \sim a (z / \ell_z)$$

$$\vartheta_1 \sim \varphi_z \quad ; \quad \vartheta_2 \sim \varphi_z (z / \ell_z)$$

(265)

Here and subsequently it is assumed that $z \gg a$ and $\ell_z \gg a$. The latter guarantees small angle scattering, $\Theta \ll 1$. 
The suggested criteria for validity of geometrical optics, \( E \gg \alpha \ll 1 \) (Part I; Section B) translates to \( z \ll z_o \) and \( \varphi_z \gg 1 \). The first inequality is a necessary condition for smallness of diffraction. For a single blob of size \( a \) in the \( z = 0 \) plane, \( z < z_o = \frac{2\pi a^2}{\lambda} \) is the Fresnel domain of the blob. In the limit \( z \ll z_o \) the blob casts a sharp shadow, but for \( z \ll z_o \) the shadow becomes fuzzy and disappears because of diffraction. The second inequality is a refinement of the usual vague condition (Landau and Lifshitz, 1962) that geometrical optics holds when the eikonal is sufficiently large. Here the eikonal \( \psi = kz + \varphi \) is always large because \( kz \) is.

Criteria for the thin screen become simply \( z \ll l_z \) and \( \varphi_z \gg 1 \). Those for the Born (or Rytov) approximation are \( z \ll l_z \) and \( \varphi_z \ll 1 \). Apparently the last condition on the distance \( z \) is the least stringent of the three because \( l_z \gg z_o \) for \( \varphi_z \ll 1 \).

The domains of suggested applicability of the three approximate solutions of the wave equation (21) are summarized in Figure (19).

The variables listed in (265) are fundamental in subsequent sections. These occur in certain new combinations so that it is useful to summarize their parameter dependences:
FIGURE 19: Wave Propagation Régimes.
\[ \begin{array}{l}
\phi_1 \sim \gamma \lambda a^{1/2} z^{1/2} \\
\phi_2 \sim \gamma^2 \lambda^3 a^{-1} z^2 \\
l_z \sim \gamma^{-1} \lambda^{-2} a^{3/2} z^{-1/2} \\
\langle \Theta^2(z) \rangle^{1/2} \sim \gamma \lambda^2 a^{-1/2} z^{1/2} \\
\langle X^2(z) \rangle^{1/2} \sim \gamma \lambda^2 a^{-1/2} z^{3/2}
\end{array} \] 

\[(266)\]

Where \( \gamma = r_e \cdot \frac{\delta N^2}{a} \) has been introduced to simplify the notation in the above.
B. Geometrical Optics of the Thick Screen:

In this Section we investigate certain aspects of rays with increasing distance \( z \) which reveal a distinct transition in behaviour in going from the domain \( z < l_z \) to that where \( z > l_z \). The latter domain is termed the 'thick screen.' First with reference to the preceding section, assume that geometrical optics holds: \( z \ll z_0 \) and \( \phi_z \gg 1 \), which are only necessary conditions. Then from the dependence of the ray displacement \( \tilde{X} \) and the ray phase \( \phi \) on distance \( z \), distinction may be made between the domains \( z < l_z \) and \( z > l_z \). The typical ray displacement \( \left< \tilde{X}^2 \right>^{1/2} \) becomes larger than the scale size \( a \) for \( z > l_z \). Thus two parallel rays incident on \( z = 0 \) and initially separated by slightly more than a correlation length \( a \) will wander independently and may cross if the propagation distance is larger than \( l_z \). At the point of crossing geometrical optics is invalid. Including diffraction, the point becomes a finite sized 'focal spot' of finite intensity. But far away from the spot, for larger \( z \), geometrical optics may be re-instituted. As far as the ray-motion is concerned the crossing may be ignored.

The ray phase \( \phi \) of equation (257) was split into two contributions, \( \phi_1 \) and \( \phi_2 \). At a distance \( z \sim l_z \) these phases are roughly equal (leading to the possibility of focal spots); at larger distances the path length phase \( \phi_2 \sim (z/l_z)\phi_1 \) dominates.
Below we treat statistics of the geometrical optics intensity (§1), and then a related quantity, the transverse distance between two rays incident on \( z = 0 \) in (§2). A qualitative theory of wave propagation in a thick medium is constructed from the ray optics results (§3). The theory is developed through treatment of three problems: (a) the dependence of the emergent wave field on the angular size of the incident wave (§4); (b) the time dependence of the wave due to turbulent velocities of plasma turbules (§5); and (c) wavelength dependence of the emergent wave (§6). A wave optics model of the thick screen is discussed in the next section, and certain results obtained from the model prompted correction of earlier conjectures for (a) and (b), as mentioned below in (§4) and (§5).
§1. **Intensity**

The geometrical optics intensity is inversely proportional to the area enclosed by a narrow tube of rays. Thus by applying Gauss' theorem to the volume enclosed by two small disc-like surfaces lying on nearby wavefronts $\psi = c_1$ and $\psi = c_2$ and the surface of the frustum of the cone formed by rays connecting edges of the discs one may obtain,

$$I(s) = \exp \left[ -2R(s) \right] \quad ; \quad R(s) = \frac{1}{2} \int_0^s \frac{d}{ds'} \left( \nabla \cdot \frac{d}{ds'} X(s') \right)$$  \hspace{1cm} (267)

where the incident wave intensity is taken to be unity. Introducing $X(z)$ of equation (252) and retaining only the lowest order terms in $\mu$ in $R$ (namely $ds/dz = 1$) one finds,

$$I(z) = \exp \left[ -2R(z) \right] \quad ; \quad R(z) = \frac{1}{2} \int_0^Z \nabla^2 \mu [X(z), z']$$  \hspace{1cm} (268)

which is also derived by Chernov (1960), page 32, and by others. Note that $R(z)$ corresponds to an imaginary correction to the real phase $\varnothing$ and thus represents a small contribution from term (a) of equation (23).

The derivation of (267) assumes that rays making up the tube do not cross. Therefore assume that the distance $z$ is small enough for this to be the case. Define $J = \ln \langle I \rangle$. The average $\langle J \rangle$ is zero. The second moment is given by
\[ \langle J^2(z) \rangle = \iiint d\zeta \Delta z (z-z')(z-z'') \langle \frac{\partial^2}{\partial x_1^2} \mu(\chi(z'), \zeta') \frac{\partial^2}{\partial x_2^2} \mu(\chi(z''), \zeta'') \rangle \]  

(269)

Using methods of the previous section and assuming \( z \gg a \), equation (269) may be reduced to

\[ \langle J^2(z) \rangle = \frac{2 \mu^2 z^3}{3} \left\{ \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} \int_0^\infty dz' \mathcal{Q}_N(x,z') \right\}_{x=0} \]  

(270)

For isotropic \( \mathcal{Q}_N \),

\[ \langle J^2(z) \rangle = \frac{16 \mu^2 z^3}{3} \left\{ \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} \left( \frac{1}{z'} \frac{\partial}{\partial z'} \mathcal{Q}_N(z') \right) \right\} \]  

(271)

and finally assuming \( \mathcal{Q}_N = \exp(-z^2/2a^2) \),

\[ \langle J^2(z) \rangle = \frac{8}{3} \left( \frac{z}{\ell_Z} \right)^2 \quad ; \quad \ell_Z = k a^2 / \phi_z \]  

(272)

The salient feature of (272) is that \( \langle J^2 \rangle \) is small for \( z \ll \ell_Z \), but of order unity for \( z \sim \ell_Z \). This shows that strong intensity variations set in at distances \( z \gtrsim \ell_Z \) in accord with the mentioned possibility of rays crossing in this domain.
§2. Separation of Two Rays:

For the domain \( z \gg \xi \), consider the transverse distance between two nearby initially parallel rays incident on \( z = 0 \) with specified separation \( \Delta X \), \( \Delta X(z) = X(z) - X'(z) \), with \( \Delta X(z=0) = \Delta X_0 \). For ray motion in one-dimension \((\Delta X = (\Delta X_x, 0))\) \( \Delta X_x \) is simply related to the intensity by \( I \propto (\Delta X_x)^{-1} \) for very small \( \Delta X_0 \), except at the points where the two rays cross, \( \Delta X(z) = 0 \). The advantage in dealing with \( \Delta X(z) \) rather than the intensity is evident. From equation (252),

\[
\begin{align*}
X(z) &= \frac{1}{a} \Delta X_0 + \int_0^z dz' (z-z') \frac{\partial}{\partial z} \mu \left[ X(z'), z' \right] \\
X'(z) &= -\frac{1}{a} \Delta X_0 + \int_0^z dz' (z-z') \frac{\partial}{\partial z} \mu \left[ X'(z'), z' \right]
\end{align*}
\]

Forming the difference gives

\[
\Delta X(z) = \Delta X_0 + \int_0^z dz' (z-z') \frac{\partial}{\partial z} \left\{ \mu \left[ X(z'), z' \right] - \mu \left[ X'(z'), z' \right] \right\}
\]

If now it is assumed that the refractive index \( \mu \left[ X, z \right] \) is completely continuous, the two rays collapse to a single ray continuously as \( \Delta X_0 \to 0 \). That is, \( |\Delta X(z)| < \text{const.} \) \( * |\Delta X_0| \) for sufficiently small \( |\Delta X_0| \). For the moment assume even further that \( |\Delta X_0| \) is sufficiently small for \( |\Delta X(z)| < a \). Then a Taylor expansion of the integrand of (274) is valid:

\[
\mu \left[ X(z), z \right] = \mu \left[ \bar{X}, z \right] + \frac{1}{2} \Delta X \cdot \nabla \mu \left[ \bar{X}, z \right] \quad \text{where} \quad \bar{X} = \frac{1}{2}(X + X').
\]

This gives an inhomogeneous Volterra equation for \( \Delta X \),
\[ \Delta X(z) = \Delta X_{\text{initial}} + \int_0^z dz' (z-z') \Delta X(z') \cdot \nabla \mu \left[ \overline{X(z), z'} \right] \] (275)

By symmetry \( \langle \Delta X \rangle = \Delta X_{\text{initial}} \).

A Born expansion of (275) leads to an infinite series which is effectively divergent, involving powers of \( z/a \). No renormalization procedure has yet been found to correct this.

Twice differentiating (275) gives an equivalent differential equation,

\[ \frac{d^2}{dz^2} \Delta X(z) = \left( \Delta X(z) \cdot \nabla \right) \nabla \mu \left[ \overline{X(z), z} \right] \] (2.76)

It is seen that \( \Delta X(z) \) is slowly varying functions of \( z \) in comparison with \( \nabla \nabla \mu \); \( \Delta X(z) \) changes by itself in a distance \( \sim a \left( \mu^2 \right)^{-1/4} \sim (e^2 z)^{1/4} a^{1/4} \) whereas \( \nabla \nabla \mu \) changes by itself in a distance \( \sim a \). Therefore it is plausible to treat \( \Delta X(z) \) in the integrand of (275) as independent of \( \mu \left[ \overline{X(z), z} \right] \). This is the prime assumption of the ray optics theory of the thick screen. It is not susceptible to mathematical proof. Although Keller (1962) describes assumptions of this kind as 'dishonest,' the results which follow appear self-consistent and are consistent with wave optics results of the next section. The assumption was suggested by Professor E.E. Salpeter. However, the exponential spreading and con-
verging of rays was first proposed by the author.

Squaring and averaging (275) utilizing the prime assumption gives,

$$
\langle \Delta X^2(z) \rangle = \Delta X_o^2 + \sum_{z} dz' dz'' (z-z')(z-z'') \langle \Delta X(z') \cdot \nabla' \Delta X(z'') \cdot \nabla'' \rangle \ast 
\ast (-1)^2 \langle \mu[z(z'),z''] \mu[z(z''),z'''] \rangle \quad (277)
$$

The cross term has already been set to zero because \( \langle \mu \rangle = 0 \).

Consider the first average in the integrand of (277). The \( z' \) and \( z'' \) arguments must not be farther separated than a few correlation lengths \( a \) in order for a non-zero contribution. Thus in this average one may set \( z' = z'' \) in view of the relatively slow variations of \( \Delta X(z) \). Next consider the vector \( \Delta(z) = \Delta X(z) \cdot \Delta X_o \). \( \Delta(z) \) averages to zero and has the directional bias of \( \Delta X_o \) removed. Therefore \( \Delta(z) \) is assumed statistically isotropic in the \( x-y \) plane. This gives,

$$
\langle \Delta X_i \cdot \Delta X_j \rangle = \frac{1}{2} \delta_{ij} \left[ \langle \Delta X^2 \rangle - \Delta X_o^2 \right] + \Delta X_{ri} \Delta X_{rj} \quad (278)
$$

where \( i, j = 1, 2 \). Terms with \( i \neq j \) from the first average do not contribute to (277) for cases of isotropic \( \mathcal{O}_N \) and further inspection shows that the first average in (277) may be replaced by \( \frac{1}{2} \langle \Delta X^2 \rangle \nabla^2 \). Thus equation (277) may be written as

$$
\hat{f}(z) = 1 + \frac{1}{2} \kappa^3 \int_0^z dz' (z-z')^2 \hat{f}(z') \quad \hat{f}(z) \quad (279)
$$
where

\[ f(z) = \frac{\langle \Delta x^2(z) \rangle}{\Delta x_0^2} \quad \text{and} \quad \mathcal{K}^3 = 2\mu^2 \left\{ \int_0^{\infty} dz \int_0^1 d\xi \xi^2 \phi_N(z, \xi) \right\}_{\xi = 0} \]

From equations (270) and (271) for isotropic \( \phi_N \),

\[ \mathcal{K}^3 = 16 \mu^2 \int_0^{\infty} dz' \int_0^1 \frac{1}{z'} \frac{d}{dz'} \left( \frac{1}{z'} \frac{d}{dz'}, \phi_N(z') \right) \]

For the Gaussian spectrum or \( \phi_N = \exp(-z^2/2a^2) \),

\[ \mathcal{K}^3 = \frac{8}{z' \mathcal{L}_z^2} \quad \text{with} \quad \mathcal{L}_z = ka^2/\phi_z \]

\( \mathcal{L}_z \) is the characteristic distance for appreciable changes in \( \Delta x(z) \). In fact \( \mathcal{L}_z \) is roughly the distance to the first ray crossing. Let \( z_\text{x} \) be the distance to the first crossing; then \( z_\text{x} = \mathcal{L}_z z_\text{x} \) or \( z_\text{x}^3 = \mathcal{L}_z^2 z_\text{x} \), the right side of the latter expression being independent of \( z_\text{x} \) by the third equation in (266). The distance between successive crossing \( \delta z \) remains at roughly \( \delta z \sim z_\text{x} \sim \mathcal{L}_z^{-1} \). Note that \( \delta z \) is independent of \( \Delta x_0 \).

Thrice differentiating \( f(z) \) in equation (279) gives,

\[ \frac{d^3}{dz^3} f(z) = \mathcal{K}^3 f(z) \]

with boundary conditions \( f(z=0) = 1, \frac{df}{dz} f(z=0) = 0, \) and \( \frac{d^2}{dz^2} f(z=0) = 0 \). The general solution to (282) is \( f(z) \propto \exp(\lambda_\infty \mathcal{K} z) \), where \( \lambda_\infty^3 = 1 \) or \( \lambda_1 = 1, \lambda_2 = -\frac{1}{2} + 3^{3/2}i/2 \), and \( \lambda_3 = -\frac{1}{2} - 3^{3/2}i/2 \), are the three roots of unity. The
boundary conditions are satisfied by,

\[
\{ f(z) = \frac{1}{3} \left\{ e^{\beta z} + 2 e^{-\frac{1}{2} \beta z} \cos \left( \frac{\sqrt{3}}{2} \beta z \right) \right\} \}
\]

Thus \( f(z) \) increases monotonically from unity. For \( z \gg \beta^{-1} \),

\[
\langle \Delta X^2(z) \rangle = \frac{1}{3} \Delta X_0^2 e^{\beta z}
\]

(284)

The rays typically spread apart exponentially with distance in this limit. The separation continues according to (284) up to \( \langle \Delta X^2 \rangle^\beta \sim a \). For larger distances (284) is invalid.

Let us examine the case where the ray separation is not restricted to less than \( a \). For this the probability density of \( \Delta \) is needed. Fortunately no calculation of moments is required because \( \Delta = X - X' - \Delta X_0 \) is a linear combination of Gaussian random variables (see equation 256). Any linear combination of Gaussian variables is also a Gaussian variable. Hence

\[
W(\Delta) = \left( \frac{2 \pi \sigma^2}{2} \right)^{-\frac{1}{2}} \exp \left\{ - \frac{\Delta^2}{2 \sigma^2} \right\}
\]

(285)

\[
\int_{-\infty}^{\infty} d\Delta x \int_{-\infty}^{\infty} d\Delta y \quad W(\Delta x, \Delta y) = 1
\]

where \( \langle \Delta^2(z) \rangle = 2 \sigma^2(z) \). The analogue of equation (275) for arbitrary ray separation is:

\[
\Delta X(z) = \Delta X_0 + 2 \int_{0}^{z} dz' \frac{z'}{z} \left\{ \sinh \left( \frac{1}{2} \Delta X(z') \nabla_Y \right) \nabla_Y \mu[Y, z] \right\}
\]

(286)

\( Y = X(z') \)
Squaring and averaging (286) according to the prime assumption gives,
\[
\left\langle \Delta X^2(z) \right\rangle = \Delta X^2 + \sum_{o} \int \int (z-z') (z-z'') \left\{ 2 - e^{-\Delta X^2(z') \rho_{z'}} - e^{-\Delta X^2(z'') \rho_{z''}} \right\} * (-1) \rho_{z'}^{2} \left\langle \mu_{z',z''} \mu_{z',z''} \right\rangle \}
\]
\[
(287)
\]
Utilizing the probability density (285),
\[
\sigma^2(z) = -2 \mu^2 \sum_{o} (z-z')^2 \left\{ 1 - e^{-\frac{1}{2} \sigma^2(z') \rho_{z'}^2} \rho_{z'} \Delta X^2 + \right\} \]
\[
- e^{-\frac{1}{2} \sigma^2(z') \rho_{z'}^2} \rho_{z'} \Delta X^2 \sum_{o} \int (z-z'') \rho_{z''} \}
\]
\[
(288)
\]
Finally, introducing \( \Phi_z \) as defined in equation (36), thrice differentiating (288), and invoking the isotropy of \( \Phi_z(\varphi) \) or \( \rho_N \), one finds,
\[
\frac{d}{dz} \sigma^2(z) = 4 \pi \sum_{o} \int \int \rho (z-z') \left[ 1 - e^{-\frac{1}{2} \rho (z-z')^2} \right] d\rho \]
\[
\sigma^2(z) = \left| \Delta X_0 \right| . \text{ The boundary conditions on (289) are} \sigma^2(z=0)=0, \ d/\rho \sigma^2(z=0)=0, \text{ and } d^2/\rho \sigma^2(z=0)=0. \]
As a check on (289) for small ray separation, \( \Delta X_0 \ll a \) and \( \sigma(z) \ll a \), the second bracketed factor of the integrand may be expanded to give,
\[ \frac{d^3}{dz^3} \langle \Delta X^2(z) \rangle = \mathcal{K}^3 \langle \Delta X^2(z) \rangle \]

\[ \mathcal{K}^3 = 2\pi \int_0^\infty q^5 dq \left[ \frac{1}{k^2z} \Phi_z(q) \right] \]

This is identical to equations (282) and (280). A further check on (289) is obtained in the limit \( \sigma \gg a \) where \( \exp(-\frac{1}{2} \sigma^2 q^2) \sim 0 \). Then,

\[ \frac{d^3}{dz^3} \sigma^2(z) = 4\pi \int_0^\infty q^3 dq \left[ \frac{1}{k^2z} \Phi_z(q) \right] \]

\[ = \frac{d^3}{dz^3} \langle \Delta X^2(z) \rangle = \text{constant} \]

In this limit the two rays move independently: \( 2 \sigma^2 = \langle (X-X')^2 - \Delta X^2_o \rangle \sim \langle X^2 \rangle + \langle X'^2 \rangle \sim 2 \langle X^2 \rangle \).

Equation (289) is easily evaluated for the Gaussian spectrum \( F_G \):

\[ \frac{d^3}{dz^3} \sum = \frac{\mathcal{K}^3}{2} \left\{ 1 - \frac{1}{(1+\Sigma)^2} \left( 1 - \frac{\Sigma}{a(1+\Sigma)} \right) \right\} \]

\[ \sum \equiv \frac{\sigma^2(z)}{a^2} ; \quad \Sigma \equiv \frac{\Delta X^2_o}{a^2} \]

where \( \mathcal{K} \) is the same as given in equation (281). The \( z \)-integral of (292) is not obvious and some 'arm waving' will
have to suffice. In all cases of interest $\xi \ll 1$ otherwise the two rays move independently. For small $z$, $\Sigma \ll 1$ up to $z \lesssim z_1 = \kappa^{-1} \ln_e (3/\xi)$, where $\Sigma \lesssim 1$, from equation (284).

For $z \gg z_1$ we may set $\xi = 0$ in (292) and obtain approximately

$$\frac{d^3}{dz^3} \Sigma = \frac{1}{2} \kappa^3$$

Matching boundary values at $z = z_1$ gives the crude estimate

$$\Sigma(z) \approx \frac{1}{2} \langle \frac{\Delta x^2(z)}{\Delta \xi^2} \rangle \approx 1 + \kappa(z-z_1) + \frac{1}{2} \kappa^2(z-z_1)^2 + \frac{1}{12} \kappa^3(z-z_1)^3 \quad (293)$$

For $z - z_1 \gg \kappa^{-1}$,

$$\langle \frac{\Delta x^2(z)}{\Delta \xi^2} \rangle \approx 2 \langle \frac{\Delta x^2(z)}{\Delta \xi^2} \rangle \left[ 1 - \frac{1}{\kappa z} \ell n_e \left( \frac{3 \ell a^2}{\Delta x_o^2} \right) \right]^3 \quad (294)$$

and finally for $z \gg z_1$ the two rays move independently.

The exponential spreading of two rays is illustrated in Figure (20).

As a slight digression on an interesting aspect of the ray spreading consider a sequence of given parallel vectors

$$\frac{\Delta X}{\Delta \xi_{\alpha}} \text{ end to end along the } x\text{-axis in the } z=0 \text{ plane.}$$

Choose the lengths of these as $|\Delta X_{\alpha}| = 3^{\frac{1}{2}} a \exp(-\kappa L/2) \ll a$, and the number of the vectors as $\alpha = 1, 2, \ldots, N = 3^{-\frac{1}{2}} \exp(\frac{1}{2} \kappa L)$. Then two rays incident parallel on $z = 0$ at each end of say $\Delta X_{\alpha}$ will typically expand exponentially to a separation $|\Delta X| (z = L) \gg a$ on the $z = L$ plane. The sequence (all $\alpha$) of the vectors $\Delta X_{\alpha}(L)$ form a chain in the $z = L$ plane with surprising properties. At first sight it might be
FIGURE 20: Ray Spreading.

\[ \Delta X(z) \approx \frac{\Delta X_0}{\sqrt{3}} e^{\frac{Kz}{a}} \]
guessed that the vector \( \mathbf{v}_n = \sum_{\alpha=1}^{N} \Delta X_{\alpha}(L) \) would have a length corresponding to a random walk, that is, \( |\mathbf{v}_n| \sim n^{\frac{1}{2}} a \). This cannot be because for \( n = N \), \( |\mathbf{v}_n| \sim a 3^{-\frac{1}{2}} \exp\left(\frac{1}{2} \kappa L\right) \) is much larger than the separation of two rays with \( |\Delta X_{\alpha}| > a \), \( \left(\frac{\Delta X^2(L)}{a} \right)^{\frac{1}{2}} \sim (\kappa L)^{3/2} a \), from equation (294). The alternative is that there is correlation between different \( \Delta X_{\alpha}(L) \) which limits the magnitude of \( \mathbf{v}_n \). From the discussion leading to equation (294), one may obtain an approximate expression \( |\mathbf{v}_n| \sim a \left(1 + (\ln e n^2)^{3/2}\right) \) which is valid at the limits of \( n = 1 \) and \( n \gg 1 \). For large \( n = N \) the chain of \( \Delta X_{\alpha}(L) \) vectors has the remarkable property that the length along the chain, roughly \( Na \sim a 3^{-\frac{1}{2}} \exp\left(\frac{1}{2} \kappa L\right) \), can be made longer than the distance between the ends, \( |\mathbf{v}_N| \sim (\kappa L)^{3/2} a \), by an exponentially large factor, \( \exp\left(\frac{1}{2} \kappa L\right) \). A suggested realization of the chain of vectors is shown in Figure (21) as a curve. Although as yet we have no proof, it is clear that this curve must cross back over itself many times. The existence of crossing points shows that there are pairs of rays for which the separation \( \Delta X(z) \) decreases with increasing \( z \).

The longitudinal correlation of the ray separation \( \Delta X(z) \) is of interest. For simplicity consider \( \langle \Delta X(z) \cdot \Delta X(z') \rangle \) with \( \Delta X(z) \) and \( \Delta X(z') \) of magnitude less than a correlation length \( a \). Then a trivial calculation gives,

\[
\langle \Delta X(z) \cdot \Delta X(z') \rangle = \langle \Delta X^2(z) \rangle \quad \text{for} \quad z \leq z' \quad (295a)
\]
FIGURE 21: Sequence of $\Delta X(L)$ Vectors.
The normalized correlation for \( z, z' \gg \mathcal{K}^{-1} \) is:

\[
\left< \Delta X(z) \cdot \Delta X(z') \right> \left( \Delta X^2(z) \right)^{-\frac{1}{2}} \left( \Delta X^2(z') \right)^{-\frac{1}{2}} = e^{-\frac{1}{2} \mathcal{K} |z-z'|}
\]

(295b)

Of course, from (295) the cause of the decorrelation is not separated between the possibilities of twisting (about the z-axis) and/or changes in magnitude (and sign) of \( \Delta X(z) \). However, the latter alone is possible because as already noted \( \Delta X(z) \) passes through zero typically every distance \( \mathcal{K}^{-1} \) along the z-axis. The twisting has not been investigated.
§3. **Thick Screen Amplitude:**

The exponential spreading of nearby rays suggested a heuristic theory for wave propagation in a thick medium. On the far boundary of the plasma slab on \( z = L \) consider the wave to be a superposition of a large number of beams or bundles of correlated rays. Each beam is identified as having a diameter \( \langle A x^2(z) \rangle^{1/2} \approx a \) on \( z = L \), which is roughly the maximum permitted for there to be strong correlation between rays constituting a beam. The number of beams incident at an arbitrary point on the \( z = L \) plane is estimated as \( N \sim 1/3 \exp(\kappa L) \gg 1 \), from equation (284). The instantaneous wave amplitude on \( z = L \) is expressed as,

\[
A(x) \sim \sum_{j=1}^{N} \sqrt{I_j(x)} \exp(i \varphi_j(x))
\]

(296)

where \( \varphi_j(x) \) is the ray phase of equation (257) and \( I_j(x) \) is the intensity of the \( j \)th beam. Figure (22) shows the ray picture of equation (296). The representation of the amplitude by an expression of the form of (296) was first proposed by Prof. E.E. Salpeter.

Phases and intensities of different beams are assumed independent because rays of different beams have widely different paths. Rays arriving at a particular point on \( z = L \) typically come from points on \( z = 0 \) separated by transverse distances \( \sim (\kappa L)^{3/2} a \). The intensity \( I_j(x) \) varies by itself for a change in \( x \) of the order of \( a \). Of course, the beam phase \( \varphi_j = \varphi_{1j} + \varphi_{2j} \) gives the strongest dependence.
FIGURE 22: Thick Screen Amplitude.
dence of $A(x)$. From equation (257), the change in $\phi_{1j}$, denoted $\Delta \phi_{1j}$, corresponding to $\Delta X_j(z)$ is:

$$\Delta \phi_{1j} = k \int_0^z dz' \Delta X_j(z') \cdot \nabla \mu \left[ \overline{\theta}(z), z' \right]$$

(297)

The mean-square is:

$$\langle \left[ \Delta \phi_{1j} \right]^2 \rangle = K_1 \int_0^z dz' \langle \Delta X_j^2(z') \rangle$$

$$K_1 = -2 k^2 \frac{1}{\mu} \int_0^\infty dz' \frac{1}{z'} \frac{\partial^2}{\partial z'} \phi_N(z') \quad (\text{isotropic } \phi_N)$$

$$= \frac{\phi_L^2}{L} a^2 \quad (\text{Gaussian } \phi_N)$$

(298)

Introducing $\langle \Delta X_j^2(z) \rangle$ from equation (284), one finds that the typical change of $\phi_{1j}$ across a beam is of the order of $(\ell_L / L)^{1/3} \phi_L$ on the $z = L$ plane.

The path length phase $\phi_{2j}(\overline{x})$ change across a beam is denoted $\Delta \phi_{2j}$. It may be expressed as:

$$\Delta \phi_{2j} = k \int_0^z dz' \Delta X_j(z') \cdot \nabla \mu \left[ \overline{\theta}(z), z' \right]$$

$$\overline{\theta}(z) = \int_0^z dz' \nabla \mu \left[ \overline{\theta}(z), z' \right]$$

$$\Delta \theta(z) = \int_0^z dz' \Delta X_j(z') \cdot \nabla \mu \left[ \overline{\theta}(z), z' \right]$$

(299)

We have $\langle \Delta \phi_2 \rangle = 0$ by symmetry. The mean-square is given by,
\[
\langle (\Delta \phi^2) \rangle = k^2 \int_0^\infty \frac{dz'}{z'} \int_0^{z'} \frac{dz''}{z''} \left\{ \int_0^{z_1} \frac{dz_1}{z_1} \int_0^{z_2} \frac{dz_2}{z_2} \int_0^{z_3} \frac{dz_3}{z_3} \int_0^{z_4} \frac{dz_4}{z_4} \right\} \langle \Delta X(z_1) \cdot \nabla_1 \Delta X(z_3) \cdot \nabla_3 \rangle \langle \nabla \cdot \nabla_2 \nabla \cdot \nabla_4 \rangle < \mu(i) \mu(2) \mu(3) \mu(4) > \]

(300)

The average of the \( \mu \)'s is split in the usual way into averages of pairs, \( \propto \rho_N(i2) \rho_N(34) + \rho_N(i3) \rho_N(24) + \rho_N(i4) \rho_N(23) \). Of the three terms, the first and last are zero because each involves a derivative of odd order of \( \rho_N \), which are zero for all Gaussian-like \( \rho_N \). The remaining term gives,

\[
\langle (\Delta \phi^2) \rangle = K_2 \int_0^\infty \frac{dz'}{z'} \int_0^{z'} \frac{dz''}{z''} \left\{ \int_0^{z_1} \frac{dz_1}{z_1} \int_0^{z_2} \frac{dz_2}{z_2} \right\} \langle \Delta X^2(z_2) \rangle
\]

\[
= \frac{1}{3} K_2 \int_0^\infty \frac{dz'}{(z-z')^2(z+2z')} \langle \Delta X^2(z') \rangle
\]

(301)

\[
K_2 = -8 k^2 \bar{\rho}^2 \int_0^\infty \frac{dz'}{z'} \rho_N(z') \left( \int_0^\infty \frac{dz'}{z'} \frac{2}{z'} \rho_N(z') \right)
\]

\[
\text{isotropic } \rho_N
\]

\[
K_2 = 2 \phi_z^2 / \ell_z^2 \int_0^\infty \frac{dz}{z} \int_0^\infty \frac{dz'}{z'} \left( \int_0^\infty \frac{dz''}{z''} \frac{2}{z''} \rho_N(z) \right)
\]

\( \text{Gaussian } \rho_N \)

where the brackets \{z', z''\} denote the smaller of z' or z''. Note the lack of symmetry of (301) to the replacement \( z \to L - z \). This is because for small z, \( \rho_N \) and \( \Delta \phi \) are small, whereas at large z both are large. Substitution of \( \langle \Delta X^2(z_2) \rangle \) from equation (284) for \( z_2 \gg K_2^{-1} \)
and \( z = L \) gives,

\[
\left \langle \left( \Delta \phi \right)^2 \right \rangle^{1/2} = \frac{1}{\sqrt{a}} \phi_L \left \langle \Delta \chi^2 (L) \right \rangle^{1/2} / a
\]  

(302)

for Gaussian \( \xi_N \). Thus the path length phase change across a beam is \( \sim \phi_L \) on \( z = L \).

The change in the ray angle \( \Theta_j (z) \) across a beam, denoted \( \Delta \Theta \), may be evaluated as,

\[
\left \langle \Delta \Theta^2 \right \rangle = \frac{1}{\kappa} \chi^2 \int_0^Z dz' \left \langle \Delta \chi^2 (z') \right \rangle
\]  

(303)

On \( z = L \) the change is typically \( \Delta \Theta \sim (\ell_L / L)^{1/3} (a / \ell_L) \) across a beam.

Differences of \( \phi_1 \), \( \phi_2 \), and \( \Theta_j \) between different beams are much larger than changes across a single beam. The differences between these quantities for different beams are \( \sim \phi_L \), \((L/\ell_L) \phi_L \), and \( a / \ell_L \), respectively. However, the change in path-length phase across a single beam determines the characteristic scale of variation of the amplitude \( A(\chi) \) of equation (296). \( A(\chi) \) changes by itself in a transverse distance of the order of the distance that the path length phase of a beam changes by unity. This is a distance \( \sim a / \phi_L \), which is much shorter than the distance over which \( \phi_1 \) changes by unity. \( a / \phi_L \) is formally the same as the amplitude correlation length for the thin screen in the \( \phi_L \gg 1 \) limit; (see Part I; D; \( \xi_1 \)). The angle probability density for the thick screen is given by (251) which is iden-
tical to that for the thin screen for \( \varphi_L \gg 1 \) (equation 64). Alternatively, the amplitude correlation functions are the same.
§4. Angle Dependence of Amplitude:

One early result of the ray theory of the thick screen dealt with scintillations of finite diameter sources. However, for simplicity consider two distant point sources separated by an angle \( \chi \). For small enough \( \chi \), the sources scintillate as one point source; for larger \( \chi \), the scintillations are quenched. In the case of the thin screen with increasing \( \chi \), the sources cease to act as a point source when the line of sights to the two is separated by more than an amplitude correlation length \( a/\varphi_L \) at the screen. Thus for observations at a distance \( L \) from the thin screen, scintillations of sources larger than \( \chi \approx a/\varphi_L \) are quenched.

For the thick screen consider two rays incident at the same point on \( z = 0 \), but separated in angle by \( \chi \). As before denote the ray separation \( \Delta X = \bar{X} - \bar{X}' \), assumed smaller than \( a \). Then,

\[
\Delta X(z) = \chi z + \int_0^z dz' (z-z') \Delta X(z') \cdot \varphi \mu[\bar{X}(z'), z']
\]

(304)

with \( \langle \Delta X(z) \rangle = \chi z \). Squaring and averaging (304) gives,

\[
\frac{1}{3} \frac{d}{dz^3} \langle \Delta X^2(z) \rangle = \chi^3 \langle \Delta X^2(z) \rangle
\]

(305)

with boundary conditions \( \langle \Delta X^2(z=0) \rangle = 0 \), \( d/dz \langle \Delta X^2(z=0) \rangle = 0 \), and \( d^2/dz^2 \langle \Delta X^2(z=0) \rangle = 2 \chi^2 \). The solution to (305) corresponding to (283) is:
\[ \langle \Delta X^2(z) \rangle = \frac{2}{3} \left( \frac{\chi}{\kappa} \right)^2 \left[ e^{\kappa z} - e^{-\frac{\kappa z}{2}} \left( \cos \left( \frac{\pi}{2} \kappa z \right) + \frac{\sqrt{3}}{3} \sin \left( \frac{\pi}{2} \kappa z \right) \right) \right] \]  

Equation (306) increases monotonically to 
\[ \langle \Delta X^2 \rangle \overset{\text{for } \kappa z \gg 1}{\sim} \frac{\kappa}{\kappa} e^{\frac{1}{2} \kappa z} \]

for \( \kappa z \gg 1 \).

A general condition for strong or unit cross-correlation between amplitudes of the two sources is that for each beam from one source a beam from the second source can be found such that the total optical path fluctuations along the two beams is less than say a radian. In early development of the ray optics of the thick screen it was thought incorrectly that beams constituting \( A(x) \) of equation (296) moved, as \( \chi \) was varied, with their point of origin on \( z = 0 \) fixed.

According to this, changing \( \chi = 0 \) to \( \chi \) typically moved the \( j \)th beam by a distance \( \sim \frac{\chi}{\kappa} \exp \left( \frac{1}{2} \kappa L \right) \) on \( z = L \). Rays of the two beams to the two sources would be typically separated by \( \langle \Delta X^2(z) \rangle \overset{\chi}{\sim} \) as given by (306). The path length difference in this case is given by

\[ \Delta \phi \overset{\chi}{\sim} (z) = \kappa \int_0^z dz' \bar{\theta}(z') \cdot \left( \chi + \Delta \theta(z') \right) \]  

(307)

where \( \bar{\theta}(z) \) and \( \Delta \theta(z) \) are given by equation (299).

\[ \langle \Delta \phi \overset{\chi}{\sim} \rangle = 0 \], but the mean-square has an additional term,
\[ \langle (\Delta \phi_2 \chi(z))^2 \rangle = \frac{1}{3} \frac{z^3}{L a^2} \phi_L^2 \chi^2 + \langle (\Delta \phi_2(z))^2 \rangle \]  

(308)

where \( \langle (\Delta \phi_2)^2 \rangle \) is given by equation (301). Here the extra term in (308) is insignificant. Setting \( \langle (\Delta \phi_2)^2 \rangle \) to unity gives \( \chi = \chi_\ast \sim (a/L_r)(kL)\exp(-\frac{1}{2}kL) \) as the (incorrect) condition for point source behaviour. This \( \chi_\ast \) is much smaller than the thin screen value if \( kL \gg 1 \).

Although invalidity of the early ray optics conjecture for \( \chi_\ast \) was deduced from a model discussed in the next section, the error involved can be elucidated with ray tracing methods, and a correct order of magnitude estimate for \( \chi_\ast \) obtained.

Previously we assumed \( \Delta X(z=0) \) or \( \Delta \phi(z=0) \) specified, non-random, boundary values on the ray equation. This lead to equation (292) and (289). The asymmetry of the \( \lambda \) roots of equation (282) about the imaginary axis, growing solutions but no non-oscillatory damping ones, may be traced to the fact that the boundary conditions were set on \( z = 0 \). The incident rays were assumed perfectly columnned. All positive definite (as for all reasonable boundary conditions) of equation (282) are exponentially growing for large enough \( z \). This spreading of rays is the most probable occurrence (equation 285) and there is a definite irreversibility implied. But not all nearby rays spread apart with \( z \), as may be demonstrated
by a different formulation of the boundary conditions on the
ray equation.

Consider anew the two sources with specified angular
separation $\chi$. Assume for simplicity one-dimensional ray
motion, $x = (x, 0)$, etc. Let $\Delta x(z)$ denote the distance
between two rays, one from one source and the other from the
other. However, do not specify $\Delta x(z=0)$. Vaguely expressed
we want to allow $\Delta x(0)$ to vary from one realization to
the next in order to minimize the optical path length fluctua-
tions along the two rays subject to the constraint $\Delta \theta(z=0)=
= \chi$. This is closer to the 'right' question to ask for deter-
mining phase fluctuations between two source than the prior
discussion of $\chi$. Of course, $\Delta x(z=0)$ becomes a random
variable.

Let $x_R(z)$ denote a reference ray for which $\theta_R(z=0) =
0$ and $x_R(z=0)$ is arbitrary. On $z = L$, $\theta_R = \theta_R(z=L)$ and
$x_R = x_R(z=L)$. Consider a second ray $x(\zeta)$ parameterized
by $\zeta = L - z$, where $\zeta$ is termed the 'backwards' z-direction.
Choose $x(\zeta=0) = x_R(z=L)$ and $-\frac{d}{d\zeta} x(\zeta=0) = \theta_R(z=L) +
+ \Delta \theta_L \cdot \Delta \theta_L$ is specified. Denote $\Delta x(\zeta) = x(\zeta) - x_R(L-\zeta)$
and assume $|\Delta x(\zeta)| < a$ for $0 \leq \zeta \leq L$ and small enough
$\Delta \theta_L$. Then,

$$\Delta x(\zeta) = -\zeta \Delta \theta_L + \int_0^\zeta d\zeta' (\zeta - \zeta') \Delta x(\zeta') \frac{\mu}{xx} \left[ x_R(L-\zeta'), L-\zeta' \right]$$ (309)

where $\frac{\mu}{xx} = d^2/dx^2 \mu$. Equation (309) is of the form of
(275) except that $x_R$ replaces $\overline{x}$ as an argument of $\mu$. 
Thus in averages of (309), the mean-square in particular, $X_R$, which is determined by $\mu$, determines values of the function $\mu_{xx}$ for different realizations. Nevertheless, it is again true for (309) as for (275) that the rapid variations of $\mu_{xx}$ compared with $\Delta X(\xi)$ permit separation of averages of $\mu_{xx}$ and $\Delta X(\xi)$ factors. For example, $\langle \Delta X(\xi) \rangle = -\xi \Delta \Theta_L$, and,

$$\langle \Delta X^2(\xi) \rangle = \xi^2 \Delta \Theta_L^2 + \frac{1}{2} \kappa^3 \int_0^\xi (\xi'-\xi)^2 \langle \Delta X^2(\xi') \rangle$$  

(310)

where $\kappa^3$ is given by the one dimensional analogue of (280). Equivalent to (310) is

$$\frac{d^3}{d\xi^3} \langle \Delta X^2(\xi) \rangle = \kappa^3 \langle \Delta X^2(\xi) \rangle$$  

(311)

with boundary conditions $\langle \Delta X^2(\xi=0) \rangle = 0$, $\frac{d}{d\xi} \Delta X^2(\xi=0)=0$, and $\frac{d^2}{d\xi^2} \langle \Delta X^2(\xi=0) \rangle = 2 \Delta \Theta_L^2$. The exact solution is (306), but for $\xi \gg \kappa^{-1}$,

$$\langle \Delta X^2(\xi) \rangle = \frac{2}{3} \left( \frac{\Delta \Theta_L}{\kappa} \right)^2 e^{\kappa \xi}$$  

(312)

Thus the paradoxical fact that there are pairs of rays which spread apart exponentially in the backwards direction. The role of such rays is crucial in determining the critical angle $\kappa_+$.  

By varying $\Delta \Theta_L$ is it generally possible to obtain $\kappa = \Delta \Theta_L (= \Delta \Theta(z=0) = \Delta \Theta(\xi=L))$? The reference ray comes from one source and we want the second ray from the second
source. First let us find a typical value of $\Delta \Theta_o$ for the ray pair described by $(3|2)$. For example the r.m.s.,

$$\langle \Delta \Theta_o^2 \rangle = \Delta \Theta_L^2 + \frac{1}{2} \kappa^3 \int_0^L \langle \Delta X^2(\gamma) \rangle \left( \frac{L}{\gamma} \right)$$

(313)

where $\langle \Delta \Theta \rangle = \Delta \Theta_L$ by symmetry. Substitution of (312) gives,

$$\langle \Delta \Theta_o^2 \rangle = \frac{1}{3} \Delta \Theta_L^2 e^{\kappa L}$$

(314)

Thus any realization of the function $\Delta \Theta_o / \Delta \Theta_L$ is linear with exactly zero intercept. Typically the function is of the form $\Delta \Theta_o \approx 3^{-\frac{1}{2}} \Delta \Theta_L \exp(\frac{1}{2} \kappa L)$. The typical value of $\Delta \Theta_L$ which gives $\kappa = \Delta \Theta_o$ is,

$$\Delta \Theta_L \sim \sqrt{3} \kappa e^{-\kappa L/2}$$

(315)

Is it also possible to choose $\Delta \Theta_L$ so that $\kappa = \Delta \Theta_o$ in the case of two dimensional ray motion? We have no proof, but there would appear to be no difficulty.

For any ray arriving at some point on $z = L$ from one source a second ray can be found at the same point on $z = L$ going back to the second source with the typical separation of the two rays given by (312). Assume that the N beams of equation (296) are from one source. Then N beams of the kind constituting (296) from the second source may be constructed utilizing the pairs of rays which spread apart in the backwards direction as pilot rays. In contradistinction
to the earlier incorrect view, the $z = L$ end of each beam is fixed and the narrow end is allowed to 'wag'. This is shown in Figure (23). The path length fluctuations $\Delta \rho_2 \chi$ (which are larger than $\Delta \rho_1$) along two of the paired rays is still given by equation (308), the two terms of which are now roughly equal. Setting $\langle \Delta \rho_2^2 \chi \rangle$ to unity gives,

$$\chi_* \sim \frac{a}{L \phi_L} \tag{316}$$

Aside from numerical factors this is identical to the thin-screen result.

The foregoing derivation of $\chi_*$ gives the typical separation of the correlated rays from the two sources,

$$\langle \Delta x^2 (z=0) \rangle^{1/2} \sim \chi L \left( \frac{L_L}{L} \right)^{3/2} \tag{317}$$

The average separation, $\sim (\chi / \chi') \exp(-\frac{1}{2} \chi L)$, is practically zero. For the thin screen the ray separation corresponding to (317), $\chi L$, is larger.
FIGURE 23: Two Source Beam Amplitudes; (a) Early Incorrect View. (b) Correct View.
In treating two sources, pairs of rays were found which converge exponentially with increasing \( z \). There would appear to be a contradiction with previous statements that exponential spreading is typical or most probable. To investigate this first consider a single source, an incident plane wave, and one-dimensional ray motion. In this situation rays converging exponentially do not occur in a typical realization. For a demonstration assume boundary values,

\[
\Delta X_o = \Delta X(z=0) \quad \text{random} \\
\Delta \Theta_o = \Delta \Theta(z=0) \quad \text{random (but may be made } = 0 \text{)} \\
\Delta X_L = \Delta X(z=L) \quad \text{specified} \\
\Delta \Theta_L = \Delta \Theta(z=L) \quad \text{specified (so that } \Delta \Theta_o = 0 \text{)}
\]

For a particular realization, \( \Delta X_o \) and \( \Delta \Theta_o \) depend linearly on the boundary values,

\[
\begin{pmatrix}
\Delta X_o \\
\Delta \Theta_o
\end{pmatrix}
= 
\begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix}
\begin{pmatrix}
\Delta X_L \\
\Delta \Theta_L
\end{pmatrix}
\tag{318}
\]

Where the \( B_j \) coefficients are given by

\[
B_j = \mathcal{C}_j(\xi = L)
\]

and where

\[
\begin{aligned}
\mathcal{C}_1(\xi) &= 1 + \int_0^\xi d\xi' (\xi - \xi') \mathcal{L}_1(\xi') \mu_{xx} \left[ \chi_R(L-\xi'), L-\xi \right] \\
\mathcal{C}_2(\xi) &= -\xi + \int_0^\xi d\xi' (\xi - \xi') \mathcal{L}_2(\xi') \mu_{xx} \\
\mathcal{C}_3(\xi) &= -\int_0^\xi d\xi' \mathcal{L}_1(\xi') \mu_{xx} \\
\mathcal{C}_4(\xi) &= 1 - \int_0^\xi d\xi' \mathcal{L}_2(\xi') \mu_{xx}
\end{aligned}
\tag{319}
\]
The different quadratic averages of the $B_j$'s, obtained from equation (310) for $\kappa L \gg 1$, are:

\[
\begin{pmatrix}
\mathcal{B}_1 & \kappa \mathcal{B}_2 & \kappa^{-1} \mathcal{B}_3 & \mathcal{B}_4 \\
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\
\frac{2}{3} & \frac{1}{6} & -\frac{1}{3} & \\
\frac{1}{6} & \frac{1}{6} & \\
\frac{1}{3} & \\
\end{pmatrix}
\times e^{\kappa L} \quad (320)
\]

The $B_j$ coefficients are thus typically large, being proportional to $\exp(\frac{1}{2}\kappa L)$. But they are not large independently.

The determinant of the $B$ matrix, $J = \mathcal{B}_1 B_4 - \mathcal{B}_2 B_3 = \frac{\partial (\Delta X_0, \Delta \Theta_0)}{\partial (\Delta X_L, \Delta \Theta_L)}$, is exactly unity as may be proved by showing $d/d\xi \{ \mathcal{B}_1 B_4 - \mathcal{B}_2 B_3 \} = 0$. This is a statement of Liouville's theorem and also the method of its proof (Khinchin, 1949).

The forward $B$ transform is:

\[
\begin{pmatrix}
\Delta X_L \\
\Delta \Theta_L \\
\end{pmatrix}
= \mathcal{B}' \begin{pmatrix}
\Delta X_0 \\
\Delta \Theta_0 \\
\end{pmatrix} \quad (321a)
\]

Because the determinant $|B_0| = 1$,

\[
\mathcal{B}' = \mathcal{B}^{-1} = \begin{pmatrix}
B_4 & -B_2 \\
-B_3 & B_1 \\
\end{pmatrix} \quad (321b)
\]
Equations for the $\mathcal{B}_j^\prime(z)$, where $B^\prime_j = \mathcal{B}_j^\prime(z=L)$, are:

\[
\begin{align*}
\mathcal{B}_1^\prime(z) &= 1 + \int_0^z dz'(z-z') \mathcal{B}_1^\prime(z') \mu_{xx} \left[ X_R(z'), z' \right] \\
\mathcal{B}_2^\prime(z) &= z + \int_0^z dz'(z-z') \mathcal{B}_2^\prime(z') \mu_{xx} \\
\mathcal{B}_3^\prime(z) &= \int_0^z dz' \mathcal{B}_1^\prime(z') \mu_{xx} \\
\mathcal{B}_4^\prime(z) &= 1 + \int_0^z dz' \mathcal{B}_2^\prime(z') \mu_{xx}
\end{align*}
\]

We want to vary $A_{\theta L}$ in order to obtain columnated rays on $z = 0$, that is $A_{\theta 0} = 0$. Elimination of the second equation from (318) gives

\[
\Delta X_0 = J \Delta X_L / B_4
\]  

(323)

$J = 1$ and the typical value of $B_4$ is $\sim 3^{-1/2} \exp(\frac{1}{2} \chi L)$; therefore, $\Delta X_0 \sim 3^{1/2} \Delta X_L \exp(-\chi L/2)$. This corresponds to exponential spreading with $z$. Converging rays occur only for improbable ($\sim \exp(-\chi L/2)$) small values of $B_4$.

Only for two (or more) sources are there rays in a typical realization which converge with $z$, and in a certain sense these are improbable. A sketch representing a not improbable $B^\prime$ transformation is shown in Figure (24). The dashed square encloses a domain of phase space on $z = 0$, and the long narrow parallelogram encloses the corresponding range on $z = L$, which has the same area by Liouville's theorem. For a single source $\Delta \theta_0 = 0$, and any choice of $\Delta X_0$
FIGURE 24: Liouville's Theorem for Random Ray Motion.
transorm to a larger $\Delta X_L$. However, for two sources $\Delta \Theta_0 = \mathcal{N}$, a small fraction ($\simeq \exp(-\frac{1}{2} \kappa L)$) of $\Delta X_0$ values transform to smaller $\Delta X_L$ as depicted in the Figure.

If the transform $f'(z)$ is plotted as a function of increasing $z$, the parallelogram is seen to rotate (in one sense) while becoming narrow and elongated.
§5. Time Dependence of Amplitude:

One early result from the ray optics theory of the thick screen was on the possibility of very rapid time variations of the radio wave amplitude. In the case of a thin screen, time variations may arise for example from the frozen motion of the screen past the line of sight. At a velocity \(u\), the characteristic time scale is simply \(\tau_L \sim a/u\theta_L\) for observations at a distance \(L \gg \theta_L\) beyond the thin screen. However, for the thick screen it appeared possible to deflect rays in a beam and thereby change the beam phase \(\theta_j\) by a radian, and therefore the amplitude \(A(x)\) by itself by a rearrangement of the turbules along the initial part of the ray tube where \(|\Delta x(z)| \ll a\). The time required for a turbulent motion with velocity \(v_t\) to effect the rearrangement may be estimated (incorrectly) as \(\sim (a/v_t\theta_L)(\kappa L)^{-\frac{1}{2}} \exp(-\frac{\kappa L}{2})\). This is apparently much shorter than the thin screen time scale even for \(v_t \ll u\), if \(\kappa L \gg 1\). Below it is shown that this estimate is invalid and that a correct formula is identical to that for the thin screen apart from a numerical factor.

Consider ray motion in a medium having index variations in both space and time, \(\mu = \mu(x,y,z,t)\). We assume \(\mu(x,t)\) a homogeneous stationary random variable with auto-correlation \(\varrho_N(x,t)\), where \(\varrho_N\) is the auto-correlation of the electron number density given by equation (14), because \(\mu\) is proportional to \(\delta N\) by equation (19).
For simplicity of the ensuing formulae let us consider ray motion in just one-dimension so that \( \dot{X} = (X, 0) \), etc. Let \( X(z,t) \) denote the transverse ray coordinate at time \( t \). Then the variable of interest here is the ray separation \( \Delta X \) between initially \( (z=0) \) nearby rays at instant \( t \) and \( t+\tau \):

\[
\Delta X(z,t) = X(z,t) - X'(z,t+\tau)
\]

\[
X(z,t) = \frac{1}{2} \Delta X_0 + \frac{i}{\lambda} \Delta \Theta_0 z + \int_z^z \frac{dz' (z-z')}{\lambda} \frac{2}{\partial x} \mu [X(z'_t), z'; t]
\]

\[
X'(z,t+\tau) = -\frac{1}{2} \Delta X_0 - \frac{i}{\lambda} \Delta \Theta_0 z + \int_z^z \frac{dz' (z-z')}{\lambda} \frac{2}{\partial x} \mu [X(z'_t+\tau), z'; t+\tau]
\]

where \( \Delta X(z=0, t) = \Delta X_0 \) and \( d/\partial z \Delta X(z=0, t) = \Delta \Theta_0 \). If we assume \( |\Delta X(z,t)| < a \) and \( \tau < \tau_o \), where \( \tau_o \) is some characteristic correlation time of \( \Theta(\tau, x) \), then we may Taylor expand the difference of the second and third equations in (324) as in the derivation of (275). One finds,

\[
\Delta X(z,t) = \Delta X_0 + \Delta \Theta_0 z + \int_z^z \frac{dz' (z-z')}{\lambda} \left( \Delta X(z'_t) \frac{2}{\partial x} + \gamma \frac{2}{\partial t} \right) \mu \left[ \frac{2}{\partial x} \mu [X(z'_t), z'; t] \right)
\]

\[
\bar{X}(z,t) = \frac{1}{2} \left[ X(z,t) + X'(z,t+\tau) \right]
\]
By twice differentiating (325) one finds,

\[ \frac{d^2}{dz^2} \Delta X(z,t) = \Delta X(z,t) \mu_{xx} [\bar{x},z,t] + \zeta \mu_{xt} [\bar{x},z,t] \] (326)

where \((..)_{xt} = \frac{\partial^2}{\partial x \partial t} (..)\), etc. It is evident that the finite interval \(\zeta\) has the effect of contributing a forcing term to (326).

Invoking the prime assumption of (§2) regarding the average of the square of equations such as (325) one obtains,

\[ \begin{aligned}
\langle \Delta X'^2(z) \rangle &= \Delta X'_o^2 + 2 \Delta X'_o \Delta \theta'_o z + \Delta \theta'_o^2 z^2 + \\
&+ \frac{1}{2} \kappa^3 \left( \int_0^z dz' (z-z')^2 \langle \Delta X'^2(z') \rangle \right) + \\
&+ \frac{1}{6} (\kappa z)^3 \langle v'_t \zeta \rangle^2 
\end{aligned} \] (327)

which is the finite \(\zeta\) analogue of equation (279). For \(\zeta = 0\), equation (327) is identical to (279). \(v'^2_t\) is the mean square turbulent velocity defined by

\[ v'^2_t \equiv \int_0^\infty dz \left[ \frac{\partial^4}{\partial x^2 \partial t^2} \rho_N(x,z,t) \right] \bigg|_{x=0}^{x=0} \int_0^z dz \left[ \frac{\partial^4}{\partial x^4} \rho_N(x,z,t) \right] \bigg|_{x=0}^{x=0} \] (328)

and \(\kappa^3\) is defined by

\[ \kappa^3 \equiv 4 \mu^2 \int_0^\infty dz \left[ \frac{\partial^4}{\partial x^4} \rho_N(x,z;0) \right] \bigg|_{x=0} \] (329)
By thrice differentiating (327) one gets an equivalent differential equation,

\[
\frac{d^3}{dz^3} \langle \Delta X^2(z) \rangle - \kappa^3 \langle \Delta X^2(z) \rangle = \kappa^3 \nu_t^2 \zeta^2 \tag{330}
\]

with the boundary conditions,

\[
\begin{aligned}
\langle \Delta X^2(z=0) \rangle &= \Delta X_0^2 \\
\frac{d}{dz} \langle \Delta X^2(z=0) \rangle &= 2 \Delta X_0 \Delta \Theta_0 \\
\frac{d^2}{dz^2} \langle \Delta X^2(z=0) \rangle &= 2 \Delta \Theta_0^2 
\end{aligned} \tag{331}
\]

The general solution to (330) may be written as the sum of that to the homogeneous equation and the particular solution \( \langle \Delta X^2(z) \rangle_p = -\nu_p^2 \zeta^2 \). By matching the boundary conditions (331), the exact solution may be obtained. However, for \( \kappa z \gg 1 \) the leading exponentially large terms are:

\[
\langle \Delta X^2(z) \rangle = \frac{1}{3} \left( \Delta X_0^2 + \frac{2 \Delta X_0 \Delta \Theta_0}{\kappa} + \frac{2 \Delta \Theta_0^2}{\kappa^2} + \nu_t^2 \zeta^2 \right) e^{\kappa z} \tag{332}
\]

Analogous to the development in (§4) it is useful for understanding the ray motion to write out the instan-
taneous dependence of \( \Delta X(z,t) \) on the boundary values \( \Delta X_0 \) and \( \Delta \Theta_0 \), and the time interval \( \zeta \).

\[
\begin{pmatrix}
\Delta X(z,t) \\
\Delta \Theta(z,t)
\end{pmatrix} = \begin{pmatrix}
\ell_1'(z,t) & \ell_2'(z,t) \\
\ell_3'(z,t) & \ell_4'(z,t)
\end{pmatrix} \begin{pmatrix}
\Delta X_0 \\
\Delta \Theta_0
\end{pmatrix} + \zeta \begin{pmatrix}
d_1'(z,t) \\
d_2'(z,t)
\end{pmatrix} \tag{333}
\]
The $b'_j$ and $d'_j$ coefficients are given by the following integral equations,

\[
\begin{align*}
    b'_1(z,t) &= 1 + \int_0^z dz' (z-z') b'_1(z',t) \mu_{xx} \left[ \overline{x}(z',t), z'; t \right] \\
    b'_2(z,t) &= z + \int_0^z dz' (z-z') b'_2(z',t) \mu_{xx} \\
    b'_3(z,t) &= \int_0^z dz' b'_1(z',t) \mu_{xx} \\
    b'_4(z,t) &= 1 + \int_0^z dz' b'_3(z',t) \mu_{xx} \\
    d'_1(z,t) &= \int_0^z dz' (z-z') \mu_{xt} + \int_0^z dz' (z-z') d'_1(z',t) \mu_{xx} \\
    d'_2(z,t) &= \int_0^z dz' \mu_{xt} + \int_0^z dz' d'_1(z',t) \mu_{xx}
\end{align*}
\] (334)

The typical or r.m.s. value of the $b'_j$'s or $d'_j$'s may be read off of equation (332) and the analogous formula for the mean square angle difference $\langle \Delta \theta^2(z) \rangle$. For example, $b'_j \approx 3^{-\frac{1}{2}} \exp(\frac{1}{2} \kappa z)$ and $d'_j \approx 3^{-\frac{1}{2}} v \exp(\frac{1}{2} \kappa z)$ for $\kappa z \gg 1$.

The $b'_j$'s and $d'_j$'s of (333) and (334) are not all independent. First we have $J_1 = b'_1 b'_4 - b'_2 b'_4 = 1$ as before in (34) corresponding to Liouville's theorem. In addition, one may show that the Jacobian $J_2 = d'_1 b'_4 - d'_2 b'_4$ is given by

\[
J_2(z,t) = \int_0^z dz' b'_2(z',t) \mu_{xt}
\] (335)

For an estimate of the magnitude of $J_2$ take the r.m.s., $\langle J_2^2(z) \rangle^{\frac{1}{2}} = 2^{-\frac{1}{2}} \kappa v t \langle b'_2(z) \rangle^{\frac{1}{2}}$ for $\kappa z \gg 1$. $J_2$ is found to be of interest below.
We now have the requisite mathematics for determination of the auto-correlation time for the wave amplitude received through the thick screen. A general condition for strong or almost unit auto-correlation between the wave amplitude at time \( t \) at a point on \( z = L \) with that at a later instant \( t + \gamma \) at the same spatial point is that for each beam at time \( t \) a different beam at time \( t + \gamma \) can be found such that the optical path fluctuations along the two beams is less than say a radian. The relative configuration of rays constituting the two beams is found by inspection.

As in (64) it is useful to consider the rays parameterized by the backwards \( z \) direction, \( \zeta = L - z \). Let \( X_R(\zeta,t) \) denote a reference ray for which \( \Theta_R(\zeta=L,t)=0 \) and \( X_R(\zeta=L,t) = \) arbitrary. Then consider all other rays at a later instant \( t + \gamma \) which have \( X(\zeta=0,t+\gamma) = X_R(\zeta=0,t) \), but with \( \Theta(\zeta=0,t+\gamma) = -d/d\zeta X(\zeta=0,t+\gamma) = \Theta_R(\zeta=0,t)+ \Delta \Theta_L \), with \( \Delta \Theta_L \) unspecified for the moment. The equations governing \( \Delta X(\zeta,t) = X(\zeta,t+\gamma) - X_R(\zeta,t) \) and \( \Delta \Theta(\zeta,t) = \Theta(\zeta,t+\gamma) - \Theta_R(\zeta,t) \) are:

\[
\begin{align*}
\Delta X(\zeta,t) &= \phi_2(\zeta,t) \Delta \Theta_L + d_1(\zeta,t) \gamma \\
\Delta \Theta(\zeta,t) &= \phi_4(\zeta,t) \Delta \Theta_L + d_2(\zeta,t) \gamma
\end{align*}
\] (336)

These are the backwards analogue of (333).

We want to vary \( \Delta \Theta_L \) in order to obtain columnated rays on \( z = 0 \); that is, \( \Delta \Theta(\zeta=L,t) = 0 \). From (336) this implies,
\[ \Delta X(\xi = L, t) = \frac{J_2(\xi = L, t)}{b_4(\xi = L, t)} \cdot \xi \]  

For typical values of \( J_2 \) and \( b_4 \) one finds \( \Delta X(\xi = L, t) \approx \xi v_t \xi \), for \( KL \gg 1 \). Thus for finite \( \xi \) the ray separation is not exponentially large in \((KL)\). In fact, \( v_t \xi \) is roughly the transverse distance a ray is 'convected' in the case of uniform motion of a thin screen at velocity \( v_t \).

For the above pair of rays one may evaluate the mean square phase difference. The time interval \( \xi = \xi_L \) for which the difference is say a radian is the amplitude autocorrelation time. One finds,

\[ \xi_L \sim \frac{a}{v_t \phi_L} \]  

Aside from an unknown numerical factor (338) agrees with the time scale for the case of a thin screen having an r.m.s. phase shift \( \phi_L \) for observations at a distance \( L \gg \ell_L \) beyond the screen.
§ 6. Wavelength Dependence of Amplitude:

Analogous to the development in (§ 5) one may derive equations governing the separation of two rays, one at wavelength \( \lambda - \frac{\Delta \lambda}{2} \) and the other at \( \lambda + \frac{\Delta \lambda}{2} \), with \( \frac{\Delta \lambda}{\lambda} \ll 1 \). The characteristic wavelength \( \Delta \lambda \) of the amplitude is that which gives path length fluctuations along the two rays of say a radian.

As in (§ 5) it is convenient to simplify the formulae by assuming one-dimensional ray motion, \( \mathbf{x} = (X, 0) \), etc. The equation for \( \Delta X(z, \lambda) = X(z, \lambda + \frac{\Delta \lambda}{2}) - X(z, \lambda - \frac{\Delta \lambda}{2}) \) is obtained by noting that \( \mu \ll \lambda^2 \) from equation (19). One finds,

\[
\Delta X(z, \lambda) = \Delta X_o + \Delta \Theta_o z + \left( \frac{z}{\o} \right) \Delta X(z', \lambda) \mu \left[ X(z', \lambda), \frac{z}{\lambda} \right] - 2 \frac{\Delta \lambda}{\lambda} \left( \frac{z}{\o} \right) \mu \left[ X(z', \lambda), 1 \right] \tag{339}
\]

\[
\bar{X}(z, \lambda) = \frac{1}{2} \left[ X(z, \lambda + \frac{\Delta \lambda}{2}) + X(z, \lambda - \frac{\Delta \lambda}{2}) \right]
\]

Averaging (339) according to the prime assumption of (§ 2) one obtains,

\[
\left\langle \Delta X^2(z) \right\rangle = \Delta X_o^2 + 2 \Delta X_o \Delta \Theta_o z + \Delta \Theta_o^2 z^2 + \frac{1}{\o} \frac{z}{\o} \left( \Delta \Theta_o z \right)^2 \left\langle \Delta X^2(z') \right\rangle + \frac{1}{6} \left( \frac{\Delta \lambda}{\lambda} \right)^2 \left( \kappa z \right)^3 \frac{z^3}{\o^2} \tag{340}
\]
where $K^3$ is given by (329) and the scale size $a_\lambda$ by,

$$a_\lambda^2 = -4 \frac{\int_0^\infty dz' \left[ \frac{\partial^2}{\partial x^2} \xi_N(x, z') \right]_{x=0}}{\int_0^\infty dz' \left[ \frac{\partial^4}{\partial x^4} \xi^*_N(x, z') \right]_{x=0}}$$  \hspace{1cm} (341)

One immediately notices that equation (340) is exactly of the form of (327) for $\langle \Delta x^2(z) \rangle$ in the case of time variations of the amplitude under the replacement $a_\lambda \rightarrow v_t \tau$. Therefore, we may use the ray separation of (65) for the present problem of wavelength correlation.

The path length phase difference is here not given by equation (299), because there is an additional contribution due to the explicit wavelength dependence. Replacing (299) is:

$$\Delta \phi = k \int_0^z \Delta \theta(z') \overline{\theta}(z') + \frac{3 \Delta \lambda}{\lambda} k \int_0^z \Delta \theta(z') \overline{\theta}(z')$$  \hspace{1cm} (342)

where $\overline{\theta}(z)$ is given by (299) at wavelength $\lambda$, and $\Delta \theta(z)$ is the angle difference between the rays at $\lambda + \frac{\Delta \lambda}{2}$ and $\lambda - \frac{\Delta \lambda}{2}$. Introducing in (342) the ray separation of (65) one finds that the second term on the right dominates for $\chi L \gg 1$. The second term set to unity gives,

$$\frac{\Delta \lambda L}{\lambda} \sim \left( k \int \langle \theta^2(L) \rangle \right)^{-1}$$  \hspace{1cm} (343)
Aside from an unknown numerical factor (343) agrees with the characteristic wavelength $\Delta \lambda_L$ for the case of a thin screen having a mean-square scattering angle $\langle \Theta^2(L) \rangle$ (or phase shift $\phi_L$ and scale size $a$) for observations a distance $L \gg \ell_L$ beyond the screen.
C. Wave Optics of the Thick Screen:

The previous section dealt with the transition in behavior of geometrical optics in going from \( z < \ell_z \) to the thick screen \( z > \gg \ell_z \). Nearby rays were found to spread apart exponentially with propagation distance in the thick screen. The thick screen wave amplitude was discussed in terms of exponentially spreading beams.

In this section a model for the thick screen is developed. It consists of an arbitrary number (\( N \)) of thin parallel phase changing screens separated (for convenience) by equal distances. The mean-square phase fluctuations introduced by one of the thin screens is identified with \( \frac{\phi^2}{\ell} / N \) of the continuous medium. The model is not equivalent to the thick screen; however, for a sufficiently large number of screens it does show all important features of the geometrical optics of the thick screen. In particular, the ray-displacement, ray-phase, and separation of nearby rays have dependences of the same form as for the thick screen. The model is valuable because it permits derivation of exact expressions relevant to certain aspects of the thick screen wave amplitude. Closed formulae may be derived for different amplitude correlations: (a) that between the wave fields received from two distant point sources separated by a small angle; (b) that between the wave field at one instant and that at a later instant; and (c) that between the field at one wavelength and that at another. The results for (a)-(c)
indicate no strong thick screen effects. Alternatively, effects (a)-(c), apart from numerical factors, are the same in their dependences as for the situation where the thick screen is collapsed to a thin screen with mean-square phase shift $\theta_L^2$ for observations at a distance $L \gg \ell_L$ beyond the thin screen. This was the conclusion tortuously extracted from the ray-optics theory in Section 2. However, the model discussed below provided the first theoretical proof of equivalence for (a)-(c) of a thick screen to a thin screen. The model led to abandonment of the early thick-screen conjectures mentioned in Section (3).

Experimental evidence that the angle dependence of the thick screen wave amplitude (item (a) above) was not significantly different from that of a thin screen was first obtained in unpublished work by Cole and Hewish (1967). The work by Cole and Hewish preceded our solutions to the multiple screen model. In fact, the design of their experiment, which consisted of a light wave passed through a number of thin parallel layers of plastic with known thickness variations, suggested our model. It is regrettable that Cole and Hewish have not published a description of their experiment and observations.

Even with no strong thick screen effects on (a)-(b) above, we maintain that certain distinctive features occur for wave propagation in the domain $z \gg \ell_z$. This conflicts with Uscinski (1968) who concludes that no difference in wave properties are to be expected for $z \gg \ell_z$ compared
with the case where the thick medium is collapsed to a thin layer with the same mean-squared phase shift. Uscinski it may be noted gives a rigorous evaluation of the complex amplitude correlation. Then by an assumption "out of the blue" regarding the statistics of the amplitude (his equation 2.3) the "complete" solution to the thick screen is obtained. The results of the present Section agree with those of Uscinski for the amplitude auto-correlation. But Uscinski's assumption on the amplitude statistics in our opinion has no basis.

At the end of this Section we discuss a situation where an effect unique to the thick screen could be observed. The spreading of an initially columnated narrow beam of radiation is expected to show a strong dependence on $l/l_v$.

Subsection (§1) below describes the model, the geometrical optics results for it, and the spatial amplitude auto-correlation. Subsection (§2) treats the cross correlation of wave fields received from two sources separated by a small angle; (§3) treats the time auto-correlation of the amplitude; and (§4) treats the cross wavelength correlation. Subsection (§5) discusses the spreading of a narrow beam.
§1. Multiple Thin Screen Model:

The geometry of the multiple thin screen model is shown in Figure(25). For convenience the layers are assumed identical in thickness and statistical properties, although different layers are assumed statistically independent. The distance between each of the N layers is \( d \), and \( L = (N-1)d \) is the distance through the stack. The thickness of a single layer, \( \varepsilon \), is assumed sufficiently small for each layer to act as a thin phase changing screen. For this to be the case \( \varepsilon \) must be smaller than the shortest focal-length \( \ell_z \) encountered, which occurs near \( z \sim L \). Because in the thick screen limit \( \ell_z \ll z \sim L \), most of the volume between \( z = 0 \) and \( L \) is vacuum in contrast with the continuous medium of the prior section. This is the reason why the model is not equivalent to a thick screen.

The effect of a layer, say the jth, on an arbitrary incident wave (possibly constituted of many beams) is simply multiplication by \( \exp( i \phi_j^-(x) ) \), where \( \phi_j^-(x) \) is the phase shift imparted by the jth layer. If \( A_j^-(x) \) denotes the wave incident on the jth screen and \( A_j^+(x) \) the emergent wave, then

\[
A_j^+(x) = A_j^-(x) \exp[ i \phi_j^-(x) ]
\]  

(344)

where the phase shift is given by

\[
\phi_j^-(x) = k \int_{jz}^{jz + \varepsilon} dz' \mu_j^-(x, z')
\]  

(345)

The \( \phi_j^-(x) \) are assumed in general to be statistically homo-
FIGURE 25: Multiple Thin Screen Model.
geneous random variables.

For convenience in calculations we sometimes assume that \( \phi_j(x) \) has the characteristics of the thin-screen phase (equation 34) in the case of a Gaussian irregularity spectrum:

\[
\begin{align*}
\langle \phi_j(x) \phi_j(x+r) \rangle &= \phi_o^2 \mathcal{N}(r) \\
\langle \exp \left( i \phi_j \right) \rangle &= \exp \left( -\frac{i}{2} \phi_o^2 \right)
\end{align*}
\]

(346)

where \( \phi_o \) is the r.m.s. phase-shift imparted by a single layer. \( \phi_o \) is not required to be less than unity.

The statistical independence of the layers is expressed by

\[
\langle \exp \left( i \phi_j + i \phi_\ell \right) \rangle = \langle \exp \left( i \phi_j \right) \rangle \langle \exp \left( i \phi_\ell \right) \rangle
\]

(347)

and in particular

\[
\langle \phi_j \phi_\ell \rangle = \langle \phi_j \rangle \langle \phi_\ell \rangle
\]

for \( j \neq \ell \). These relations are independent of the irregularity spectrum.

Geometrical optics as applied to a thin screen has been treated by Salpeter(1967). A necessary condition for applicability is \( \phi_o \gg 1 \) in the case of Gaussian spectra, which we assume in the discussion below. The angle relative to the z-axis through which a ray is deflected on passing through the jth screen is:
\[ \Theta_j^\mu = \frac{1}{k} \nabla \phi_j^\mu (x) \] (348)

Thus the angle of the ray emergent from the final, Nth, screen is:

\[ \Theta_N^\mu [x_N] = \frac{1}{k} \sum_{j=1}^{N} \nabla \phi_j^\mu [x_j^\mu] \] (349)

where \( x_N^\mu \) is the transverse ray-coordinate. The mean square angle is given by,

\[ \langle \Theta_N^2 \rangle = \frac{1}{k^2} \sum_{j=1}^{N} \sum_{l=1}^{N} \langle \nabla \phi_j^\mu \cdot \nabla \phi_l^\mu \rangle \]

\[ = \frac{1}{k^2} \sum_{j=1}^{N} \langle (\nabla \phi_j^\mu)^2 \rangle \]

\[ = \frac{N}{k^2} \phi_0^2 \left[ - \nabla^2 s_N(x) \right]_{x=0} \]

\[ = \frac{2N}{k^2} \phi_0^2 \left[ - \frac{1}{r} \frac{\partial}{\partial r} s_N(r) \right]_{r=0} \{ \text{isotropic} \} \]

\[ = \frac{2N}{k^2} \phi_0^2 a^{-2} \left( \text{Gaussian } s_N \right) \] (350)

where the first step utilizes the statistical independence of different screens and the second that the screens are statistically identical. If we introduce the definitions \( \phi_L^2 = N \phi_o^2 \), \( z_o = k a^2 \), and \( \ell_L = z_o / \phi_L \), then
\[
\left\langle \Omega_N^2 \right\rangle = 2 \left( \frac{a}{\ell_L} \right)^2 \quad \text{(Gaussian } S_N \text{)} \quad (351)
\]

With these definitions the mean-square scattering angle \(351\) for the model is identical to that for the continuous medium, equation \(248\).

The transverse displacement of a ray emergent from the \(N\)th screen is:
\[
X_N = d \sum_{j=1}^{N} (N-j) \left\langle \theta_j \right\rangle \left( \frac{X_j}{\ell_j} \right)
\]
(352)

Where the initial coordinate is \(X_1 = 0\). The mean-square displacement is:
\[
\left\langle X_N^2 \right\rangle = d^2 \sum_{j=1}^{N} (N-j)^2 \left\langle \theta_j^2 \right\rangle
\]
\[
= \frac{2}{3} \left( \frac{L}{\ell_L} \right)^2 a^2 \frac{N-\frac{1}{3}}{N-1} \quad \text{(Gaussian } S_N \text{)} \quad (353)
\]

The ray-displacement \(353\) is thus identical to that for the continuous medium (equation 255) for \(N > 1\).

The phase delay along a ray is again split into two parts, \(\Phi = \Phi_1 + \Phi_2\), where \(\Phi_1\) is due to index variations along the ray path and \(\Phi_2\) is due to the geometrical path length only.
\[
\begin{align*}
\phi_1 &= \sum_{j=1}^{N} \phi_j(x_j) \\
\phi_2 &= \frac{1}{2} \sum_{j=1}^{N} k d \left( \sum_{l=1}^{L} \tilde{m}_l(x_l) \right)^2
\end{align*}
\]

(354)  

(355)

The mean-square of \( \phi_1 \) is \( N\theta_o^2 \) \( (= \phi_L^2) \). Without difficulty relations identical to equations (261) and (264) may be obtained for the path length phase \( \phi_2 \) for \( N > 1 \).

The transverse distance between two rays may be derived in a way similar to that for the continuous medium. The equations for two rays incident parallel on the first screen a distance \( \Delta x_o \) apart are:

\[
\begin{align*}
X_l' &= \Delta x_o/2 + \frac{d}{k} \sum_{j=1}^{L} (l-j)d \phi_j(x_j) \\
X_l' &= -\Delta x_o/2 + \frac{d}{k} \sum_{j=1}^{L} (l-j)d \phi_j(x_j)
\end{align*}
\]

(356)

For simplicity assume that the two rays stay closer together than a correlation length \( a \). Define \( \Delta x_j = x_j - x_j' \). Then

\[
\Delta X_l = \Delta x_o + \frac{d}{k} \sum_{j=1}^{L} (l-j) \Delta x_j \cdot \sum_{j=1}^{L} \phi_j(x_j)
\]

(357)

Squaring averaging (357) according to the prime assumption of Section B(\( \xi^2) \), one gets
\( f_{\ell} = 1 + \frac{1}{\ell^2} \, \kappa^3 \sum_{j=1}^{\ell-1} (\ell-j)^2 f_j \) \hspace{1cm} (358)

where \( f_j = \frac{\langle \hat{A}_j \hat{A}_j \rangle}{\langle \hat{A}_0 \rangle} \) and

\[
\kappa^3 = \frac{d^2}{k^2} \phi_0^2 \left[ \nabla^2 \nabla^2 \phi_0 (x) \right]_{x=0}
= \frac{8d^2 \phi_0^2}{k^2} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \phi_0 (r) \right) \right]_{r=0} \quad (\text{isotropic } \phi_N)
= 8 \left( \frac{L}{L_N} \right)^2 \frac{1}{N(N-1)^2} \quad (\text{Gaussian } \phi_N) \hspace{1cm} (359)
\]

The general solution to (358) is a polynomial

\[
f_j = \sum_{\ell=0}^{j-1} a_j^\ell \left( \frac{\kappa^3}{2} \right)^\ell \quad ; \quad 1 \leq j \leq N \hspace{1cm} (360)
\]
of degree \((j-1)\) with \( a_0^j = 1 \) and \( a_{j-1}^j = 1 \), and all of the other coefficients positive. The first few solutions are: \( f_1 = 1 \); \( f_2 = 1 + 5(\kappa^3/2) + (\kappa^3/2)^2 \); and \( f_3 = 1 + 14(\kappa^3/2) + 9(\kappa^3/2)^2 + (\kappa^3/2)^3 \). A general expression for coefficients of the \( j \)th polynomial has not been found. However, by analogy with the connection between the integral equation (279) and the differential form (282) one may obtain a difference equation equivalent, with boundary conditions, to (358):
\[
\frac{f_{\ell+2}}{\ell+2} - 3 \frac{f_{\ell+1}}{\ell+1} + 3 \frac{f_{\ell}}{\ell} - \frac{f_{\ell-1}}{\ell-1} = \frac{\kappa^3}{\zeta^2} \left[ \frac{f_{\ell+1}}{\ell+1} + \frac{f_{\ell}}{\ell} \right] \quad (361)
\]

The boundary conditions may be taken as \( f_{\ell} = f_1, f_2, f_3 \).

Observe that equation (361) has exact solutions of the form \( f_{\ell} \propto \exp(\lambda_\kappa (\ell - 1)) \), where \( \lambda_\kappa \) are roots of

\[
\sinh \left( \frac{3}{2} \lambda_\kappa \kappa \right) / \cosh \left( \frac{1}{2} \lambda_\kappa \kappa \right) = \frac{\kappa^3}{\zeta^2} + 3 \tanh \left( \frac{\lambda_\kappa \kappa}{2} \right) \quad (362)
\]

For \( \kappa^3 \ll 1 \), the \( \lambda_\kappa \) roots are given by \( \lambda_1 = 1 \); \( \lambda_1 = 1 \), \( \lambda_2 = -\frac{1}{2} + i \frac{\sqrt{3}}{2} \), and \( \lambda_3 = -\frac{1}{2} - i \frac{\sqrt{3}}{2} \). Solutions in this form cannot satisfy the boundary conditions exactly, but the error is small if \( \kappa^3 \ll 1 \). Hence consider the solution,

\[
f_{\ell} = \frac{1}{3} \left[ e^{\kappa(\ell - 1)} + 2 e^{-\frac{1}{2} \kappa(\ell - 1)} \cos \left( \frac{\sqrt{3}}{2} \kappa(\ell - 1) \right) \right] \quad (363)
\]

This satisfies \( f_1 = 1 \) exactly, but \( f_{\ell} = f_2 \) and \( f_{\ell} = f_3 \) involve small errors, \( \sim \kappa^6 \) and \( \sim \kappa^9 \), respectively. These errors in the boundary values result in only a negligible error in formula (363) for \( \kappa^3 \ll 1 \). Therefore, for a sufficiently large number of screens, \( \kappa^3 \ll 1 \) implies \( N \gg \left( L/\ell_\kappa \right)^{2/3} \), and in the thick screen limit \( \kappa N \approx 2 \left( \frac{L}{\ell_\kappa} \right)^{2/3} \),

\[
\left\langle \Delta x^2 N \right\rangle = \frac{1}{3} \Delta x^2 N \exp(\kappa N) \quad (364)
\]
In this limit the multiple thin screen model implies exponential spreading of nearby rays. This corresponds exactly to the results of Section B ( §2) for the ray spreading in the case of a continuous medium. Hence we have reason to expect that the multiple screen model would exhibit any thick screen effects (or not) which occur for the continuous medium.

Below we evaluate the spatial amplitude auto-correlation for the case of a monochromatic incident plane wave. This is related, by a two dimensional Fourier transform, to the angle probability density of the emergent radiation. For the calculation assume that the irregularities of the individual screens are Gaussian or type A, but not type B or C. That is, assume that there are no position variations.

The wave amplitude at a distance \( d \) beyond the first \( j = 1 \) thin screen is given by Fresnel's integral:

\[
A_{-2}^-(x_2) = \left( \frac{k}{2 \pi d} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dx_1 e^{-i \frac{k}{2d} (x_2 - x_1)^2} e^{i \phi(x_1)} \quad (365)
\]

Just beyond the second screen the amplitude is \( A_{-2}^+(x_2) = A_{-2}^-(x_2) \cdot \exp(i \phi_{-2}(x_2)) \). In this fashion one may obtain an expression for the amplitude just beyond an arbitrary number \( N \) of layers:
\[ A_N^+ (x_{\infty}^N) = \left( \frac{k}{2\pi d} \right)^{N-1} \int_{-\infty}^{\infty} \cdots d_{x_1} \cdots d_{x_{N-1}}^N e^{-i \frac{k}{2d} \sum_{j=1}^{N} (x_j - x_{j-1})^2} \sum_{j=1}^{N} \phi_j (x_j) \] (366)

From (366) one may express the amplitude auto-correlation as,

\[ \langle A_N^+ (x_{\infty}^N) A_N^+ (x_{\infty}^N) \rangle = \left( \frac{k}{2\pi d} \right)^{2N-2} \int_{-\infty}^{\infty} \cdots d_{x_1}^N d_{x_{N-1}}^N d_{x_2}^N \cdots d_{x_N}^N \langle \exp \left\{ -i \frac{k}{2d} \sum_{j=2}^{N} (x_j - x_{j-1})^2 (x_j'^2 - x_{j-1}^2) \right\} \rangle \] (367)

Because of the statistical independence of the \( \phi_j \)'s for different values of the index we have

\[ \langle \exp \left\{ i \sum_{j=1}^{N} (\phi_j (x_j) - \phi_j (x_j')) \right\} \rangle = \prod_{j=1}^{N} \mathcal{Q}_j \left( x_j - x_j' \right) \] (368)

\[ \mathcal{Q}_j \left( x-x' \right) = \langle \exp \left\{ i \phi_j (x) - i \phi_j (x') \right\} \rangle \]

Equation (368) utilizes the assumption that the layers are identical in statistical properties and that the \( \phi_j (x) \) are homogeneous in \( x_{\infty} \).
Because of equation (368) it is possible to evaluate (367) in closed form. For this purpose introduce the coordinates

\[ r_j^* = x_j - x'_j \quad ; \quad R_j^* = \frac{1}{\alpha} (x_j + x'_j) \quad (369) \]

Then

\[ V_N(x_N) \equiv \langle A_N^t (x'_N) A_N^* (x_N) \rangle \]

\[ = \left( \frac{\kappa}{\pi d} \right)^{2N-2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d_1 d_2 \cdots d_N d_{N-2} \exp \left( -i \sum_{j=2}^{N} \frac{\kappa}{\alpha (j-1)} (R_j - R_{j-1}) (R_j - R_{j-1}) \right) \prod_{j=1}^{N} g(x_j) \quad (370) \]

The integrations over the \( R_j \) are all trivial and one finds,

\[ V_N(x) = \left[ g(x) \right]^N \quad (371) \]

Expression (371) is a reflection of the fact that the scattering angle of radiation emergent through the \( (N) \) layers is a sum of the independent angles contributed by each layer. In such a situation the probability density of the sum is given by the Fourier transform of the product of the characteristic functions (here the amplitude auto-correlation functions) for the different random variables constituting the sum. The angle density which corresponds to (371) is given by
\[ W_N(\theta) = \left( \frac{k}{2\pi} \right)^2 \int_{-\infty}^{\infty} r^2 \ e^{-ik\hat{r} \cdot \hat{r}} \left[ \mathcal{C}(\mathbf{r}) \right]^N \]  

(372)

This is the same as relation (58) for the case of a thin screen and Gaussian or type A irregularities.

For Gaussian or type A irregularities the amplitude auto-correlation of a single screen is: \( g(\mathbf{r}) = \exp\left( -\frac{1}{2} \mathcal{C}_0^2(\mathbf{r}) \right) \), where \( \mathcal{C}_0(\mathbf{r}) \) is the r.m.s. phase difference for a single screen. Hence the auto-correlation of the wave amplitude emergent through \( N \) screens is:

\[ V_N(\mathbf{r}) = \exp\left\{ -\frac{1}{2} N \mathcal{C}_0^2(\mathbf{r}) \right\} \]  

(373)

For a Gaussian spectrum, equation (373) has the form

\[ V(\mathbf{r}) = \exp\left( -\mathcal{C}_0^2(1 - \mathcal{C}_N(\mathbf{r})) \right) \], with \( \mathcal{C}_L^2 = N\mathcal{C}_0^2 \). Thus for \( \mathcal{C}_L^2 \gg 1 \), the approximation \( 1 - \mathcal{C}_N(\mathbf{r}) \approx \frac{\mathcal{C}_0^2}{2a^2} \) in \( V(\mathbf{r}) \) is valid. Insertion of this \( V \) in (372) yields an angle density identical to equation (251) derived for the continuous medium:

\[ W(\theta) = \left( 2\pi \mathcal{C}_0^2 \right)^{-1} \exp\left( -\frac{\mathcal{C}_0^2}{2} \right), \text{ where } \theta_0 = \left( a/2\pi a \right) \mathcal{C}_L. \]

For a type A spectrum, \( \mathcal{C}_0(\mathbf{r}) = C_1 \mathcal{B}_0 \left( \mathbf{r}/r_0 \right) \) with \( 0 < \delta < 1 \), for a single screen, and thus (373) takes the form

\[ V(\mathbf{r}) = \exp\left( -\frac{1}{2} C_1^2 \mathcal{B}_0^2 \left[ \frac{r}{r_0} \right]^{2\delta} \right), \text{ where } \mathcal{B} = N^{1/2} \mathcal{B}_0 \text{ is the definition of } \mathcal{B} \text{ for the stack of } N \text{ screens}. \]

The type A angle density for the stack is of the form of formula (67) for a thin screen; the typical scattering angle is:

\[ \theta_0 = \left( \frac{2\pi}{C_1} \right)^{\frac{1}{2}} \mathcal{B}_0^{\frac{1}{2}} \left( kr_0 \right)^{-1}. \]

\# This is not a unique definition because of the arbitrariness of the length scale \( r_0 \).
§2. Angle Dependence of Amplitude:

Here we are interested in the cross correlation between the wave amplitudes received from two distant point sources separated by a small angle $\chi_m$. Previously, in Part B (§4), we discussed this correlation in terms of the exponentially spreading beams of the ray optics theory. However, the main conclusion of (B; §4), that the angle dependence for the thick screen was essentially the same as for a thin screen at a distance $L$, was found with the multiple screen model from the analysis below.

Let $\exp(i kz)$ represent the incident monochromatic amplitude from one source, and $\exp(ikz + ik \chi \cdot \mathbf{x})$ the incident amplitude from the second. Then the amplitude from the first source emergent through $N$ layers is given by (366). The corresponding amplitude for the second source is:

$$A_N^+(x_m; \chi_m) = \left(\frac{k}{2 \pi d}\right)^{N-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left[i \frac{k}{2} \chi \cdot \mathbf{x} \right] \exp \left[-i \frac{k}{2d} \sum_{j=2}^{N} (x_{m,j} - x_{m,j-1})^2 \right] \exp \left[i \sum_{j=1}^{N} \Phi_j(x_{m,j}) \right]$$

(375)

Following the steps in the evaluation of the amplitude auto-correlation, one may obtain the cross correlation,

$$\langle A_N^*(x'_N) A_N^+(x_N; \chi) \rangle = \exp \left[i \frac{k}{2} \chi \cdot \chi_N \right]$$

$$\exp \left[i \frac{k}{2d} \chi^2 \frac{N-1}{2} \right] \exp \left[i \sum_{j=1}^{N} \Phi_j (r_j - (j-1)\chi_d) \right]$$

(376)
The first factor on the right of (376) is the phase of the off-axis plane wave. The second factor is the geometrical path length delay. The product of the \( g \)'s describes the screen.

For an examination of the behaviour of the product of \( g \)'s assume a Gaussian or type A spectrum so that
\[
\exp\left(-\frac{1}{2} \mathcal{Q}_0^2(r)\right).
\]
Then the cross correlation at \( r = 0 \) is proportional to
\[
\prod_{j=1}^{N} g_j(\chi) = \exp\left\{-\frac{1}{2} \sum_{j=1}^{N} \mathcal{Q}_0^2(1j\chi)\right\}
\]
\[
\exp\left\{-N \int_{-\infty}^{\infty} d^2 q \Phi_0(q) \left[1 - \frac{e^{i\mathcal{Q}_0^2 \chi L/2 \sin(q \cdot \chi L/2)}}{q \cdot \chi L/2}\right]\right\}
\]
\[
(377)
\]
where in the last expression \( \Phi_0(q) \) is the wavenumber phase spectrum for a single screen, and we have assumed \( N \gg 1 \).

For sufficiently large \( |\chi| \) the factor in the square brackets in (377) is unity and the cross correlation is
\[
\exp(-N\mathcal{Q}_0^2) = \exp(-\mathcal{Q}_L^2),
\]
for a Gaussian spectrum. For \( \mathcal{Q}_L^2 \gg 1 \), this correlation is negligibly small. For much smaller \( \chi \) such that \( \mathcal{Q}_\infty \chi L \ll 1 \) we may Taylor expand the factor in square brackets, also for a Gaussian spectrum, to obtain
\[
\prod_{j=1}^{N} q_j (j \chi \phi) = \exp \left[ -\frac{1}{6} \chi^2 L^2 \phi^2 a_o^{-2} \right]
\]

\[
a_o^{-2} = \frac{1}{2} \frac{\int_{0}^{\infty} q^3 dq \overline{\Phi}(q)}{\int_{0}^{\infty} q^2 dq \overline{\Phi}(q)} \right) \tag{378}
\]

For an exact Gaussian \( \overline{\Phi}(q) \sim \exp(-q^2a^2/2) \), the definition of \( a_o \) gives \( a_o = a \). The cross correlation falls to \( e^{-1} \) for a separation \( \chi \) of,

\[
\chi_c = \sqrt{6} \frac{a}{L \phi_L} = \sqrt{6} \frac{\lambda}{2\pi \theta_S L} \tag{379}
\]

where \( \theta_S = \frac{\lambda}{2\pi a} \phi_L \) is the characteristic scattering angle for the stack of \( N \) screens. Formula (379) is compatible with equation (316) of the ray-optics theory.

Consider (377) for the case of a type A spectrum. The r.m.s. phase difference of a single screen may be written as \( \psi_0(\tau) = C_i \beta_0 (\frac{\tau}{r_0})^\delta \), where \( 0 < \delta < 1 \). From the first equation in (377) it is evident that the largest decorrelation arises from the largest \( j = N \); that is, from the screen most distant from the observer. The cross correlation may be written as,
\[ N \prod_{j=1}^{N} g(j \chi \delta) = \exp \left\{ -\frac{1}{\alpha} C_1^2 \beta^2 \left( \frac{L}{r_0} \right)^{2\delta} (\chi^2)^{\delta} \right\} \]

\[ \hbar_N^2 \equiv \frac{1}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^{2\delta} ; \quad \hbar_N^2 \leq 1 \]

where \( \beta = N^{1/2} \beta_0 \) is the definition of \( \beta \) for the stack of \( N \) screens\(^\#\). For \( N \gg 1 \), \( \hbar_N^2 = (2\delta + 1)^{-1} \). The angular separation for which the correlation (380) is reduced to \( e^{-1} \) is:

\[ \chi_e = \left( \frac{2}{C_1^2} \right)^{1/2} \beta^{1/2} \left( \frac{r_0}{L} \right) \hbar_N^{-1/2} \]

\[ = \left( \frac{2}{C_1^2} \right)^{1/2} \frac{\lambda}{2\pi \theta_5 L} \hbar_N^{-1/2} \]

where \( \theta_5 \) is the typical scattering angle of formula (67) appropriate to the stack of \( N \) screens as defined in (\( \delta \)).

\(^\#\) This is not a unique definition because the length scale \( r_0 \) is arbitrary.
§3. Time Dependence of Amplitude:

With the multiple screen model a simple description of turbulent motion of irregularities is possible: Consider the case where the phase fluctuations of individual screens are frozen, but where each screen has its own arbitrary transverse velocity, $v_j$. Assume that conditions, such as discussed in Part II (B; §7), hold in order for retarded time variations to be unimportant. Then the time dependent amplitude $A_N^+(x_N; t)$ is given by equation (306) by the replacement $\theta_j(x_j) \rightarrow \theta_j(x_j - v_j t)$. An elementary calculation gives the amplitude time auto-correlation as

$$
\langle A_N^+(x_N, t) A_N^+(x_N, t+\tau) \rangle = \prod_{j=1}^{N} g(v_j \tau) \quad (382)
$$

For a Gaussian spectrum with $\phi_L^2 \gg 1$, equation (382) allows the approximation,

$$
\prod_{j=1}^{N} g(v_j \tau) = \exp \left[ -\frac{1}{2} \tau^2 \phi_L^2 a_o^{-2} \langle v^2 \rangle \right] \quad (383)
$$

$$
\langle v^2 \rangle \equiv \frac{1}{N} \sum_{j=1}^{N} v_j^2 \quad a_o = \text{equation (378)}.
$$

This is compatible with the ray-optics result, equation (338).

For a type A spectrum equation (382) gives,

$$
\prod_{j=1}^{N} g(v_j \tau) = \exp \left[ -\frac{1}{2} c_2 \beta^2 \left( \frac{c}{c_o} V_{Av} \right)^{2\delta} \right] \quad (384)
$$

$$
V_{Av} \equiv \left\{ \frac{1}{N} \sum_{j=1}^{N} |v_j|^{2\delta} \right\}^{1/2\delta}
$$

without approximation.
§4. Wavelength Dependence of Amplitude:

This subsection treats the correlation, \( \mathcal{N} \), between wave amplitudes at slightly different wavelengths, \( \lambda \) and \( \lambda' \) with \( \left| (\lambda - \lambda') / \lambda \right| \approx |\Delta \lambda / \lambda| \ll 1 \). For the validity of subsequent analysis it is necessary to restrict attention to Gaussian irregularities only. The reason for this is that for a type A spectrum, for example, the average shape \( \mathcal{Q}(\Delta t) \) of a narrow pulse received through a stack of \( N \) screens, which is related to \( \mathcal{N} \) by Fourier transform (equation 177a), has a simple form only if the large pulse arrival time fluctuations are first subtracted out. See Part II; B; §1 and §3.

Consider \( \mathcal{N}(\Delta k) = \langle A_N^+(x_N; k')A_N^{+x}(x_N; k'') \rangle \), where \( k' = K + \Delta k/2 \) and \( k'' = K - \Delta k/2 \) are the wavenumbers corresponding to the two wavelengths above and \( \Delta k/K = -\Delta \lambda / \lambda > 0 \). Introducing the relative coordinates \( x_j \) and \( R_j \) defined in (369) one finds,

\[
\mathcal{N}(\Delta k) = \left( \frac{K}{2 \pi d} \right)^{2N-2} \sum_{-\infty}^{\infty} \left( \prod_{i=2}^{\infty} \int R_1 \ldots \int R_{N-1} \int R_{N-1} \right) \exp \left\{ -i K \frac{1}{d} \left[ \sum_{j=2}^{N} (R_j - R_{j-1}) (x_j - x_{j-1}) \right] \right\} \exp \left\{ -i \Delta k \frac{1}{d} \left[ \sum_{j=2}^{N} \left( (R_j - R_{j-1})^2 + \frac{1}{2} (R_j - R_{j-1})^3 \right) \right] \right\} \left( \prod_{j=1}^{N} h_j(x_j) \right) h_j(x) = \langle \exp \left[ i \phi_j(x; k') - i \phi_j(x + r_j; k'') \right] \rangle
\]
For $(\Delta k/K)^2 \ll 1$, the $R_j$ integrals in (385) may be performed and one finds,

$$
\eta_N(\Delta k) = \left( \frac{K (1-i/\alpha)}{\sqrt{2\pi d \Delta k}} \right)^{2N-2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} r_1^2 \cdots r_{N-1}^2 e^{i \frac{K}{\Delta k} \sum_{j=2}^{N} (r_j-r_{j-1})^2} \prod_{j=1}^{N} h(x_j)
$$

(386)

* \exp\left( -\frac{1}{2} (\Delta k/K)^2 \varphi^2 \right) \exp\left( -\varphi^2 (1 - \varphi_N(r)) \right), \quad \text{for a Gaussian spectrum, where } \varphi \text{ is the r.m.s. phase shift of a single screen at radio wavenumber } K. \quad \text{Introducing the non-dimensional coordinates } \tilde{r}_j = K(\Delta k d)^{-1} r_j \text{ in (386) one sees that there are two possible characteristic fractional radio wavenumbers; the first factor of } h(r) \text{ substituted in (386) indicates a characteristic value } \Delta k/K \sim \varphi^{-1} L. \quad \text{The second factor in } h(r) \text{ gives } \Delta k/K \sim (kL \theta_s^2)^{-1} \text{ as the characteristic value, where } \theta_s = \frac{\lambda}{2\pi a} \phi_L \text{ is the scattering angle for the stack of screens. In the thick screen limit, } L \gg l_L, \quad kL \theta_s^2 \sim (L/l_L) \varphi_L \gg \varphi_L, \text{ and therefore (386) implies the characteristic fractional wavenumber:}

$$
\frac{\Delta k}{K} \sim (kL \theta_s^2)^{-1}
$$

(387)

This result agrees with expression (343) obtained from the ray optics theory.
§ 5. Spreading of a Narrow Beam:

Consider a wave of the form \( A_0(\hat{z}_N)\exp(ikz) \) incident on the stack of thin screens, where \( A_0(\hat{z}_N) \) is a known function which may describe a beam of incident radiation. This subsection discusses some effects the \( N \) screens may have on the beam amplitude.

Let \( S_N(\hat{z}_N, \hat{z}_N') = \left\langle A_N^*(\hat{z}_N) A_N^*(\hat{z}_N') \right\rangle \) denote the amplitude auto-correlation of the beam which emerges through \( N \) screens. Then,

\[
S_N(\hat{z}_N, \hat{z}_N') = \left( \frac{k}{2\pi d} \right)^{2N-2} \int_{-\infty}^{\infty} d\hat{x}_1 d\hat{x}_1' \ldots d\hat{x}_{N-1} d\hat{x}_{N-1}' \times \\
\times \exp \left\{ -i \frac{k}{2d} \sum_{j=2}^{N} \{ (\hat{x}_N - \hat{x}_{j-1})^2 - (\hat{x}_N' - \hat{x}_{j-1}')^2 \} \right\} \\
\times \prod_{j=1}^{N} g_j(\hat{z}_N - \hat{z}_N') \times A_0(\hat{z}_N) A_0^*(\hat{z}_N')
\]

(388)

Introducing the relative coordinates \( \hat{R}_j \) and \( \hat{r}_j \) defined in (369) one finds that all of the \( \hat{R}_j \) integrals with \( 1 < j \leq N-1 \) are trivial.

\[
S_N(\hat{R}, \hat{r}) = \left( \frac{k}{2\pi L} \right)^2 e^{-i k \hat{R} \cdot \hat{r} / L} \int_{-\infty}^{\infty} d\hat{x}_1 d\hat{x}_1' \times \\
\times \exp \left\{ -i \frac{k}{L} (\hat{x}_1 \cdot \hat{R}_1 - \hat{x}_1' \cdot \hat{R} - \hat{r} \cdot \hat{R}_1) \right\} \\
\times \prod_{j=1}^{N} g_j \left( \frac{1}{N-1} \{ (j-1) \hat{R}_1 + (N-j) \hat{r}_1 \} \right) \\
\times A_0(\hat{R}_1 + \hat{r}_1 / 2) A_0^*(\hat{R}_1 - \hat{r}_1 / 2)
\]

(389)
Observe that $\mathcal{I}_N(R, r = 0)$ is the average intensity of the beam. Let $\langle I(R) \rangle = \mathcal{I}_N(R, 0)$. Then from (389) the beam energy is conserved,

$$
\int_{-\infty}^{\infty} d^2 R \langle I(R) \rangle = \int_{-\infty}^{\infty} d^2 R \left| A_o(R, 1) \right|^2
$$

as it should be.

For an examination of equation (389) assume that the incident wave has a symmetric Gaussian form: $A_o(x) = \exp(-x^2/2a_B^2)$, where $a_B$ is the transverse scale of the incident beam. Then the average beam intensity may be written as,

$$
\langle I(R) \rangle = \left( \frac{k}{2\pi L} \right)^2 \pi a_B^2 \int_{-\infty}^{\infty} d^2 R \left\{ i \frac{k}{L} \cdot \frac{R}{R} \right\}_1 \ast \left[ \exp \left[ - \left( \frac{k^2 a_B^2}{4 L^2} + \frac{1}{4 a_B^2} \right) R^2 \right] \ast \prod_{j=1}^{N} g \left( \frac{N-j}{N-1} \cdot \frac{R}{R} \right) \right]
$$

Let us assume that the phase of a single screen is a Gaussian random variable and that it corresponds to a Gaussian spectrum of irregularities; then $g(r) = \exp(-\varphi_o^2(1 - \varphi_N^2))$. For $N\varphi_o^2 = \varphi_L^2 \gg 1$, the product of $g$'s in (391) is negligibly small unless $|\frac{r}{R_1}| \ll a$. Hence it is valid to approximate $g(r) \equiv \exp(-\varphi_o^2 r^2/2a^2)$. Thence the product of $g$'s is approximately $\exp(-1/6(\varphi_L^2 r^2/a^2))$.
for \( N \gg 1 \). Introducing this in (391) gives,

\[
\left\langle I(R) \right\rangle = \frac{a_B^2}{R_o^2} e^{\exp \left[ -\frac{R^2}{R_o^2} \right]}
\]

\[
R_o \equiv a_B \left[ 1 + \left( \frac{L}{Z_B} \right)^2 + \frac{2}{3} \frac{L^2}{Z_o Z_B} \phi_L^2 \right]^{1/2}
\]

\[
Z_B \equiv k a_B^2 \\
Z_o \equiv k a^2
\]

(392)

where \( Z_B \) is the Fresnel length for the beam and \( Z_o \) is the Fresnel length for a typical irregularity.

Some features of formula (392) are trivial: for zero phase shift \( \phi_L = 0 \) the beam widens by the factor \( (1 + (L/Z_B)^2)^{1/2} \) due to diffraction because of its finite width \( a_B \). In the geometrical optics limit, for sufficiently large \( Z_B \) and \( Z_o \), the beam's width is unchanged.

The screens have an important effect on the beam in the limit

\[
1 \ll \frac{L^2}{Z_o Z_B} \phi_L^2 \gg \left( \frac{L}{Z_B} \right)^2
\]

(393)

or

\[
\begin{align*}
(i) \phi_L^2 & \gg 1 \\
(ii) L/\ell_L & \gg a_B/a \\
(iii) a_B & \gg a/\phi_L
\end{align*}
\]

\[
\Rightarrow R_o = (2/3)^{1/2} (L/\ell_L)^{a}
\]

where \( \ell_L = ka^2/\phi_L \) is the focal length on \( z = L \). Inequality (i) was assumed in deriving (392), and it is a necessary
condition for strong scattering by the N screens. Inequality (ii) guarantees that \( R_0 \gg a_B \) for either \( a_B \gg a \) or \( a_B \ll a \). Inequality (iii) implies that the widening of the average beam by diffraction is less than that due to the screens \( \sim \lambda \pi /2 \pi a_B \ll \) scattering angle \( \sim (\lambda /2 \pi a) \theta_L \).

For \( a_B \gg a \), condition (ii) above indicates that \( L/L_L \gg 1 \) is necessary for strong alteration of the beam by scattering. The widening in this limit \( \sim L^{3/2} \) is that predicted by geometrical optics. Two rays a distance \( a_B > a \) apart wander independently transverse distances \( \sim (L/L_L)^a \), as discussed in Part A (§2).

For \( a_B \ll a \), we may have \( L/L_L \ll 1 \), by inequality (ii) above. \( L/L_L \ll 1 \) corresponds to rays of the beam having only small deflections and not crossing within the stack of screens. However, the case \( L/L_L \gg 1 \) corresponds to the thick screen where rays of the beam may cross within \( L \). In this limit, from (§1), nearby rays typically spread apart exponentially with propagation distance. Hence for a small enough initial beam width, \( a_B \ll 3^{1/2} a \exp (2(L/L_L)^{3/2}) \) the emergent beam's instantaneous width is typically \( \sim a \), according to equation (364) (and for an exact Gaussian spectrum). The instantaneous width may thus be much smaller than the average width of equation (392), \( R_0 \sim (L/L_L)a \gg a \). This comes about because the center of mass of a typical beam
is displaced by \( \sim (L/l_L) a \). The average \( \langle I(R) \rangle \) reflects the superposition of many realizations of beams, each of width \( \sim a \), but much larger transverse displacements.

For \( a_B^{3/2} a \exp(-2(L/l_L)^{2/3}) \), condition (iii) above requires that the r.m.s. phase shift be large enough for \( \Theta_L \gg \exp(2(L/l_L)^{2/3}) \). This is a requirement on the applicability of geometrical optics to narrow beams. A similar, slightly stronger, condition on \( \Theta_L \) may be gotten by requiring that the diffraction angle on the initial part of the beam, \( \lambda/2\pi a_B \), be less than the angle of geometrical widening of the beam, \( \lambda a_B \). This implies \( \Theta_L \gg (L/l_L)^{3/5} \exp(12/5(L/l_L)^{2/3}) \).

As a numerical illustration on the possibility of observing the exponential beam widening, assume \( L/l_L = 5 \). Then one needs an initial beam width \( a_B \ll a/200 \). The r.m.s. phase shift must be \( \Theta_L \gg 3000 \). The Fresnel lengths are \( Z_o \gg 600 \) \( L \) and \( Z_B \gg L/70 \).

In the case of a thin screen with \( \Theta_L^2 \gg 1 \) and observations at a distance \( L \gg l_L \) of a narrow beam, some of the abovementioned effects also occur. For \( a_B \ll a \), the instantaneous location of the beam in the \( z = L \) plane is typically displaced by distances \( \sim (L/l_L)a \) much larger than its width \( \sim (L/l_L)a_B \). However, the thin screen beam width does not show the exponential dependence on \( (L/l_L) \), which is characteristic of the thick screen.
PART IV

Interplanetary-Interstellar

Scintillations
Introduction to Part IV:

The close approach to the sun of several pulsars (for example, CP 0950, NP 0527, and PSR 2045-16) is of interest for among other reasons for the possibility for studying their interplanetary-interstellar scintillations. Pulsars display a variety of intensity variations (Lovelace and Craft, 1968), but one component or time scale of the variations is thought to be interstellar scintillations (Rickett, 1969; Scheuer, 1968, Salpeter, 1969). At a radio wavelength \( \lambda \) the received pulsar radiation should have a small but finite angular size \( \theta_5 \) (\( \approx 0.002'' \) sec of arc for \( \lambda \approx 70 \) cm for CP 0950) due to angle scattering by the interstellar medium.

Interplanetary scintillations provide a means of measuring the angular diameters of radio sources (see Little and Hewish, 1966; Cohen et al., 1967), by observing the loss of power in the scintillations with decreasing solar elongation; at meter wavelengths sizes of continuous radio sources of the order of \( \gtrsim 0.05'' \) sec of arc have been determined, but apparently sources, or components of sources with a significant fraction of the flux, having sizes much smaller than 0.05'' are rare (Harris et al., 1970). The angular size of a 'point' source, a pulsar for example, arising from scattering in a plasma is of a different nature from that of a true finite diameter source, because for angle scattering the wave amplitudes from different directions are not obviously incoherent. However, if we assume for the moment (and verify later)
that the pulsar angle sizes affect interplanetary scintillations in a way roughly similar to that of continuous sources with finite sizes, then pulsars offer a means for observing interplanetary scintillations over a range of elongations where the continuous source scintillations are quenched by their large sizes \((0.05'' \gg \theta_s)\). Such observations offer a method for studying very strong interplanetary scintillations, because the continuous sources are known to be quenched for only moderately strong scintillations (Cohen et al., 1967). If we assume further that pulsar scintillations will be quenched for small enough elongations because of their finite angle sizes, the determination of the elongation where the quenching sets in would give the value of the angle \(\theta_s\).

In Section A the two thin-screen model for interplanetary-interstellar scintillations is described; B discusses qualitatively and C quantitatively the wave amplitude received through both screens; D considers application of the theory in the temporal domain; E discusses the cross wavelength correlation, with in F application of these results; G treats application of the theory to available observations on the pulsar CP 0950. In H finite diameter incoherent sources are considered. Section I examines the validity of the assumption that pulsars may be treated as point sources. Section J points out modifications of the formulae of previous sections required for sources at finite distances.
A. The Model:

Consider a model for the interplanetary-interstellar scintillations of a point source consisting of two thin phase changing screens, one representing the interstellar medium at a distance \( Z \) (\( \sim 20 \) to \( 2000 \) pc) from the earth, and the other the interplanetary medium at a distance \( \gamma \) (\( \pm 1 \) a.u.) from the earth. The geometry of the model is sketched in Figure (26). The screens are a first approximation to the actual media, where random phase changes are introduced by electron density irregularities along ray paths having lengths comparable to the distances \( Z \) or \( \gamma \). Phase variations introduced first by the \( Z \) screen and then by the \( \gamma \) screen cause angle scattering which may give rise to an irregular diffraction pattern at the earth. In the present study we consider average properties of the diffraction pattern on the ground at some instant. However, scintillation observations are usually of intensity as a function of time as recorded at a single antenna, which depends on both the uniform and random velocities of irregularities past the line of sight. Here we consider only the case of uniform velocity, where there is a simple relation between the instantaneous diffraction pattern and the intensity at a single point as a function of time.

Either of the screens alone it is assumed could give strong modulation in the sense of a Rayleigh amplitude distribution on the ground. This limit of scintillations for a single screen has been developed by Pisareva (1959), Mercier
FIGURE 26: Geometry of Two Thin Screen Model.
(1962), Salpeter (1967), and Little (1968). Previously Scheuer (1968) considered two screen models for interstellar scintillations, and he has pointed out some of the interesting features which are discussed below.

Assume a plane wave of radio wavelength \( \lambda \) (wavenumber \( k = 2\pi/\lambda \)) from a point source propagating in the \( z \)-direction and incident on the distant screen, the \( Z \) screen. The screen gives a random phase modulation \( \exp(i\varphi_Z(x)) \) to the wave, where \( \varphi_Z(x) \) is the phase and \( x = (x,y) \). It is convenient to expand \( \exp(i\varphi_Z) \) in a finite two-dimensional Fourier transform because the expansion may be readily interpreted in terms of an angle probability density of radiation transmitted by the screen (see Part II; B; §1). We will need values of \( \exp(i\varphi_Z) \) over that area of the \( Z \) screen, say a square of size \( \mathcal{A} \) by \( \mathcal{A} \), which contributes to the wave received at the earth. The mesh size for \( x \), \( (\Delta x, \Delta y = \Delta x) \) where \( \Delta x = \mathcal{L}/N \) with \( N \gg 1 \), should be small enough so that \( \exp(i\varphi_Z) \) changes typically by a small amount compared with unity in a distance \( \sim \Delta x \). Hence the expansion may be written as,

\[
\exp[i\varphi_Z(x)] = \sum_n a_n \exp[i k \sum_n \bar{n} \cdot \bar{x}]
\]

(394)

where \( \bar{\theta}_n = (n_x \Delta \theta, n_y \Delta \theta) \), \( \bar{x} = (j_x \Delta x, j_y \Delta x) \) and \( n = (n_x, n_y) \), \( j = (j_x, j_y) \) with \( n, j = (0,0), (0,\pm 1), (\pm 1,0), (\pm 1,\pm 1) \), etc. The sum in (394) has \( N^2 \) terms. The angle
increment is \( \Delta \Theta = (kL)^{-1} \) and this is the angular diameter of a single term or beam. \( \Theta_n \) is the scattering angle for the \( n \)th term relative to the \( z \)-axis, \( a_n \) is the scattering amplitude, and \( |a_n|^2 \) is the corresponding probability so that \( \sum_n |a_n|^2 = 1 \).

Let us assume that the angle scattering by the \( Z \) screen is characterized by a typical angle \( \Theta_z \), which is very much smaller than a radian. This angle is determined by the distance over which \( \exp(i\Theta Z) \) typically changes by unity. Let this distance be denoted \( r_z \), then \( r_z \approx \frac{\lambda}{2\pi \Theta_z} \). Because \( \Delta x < r_z \), \( N = \mathcal{L}/\Delta x \gg \Theta_z/\Delta \Theta \). For \( |\Theta_n| \gg \Theta_z \), \( N^2 |a_n|^2 \approx 1 \), but for \( |\Theta_n| \gg \Theta_z \), \( N^2 |a_n|^2 \ll 1 \).

Similarly for a plane wave propagating parallel to the \( z \)-axis and incident normally on the near or \( z \)-screen, let us assume a phase modulation \( \exp(i\Theta Z(x)) \) and

\[
\exp[i\Theta Z(x)] = \sum_m b_m \exp[i k \Theta_m \cdot \hat{x}] \quad (395)
\]

Let \( \Theta_z \) denote the typical scattering angle for the \( z \) screen, and \( r_z \) the distance over which there is an appreciable change in \( \exp(i\Theta Z) \). Then \( r_z \approx \frac{\lambda}{2\pi \Theta_z} \).

For the moment neglect the \( z \) screen. Then at the earth the wave amplitude on the ground denoted \( A_z(x) \) may be obtained from equation (394) by including in each term the phase delay \( \frac{1}{2} k Z \Theta_z^2 \hat{a}_{\Theta n} \) due to the extra geometrical path length for a tilted beam. The \( \hat{a}_{\Theta} = 0 \) path length \( k Z \) is
ignored. Hence,

\[ A_Z(x) = \sum_n a_n e^{i\varphi_n(x)} ; \quad \varphi_n(x) = k \Theta_n \cdot x - \frac{i}{2} k Z \Theta_n^2 \]  

(396)

The intensity \( I_Z(x) = |A_Z(x)|^2 \) may be written thusly,

\[ I_Z(x) = 1 + \sum_{n \neq n'} a_n a_n^* \exp \left[ i k (\Theta_n \cdot \Theta_{n'}) \cdot x - \frac{i k Z}{2} (\Theta_n^2 - \Theta_{n'}^2) \right] \]  

(397)

where the interference terms have been isolated in the sum.

The spatial average of the intensity, \( \langle I_Z(x) \rangle = 1 \). With \( Z = 0 \), the sum in equation (397) is zero. Hence a necessary condition for strong intensity variations is that the typical path lengthening phase \( \frac{1}{2} k Z \Theta_n^2 \) be larger than a radian. However, we will be interested in addition in only cases where the beams received at the earth combine with essentially random phases, and thus cases where the amplitude \( A_Z(x) \) has a Rayleigh distribution. The Rayleigh distribution it is noted is equivalent to an exponential probability density for \( I_Z \):

\[ W_Z(I_Z) = \exp(-I_Z) ; \quad \int_0^\infty dI_Z W_Z(I_Z) = 1. \]

Salpeter (1967) has shown that the Rayleigh distribution holds when the observation distance \( Z \) is much larger than a typical focal length for the irregularities of the screen. Alternatively, we may require that the relative path lengthening phase \( \frac{1}{2} k Z \Theta_n^2 \) exceed by a large factor the phase difference between any two points on the screen separated by distances less than \( |\Theta Z| \).

Without going into the details here we shall assume that the **Rayleigh limit** occurs when \( \Theta \gg \frac{1}{2\pi \sqrt{Z}} \). The Rayleigh limit for
for a single screen is discussed in detail in Part II for Gaussian irregularities (§1) and for type A irregularities (§3).

Note that in equation (397) the most rapid variations with \( \mathcal{A} \) come from the most widely separated angles, \( \max(\left| \mathcal{A}_n - \mathcal{A}_n' \right|) \). This implies a typical distance scale for variations of the amplitude \( A_Z(\mathcal{A}) \) of \( r_Z \sim \frac{\lambda}{2\pi \Theta_s} \). The largest relative phase delay, \( \max(\frac{1}{2} k Z \left| (\mathcal{A}_n^2 - \mathcal{A}_n' \Theta_s^2) \right|) \) is of order \( \frac{1}{2} k Z \Theta_s^2 \) and corresponds to a time delay of about \( \Delta t_Z \approx \frac{1}{2}(Z/c) \Theta_s^2 \). The delay \( \Delta t_Z \) occurs in terms of (397) having rapid spatial dependence with the distance scale \( r_Z \). Terms varying more slowly with \( \mathcal{A} \) arise from interference of beams with smaller angle differences \( \left| \mathcal{A}_n - \mathcal{A}_n' \right| \), and for very small angle differences \( \frac{\left| \mathcal{A}_n - \mathcal{A}_n' \right|}{\Theta_s} = \varepsilon \ll 1 \), focusing may occur (see discussion of focusing by Salpeter, 1967).

However, the number of such terms is quite small of the order of \( \varepsilon^2 \) of the total, and therefore focusing may be neglected in the Rayleigh limit.

The area of the \( Z \) screen which must be included in expansion (394) is simply \( \mathcal{L} \sim (\Theta_s Z)^2 \) when only the \( Z \) screen is present. From this it follows that the number of beams, \( N^2 \sim (k Z \Theta_s^2)^2 \), is large for \( \Theta_s \gg \frac{1}{2\pi \sqrt{Z}} \). However, note that the properties of \( I_Z(\mathcal{A}) \), namely the distance scale \( r_Z \) and time delay \( \Delta t_Z \), remain unchanged if the distance \( \mathcal{L} \) is increased. When both screens are present one needs values of \( \mathcal{L} \gg \Theta_s Z \).
Similar features are assumed for the $\gamma$ screen when the distant $Z$ screen is absent. The wave amplitude on the ground $A_\gamma(x)$ due to a plane wave incident normally on the $\gamma$ screen is:

$$A_\gamma(x) = \sum_m b_m e^{i\psi_m(x)}; \psi_m(x) = k_\gamma x - \frac{x^2}{2} k_\gamma \theta_m^2$$

(398)

The intensity on the ground $I_\gamma(x) = |A_\gamma(x)|^2$ averages to unity. The Rayleigh limit is assumed to hold also for the $\gamma$ screen alone viewed at the earth when the typical scattering angle $\theta_s \approx \frac{\lambda}{2\pi \gamma}$. The intensity $I_\gamma$ then also has an exponential probability density $W_\gamma(I_\gamma) = \exp(-I_\gamma)$; $\int_0^\infty dI_\gamma W_\gamma(I_\gamma) = 1$.

The distance scale for $I_\gamma(x)$ is

$$r_\gamma \approx \frac{\lambda}{2\pi \theta_s},$$

and the characteristic time delay is

$$\Delta t_\gamma = \frac{1}{2} \left( \frac{\gamma}{c} \right) \theta_s^2.$$
B. Amplitude through both Screens:

For simplicity for the moment assume that there are only two scattering amplitudes for each screen: \( a_1 \) and \( a_2 \) for the \( z \) screen and \( b_1 \) and \( b_2 \) for the \( \bar{z} \) screen. Hence at the earth there will be four plane wave beams with angles \( \Theta_1 + \Theta_1, \Theta_1 + \Theta_2, \Theta_2 + \Theta_1, \) and \( \Theta_2 + \Theta_2 \) relative to the \( z \)-axis. This is illustrated in Figure (27). The wave amplitude at the earth is obtained by including in each term the phase delay due to the geometrical path length,

\[
A(x) = a_1 b_1 e^{i \Psi_{11}} + a_1 b_2 e^{i \Psi_{12}} + a_2 b_1 e^{i \Psi_{21}} + a_2 b_2 e^{i \Psi_{22}}
\]

\[
\psi_{nm}(x) \equiv \psi_{zn}(x) + \psi_{\bar{z}m}(x) - p_{nm}
\]

\[
p_{nm} = k \frac{\Theta_n \cdot \Theta_m}{\lambda}
\]

where \( n,m = 1,2 \) and \( \psi_{zn} \) and \( \psi_{\bar{z}m} \) are given in equations (396) and (398). The phase term \( p_{nm} \) has been isolated because it alone involves parameters of both screens.

Consider first the case where the \( p_{nm} \) in equation (399) are much smaller than a radian for typical values of \( \Theta_n \) and \( \Theta_m \); \( p_{nm} \approx k \frac{\lambda}{\Theta_n \Theta_m} \ll 1 \). In this limit the amplitude in (399) may be factored, \( A(x) = A_z(x) A_{\bar{z}}(x) \), where \( A_z \) and \( A_{\bar{z}} \) are given by (396) and (398). Thus \( I(x) = |A(x)|^2 \) and

\[
I(x) = I_z(x) I_{\bar{z}}(x); \quad p_{nm} \ll 1
\]

The condition for equation (400) to hold may be expressed as

\( \Theta_{\bar{z}} \ll \frac{n}{2 \pi \Theta_{\bar{z}}} \), which implies \( \Theta_{\bar{z}} \gg \Theta_{\bar{z}} \) because of the assump-
tion $\theta_{\delta} \gg \frac{1}{2\Theta_{\delta} \lambda}$. Hence radiation received at the earth comes from a region of transverse dimension $\sim \theta_{\delta} \lambda$ on the $\bar{y}$ screen. With $\theta_{\delta} \lambda$ less than the amplitude correlation length $r_{\lambda} \sim \frac{\lambda}{2\pi \Theta_{\delta}}$ for the wave from the $Z$ screen, the secondary beams from the $\bar{y}$ screen effectively originate from a plane wave, or a point source, of variable intensity $I_{Z}(x)$. In this limit the intensity has a slowly varying component with a distance scale $r_{Z} \sim \frac{\lambda}{2\pi \Theta_{\delta}}$ due to $I_{Z}(x)$ and a rapidly varying component with a distance scale $r_{\bar{y}} \sim \frac{\lambda}{2\pi \Theta_{\delta}}$ due to $I_{\bar{y}}(x)$.

Over the larger distance $r_{Z} \gg r_{\bar{y}}$, the intensity averages to unity and the square of the intensity averages to \( \langle I^2(x) \rangle \) = $\langle I_{Z}^2(x) \rangle \langle I_{\bar{y}}^2(x) \rangle$.

It is clear that equation (400) also holds when there are many amplitudes $a_n$ and $b_m$ from both screens with the same condition $P_{nm} \ll 1$. Consequently, the mean-square intensity averages to $\langle I^2 \rangle = 2 \cdot 2 = 4$. And the intensity probability density is that of the product of $I_{Z}$ and $I_{\bar{y}}$, where each has an exponential density. Thus

\[
W(I) = \int_{0}^{\infty} dI_{Z} \int_{0}^{\infty} dI_{\bar{y}} \delta(I - I_{Z}I_{\bar{y}}) W_{Z}(I_{Z}) W_{\bar{y}}(I_{\bar{y}})
\]

\[
W(I) = 2 K_0(2\sqrt{I}) \quad ; \quad \int_{0}^{\infty} dI W(I) = 1 ; \quad P_{nm} \ll 1
\]

where $K_0$ is a modified Bessel function of the second kind. An expression corresponding to $W(I)$ was first obtained by Scheuer (1968).
Properties of the wave amplitude of equation (399) for arbitrary value of \( P_{nm} \) is less simple. However, it is helpful to write the intensity in a particular form,

\[
I(\vec{x}) = 1 + \sum_n |a_n|^2 \left\{ I_z(\vec{x} - \vec{\zeta}_n) - 1 \right\} + \sum_m |b_m|^2 \left\{ I_z(\vec{x} - \vec{\zeta}_m) - 1 \right\} + \sum_{n \neq n', m \neq m'} a_n a^*_n b_m b^*_m \exp\left[ i (\psi_{nm}(\vec{x}) - \psi_{n'm'}(\vec{x})) \right] \tag{402}
\]

which is of course valid for any number of scattering amplitudes \( a_n \) and \( b_m \). The first sum isolates terms characteristic of the \( \vec{z} \) screen; these have a spatial dependence of the form \( \cos(k(\vec{\theta}_n \cdot \vec{x}) \cdot \vec{x}) \), and therefore vary significantly over a distance \( \sim r_\vec{z} \). The second sum contains terms characteristic of the \( \vec{Z} \) screen, which have a distance scale of order \( r_\vec{Z} \). And the third sum consists entirely of cross terms the spatial dependence of which is determined by the larger of \( \Theta_5 \) or \( \Theta_s \); that is, if say \( \Theta_s \gg \Theta_5 \) then the distance scale of the third sum is \( \sim r_\vec{z} \sim \frac{\lambda}{2\pi \Theta_s} \). This is because the dependence of a typical term from the third sum is of the form \( \cos(k(\vec{\theta}_n \cdot \vec{\theta}_m' + \vec{\theta}_n \cdot \vec{\theta}_n') \cdot \vec{x}) \). Of course, for \( |\vec{\theta}_n - \vec{\theta}_m'| < \Theta_s \) there are also terms with the longer distance scale \( r_\vec{Z} \sim \frac{\lambda}{2\pi \Theta_5} \), but because \( \Theta_s \gg \Theta_5 \) the number of such terms is a small fraction, \( \sim (\Theta_5/\Theta_s)^2 \), of the total.

Note that for a large number of beams from both screens the first and second sum in equation (402) average to zero for \( P_{nm} \sim k_3 \vec{\theta}_5 \vec{\theta}_5 \gg 1 \). This is because, for example \( I_z(\vec{x}) \)
in the first sum is averaged over different positions \( \theta_n \) separated by distances larger than the correlation length for \( I_{\vec{z}} \), \( r_{\vec{z}} \approx \frac{\lambda}{2\pi \theta_{\vec{z}}} \). Thus in this limit only the third sum survives. The intensity \( I(x) \) is then in some respects similar to that from a single screen. The intensity averages to unity over a distance which is the shorder of \( r_{\vec{z}} \) or \( r_{\vec{Z}} \); and the square of the intensity averages to \( \langle I^2 \rangle = 2 \) as shown subsequently. The difference compared with a single screen comes from the fact that the time delay between the interfering beams in the third sum of equation (402) may not correspond to the distance scale of the intensity. That is, if the distance scale is \( r_{\vec{z}} \) (\( \ll r_{\vec{Z}} \)), the important time delay may be \( \Delta t_{\vec{z}} \) due to the \( \vec{z} \) screen, if this is larger than the time delay for the \( \vec{Z} \) screen, \( \Delta t_{\vec{Z}} \). We return to this point below.
C. Correlation Functions:

Now it is of interest that some analytic expressions can be derived which correspond to the prior discussion. First consider the amplitude auto-correlation function $S_A(x) = \langle A(x)A^*(x+x) \rangle$ on the ground, where $A(x)$ is given by equation (399), generalized for many beams. This function is related, by a two dimensional Fourier transform, to the angle probability density for $\Theta_n + \Theta_m$. $S_A$ could be measured with an interferometer for example for observations averaged over an appropriately long interval. From equation (399) the desired average is:

$$S_A(r) = \left\langle \sum_{n,m} a_n a_n^* b_m b_m^* \exp[i(\psi_{nm}(x) - \psi_{nm'}(x+x))] \right\rangle \tag{403}$$

If the distance scales $r_\gamma$ and $r_Z$ were widely different, say $r_\gamma \ll r_Z$, then clearly the average in (403) could be done by averaging first over the short distance $r_\gamma$ which implies $m = m'$ and thus $\Theta_m = \Theta_{m'}$, and then averaging over the larger distance $r_Z$ which gives $n = n'$ and thus $\Theta_n = \Theta_{n'}$. For $\Theta_{\gamma} \sim \Theta_\gamma$, the two distances $r_\gamma$ and $r_Z$ are comparable and therefore there may be nearly commensurate $x$ variations from the $\gamma$ and $Z$ screens, that is $\Theta_m - \Theta_{m'} = \Theta_n - \Theta_{n'}$ with $n \neq n'$ and $m \neq m'$, which average exactly to zero only over distances much larger than $r_\gamma$ or $r_Z$. However, because there are many amplitudes $a_n$ and $b_m$ from both screens, the $x$ dependence from the $\gamma$ screen is inde-
pendent of that due to the \( Z \) screen over a distance \( \sim r_z \)
or \( r_Z \).

With \( n = n' \) and \( m = m' \) in equation (403) one obtains,

\[
\mathcal{G}_A (r) = \mathcal{G}_{AZ} (r) \cdot \mathcal{G}_{AZ} (r) \quad (404)
\]

where \( \mathcal{G}_{AZ} (r) \) ( \( \mathcal{G}_{AZ} (r) \) ) is the amplitude correlation for the \( Z \) ( \( Z \) ) screen alone,

\[
\mathcal{G}_{AZ} (r) = \sum_n |a_n|^2 e^{-i\omega_n \cdot r}; \quad \mathcal{G}_{AZ} (r) = \sum_m |b_m|^2 e^{-ik \omega_m \cdot r} \quad (405)
\]

The simple form of equation (404) is a consequence simply of the fact that the probability density for \( \omega_n + \omega_m \) is the convolution of the density for \( \omega_n \) and that for \( \omega_m \). The formula holds without restriction on the relative magnitudes of \( \omega_n \) and \( \omega_m \) other than that these angles be much less than a radian. With \( r = 0 \), each of the three amplitude correlation functions, \( \mathcal{G}_A \), \( \mathcal{G}_{AZ} \), \( \mathcal{G}_{AZ} \), is unity, which is a statement of energy conservation in each case.

Note that the sums in equation (405) may be written as integrals by identifying \( \sum |a_n|^2 \) with \( \int_\infty^{\infty} \omega \, W_2 (\omega) \ldots \) and \( \sum |b_m|^2 \) with \( \int_\infty^{\infty} \omega \, W_2 (\omega) \), where \( W_2 (\omega) \) ( \( W_2 (\omega) \) ) is the angle probability density for the \( Z \) ( \( Z \) ) screen. The angle densities for a single thin screen are discussed in Part I ( D ).
The intensity auto-correlation function on the ground, defined by
\( \rho_I(x) = \left\langle I(\hat{x})I(\hat{x} + \hat{x}) \right\rangle - 1 \), may be evaluated from equation (402) for the case of many beams from both screens with no restriction on the magnitude of \( P_{nm} \).

We may write \( I(\hat{x}) = 1 + I_1(\hat{x}) + I_2(\hat{x}) + I_3(\hat{x}) \), where \( I_1, I_2, \) and \( I_3 \) correspond to the three sums in (402).

\[
\begin{align*}
I_1(\hat{x}) &= \sum_n |a_n|^2 \left\{ I_{z_n}(\hat{x} - \hat{z_n}) - 1 \right\} \\
I_2(\hat{x}) &= \sum_m |b_m|^2 \left\{ I_{z_m}(\hat{x} - \hat{z_m}) - 1 \right\} \\
I_3(\hat{x}) &= \sum_{n \neq n', m \neq m'} a_n a_{n'}^* b_m b_{m'}^* \exp[i(\psi_{nm}(\hat{x}) - \psi_{n'm'}(\hat{x}))]
\end{align*}
\]

Note that \( I_1(\hat{x}) \) averages to zero over a distance \( \sim r_z \); \( I_2(\hat{x}) \) over a distance \( \sim r_Z \); and \( I_3(\hat{x}) \) over a distance the shorter of \( r_{\hat{z}} \) or \( r_Z \). It is clear that the cross terms such as \( \left\langle I_1 I_2' \right\rangle \) in the intensity correlation vanish because the \( \hat{x} \) variations of \( I_{\hat{z}} \) are independent of those of \( I_Z \) over the shorter of the distances \( r_{\hat{z}} \) or \( r_Z \). The cross terms such as \( \left\langle I_1 I_3' \right\rangle \) vanish because of the restriction \( n \neq n' \) (or \( m \neq m' \)) in the definition of \( I_3(\hat{x}) \) (also over the shorter of the distances \( r_{\hat{z}} \) or \( r_Z \)). Therefore the following relation holds in general,
\[ \rho_{\mathcal{I}}(r) = \sum_{n,n'} |a_n|^2 |a_{n'}|^2 \mathcal{G}_{\mathcal{I}Z}(r-z(x_n-x_{n'})) + \sum_{m,m'} |b_m|^2 |b_{m'}|^2 \mathcal{G}_{\mathcal{I}Z}(r-z(x_m-x_{m'})) + \left< I_{\mathcal{I}Z}(x) I_{\mathcal{I}Z}(x+r) \right> \]

(407)

where

\[ \mathcal{G}_{\mathcal{I}Z}(r) \equiv \left< I_{\mathcal{I}Z}(x) I_{\mathcal{I}Z}(x+r) \right> - 1 \]

\[ \mathcal{G}_{\mathcal{I}Z}(r) \equiv \left< I_{\mathcal{I}Z}(x) I_{\mathcal{I}Z}(x+r) \right> - 1 \]

The remaining term \( \left< I_{3} I_{3}' \right> \) is quite cumbersome, but it can be expressed in a simple form if the amplitudes \( A_{3}(x_n) \) and \( A_{3}(x_{n'}) \) are in the Rayleigh limit.

The approximation involved in the Rayleigh limit is illustrated for the case of a single screen, the \( Z \) screen. The rigorous theory for this limit is described by Mercier (1962). The intensity correlation function for the \( Z \) screen alone may be written out as,

\[ \rho_{\mathcal{I}Z}(r) = \sum_{n \neq n', j \neq j'} a_n a_{n'}^{*} a_{j} a_{j'}^{*} e^{i k (z_j - z_{j'})} x e^{i k \mathcal{Z} [z_j^2 + z_{j'}^2 - \delta_j^2 - \delta_{j'}^2]} \]

(408)

Observe that the last factor in equation (408) involving the geometrical path lengths is unity for that set of terms which have \( n = j' \) and \( n' = j \). As noted previously, a term in
the intensity (equation 397) with angles $\Theta_n$ and $\Theta_n'$ corresponds to the interference of the $n$ and $n'$ th beams. Thus the terms with $n = j'$ and $n' = j$ represent a beating of each pair of beams with itself only. One can neglect the beating of a particular pair of beams with another pair (having the same $x_{\infty}$ dependence) if the path lengthening phase difference for one pair is on the average greatly different from that of the other. That is, for sufficiently large $\frac{1}{2} k Z \Theta_5^2 \gg 1$, the pairs of beams, or Fourier components of $I_Z(x_{\infty})$, do not combine with large phases if $n = j'$ and $n' = j$. Retention in (408) of only these terms gives,

$$P_{I_Z}(x_{\infty}) = \left\{ \sum_n |a_n|^2 e^{-ik\Theta_n x_{\infty}} \right\}^2 = (P_{A_Z}(x_{\infty}))^2 \quad (409)$$

in the Rayleigh limit of the $Z$ screen. Also for the $Y$ screen in the Rayleigh limit, $\frac{1}{2} k Y \Theta_5^2 \gg 1$, one has

$$P_{I_Y}(x_{\infty}) = (P_{A_Y}(x_{\infty}))^2.$$

An approximation analogous to the Rayleigh limit for a single screen may be applied to the $I_3$ correlation function in equation (407),

$$\langle I_3(x_{\infty}) I_3(x + x_{\infty}) \rangle =$$

$$= \sum_{n \neq n', j \neq j'} a_n a_{n'}^* a_j a_j'^* b_{m} b_{m'}^* b_{\ell} b_{\ell'}^* e^{i\left\{ \psi_{nm}(0) - \psi_{n'm'}(0) + \psi_{j\ell}(r) - \psi_{j'\ell'}(r) \right\}}$$

$$n - n' = j' - j,$n - n = j' - j,$m - m' = \ell' - \ell \quad (410)$$
Consider a beam incident on the earth with amplitude $a_n b_m$ making an angle $\Omega_n + \varnothing_m$ to the z-axis. A Fourier component of the intensity has a magnitude of $a_n b_m^* a_{n'} b_{m'}^*$. All of the terms with $\Omega_n \neq \Omega_n'$ and $\varnothing_m \neq \varnothing_m'$ are included in $I_3$. If the different Fourier components of $I_3(x)\tilde{}$ combine with large path length phases, which certainly occurs when both $\frac{1}{2} k z \Theta_s^2 \gg 1$ and $\frac{1}{2} k \gamma \Theta_s^2 \gg 1$, then one need include only the terms $n = j'$, $n' = j$, $m = \ell'$, and $m' = \ell$ in equation (410). This gives,

$$\langle I_3(x) I_\tilde{3}(x + r) \rangle = \mathcal{P}_{I_3}(r) \times \mathcal{P}_{I_{\tilde{3}}}(r) \quad (411)$$

where $\mathcal{P}_{I_3}(r) = (\mathcal{P}_{I_3}(r))^2$ and $\mathcal{P}_{I_{\tilde{3}}}(r) = (\mathcal{P}_{I_{\tilde{3}}}(r))^2$ as before.

A rigorous proof of equation (411) has been done using Mercier's (1962) techniques of expanding and resumming the appropriate Fresnel integrals, but the rather lengthy proof would add nothing to our discussion.

For small values of $P_{nm} \sim k z \Theta_s\Theta_s \ll 1$ the first two terms on the right side of equation (407) become $\mathcal{P}_{I_3}(r)$ and $\mathcal{P}_{I_{\tilde{3}}}(r)$ respectively because the angle smearing for each is much less than the correlation lengths $r_{\tilde{3}}$ or $r_{3}$. Hence in this limit,

$$\mathcal{P}_{I}(r) + 1 = \left[1 + \mathcal{P}_{I_3}(r)\right]\left[1 + \mathcal{P}_{I_{\tilde{3}}}(r)\right] ; P_{nm} \ll 1 \quad (412)$$

Formula (412) follows directly from equation (400) with only the restriction $P_{nm} \ll 1$ and without the assumption that
the $\gamma$ and $Z$ screens are in the Rayleigh limit. From (412) the mean square intensity is $\langle I^2 \rangle = 1 + \mathcal{O}_I(0) = 4$. If $r_\gamma \ll r_Z$, then the correlation $\mathcal{O}_I = 3$ for $|\mathbf{r}| \ll r_\gamma$ and falls to $\mathcal{O}_I = 1$ for $r_\gamma \ll |\mathbf{r}| \ll r_Z$, and finally $\mathcal{O}_I = 0$ for $|\mathbf{r}| \gg r_Z$.

At the other extreme with $P_{nm} \gg 1$ the first two terms of equation (407) vanish because of the angle smearing and thus

$$\mathcal{O}_I(\mathbf{r}) + 1 = \mathcal{O}_{IZ}(\mathbf{r}) \times \mathcal{O}_{I\gamma}(\mathbf{r}) + 1; P_{nm} \gg 1 \quad (413)$$

Hence the mean square intensity is $\langle I^2 \rangle = 2$. The correlation $\mathcal{O}_I(\mathbf{r})$ falls from a value of unity to zero for $|\mathbf{r}|$ increased from zero to much larger than the smaller of $r_\gamma$ or $r_Z$. 
D. Considerations on Interplanetary-Interstellar Scintillations:

The correlation function \( f_I(\tau) \) represents an average of the intensities on the ground at some instant. It could be measured with many widely separated antennas. But time variations of the intensity on the ground at one point arise from the motion of the plasma irregularities, which cause the phase shifts, in the solar wind and interstellar medium across the line of sight. One simple assumption is that the irregularities in both media move with uniform translational velocities, \( \dot{u} \sim 400 \text{ km/sec} \) for the solar wind, and \( \dot{U} \sim 50 \text{ km/sec} \) for the interstellar medium, perpendicular to the line of sight. The assumption is reasonable for the solar wind, but ambiguous for the interstellar medium. Nevertheless, this approximation for the interstellar medium should give the correct qualitative behaviour, and the modulation indices will be correct (see Part II; B; §7). The intensity received at an antenna on the ground may then be characterized by two time scales, \( \gamma_\phi = \frac{r_\phi}{|\dot{u}|} \) for the interplanetary component, and \( \gamma_Z = \frac{r_Z}{|\dot{U}|} \) for the interstellar component. The frequency or power spectrum of the intensity time variations is then simply the Fourier transform (as a function of time lag \( \tau \)) of \( f_I(\tau) \), with \( \tau = u \gamma \) in the \( \phi \) correlation functions and \( \tau = U \gamma \) in the \( Z \) correlation functions. The spectrum of the intensity is sketched in Figure (28) for the two extremes of \( P_{nm} \) small and large compared with unity.
FIGURE 28: Power Spectra of Intensity Variations.
For observations of a source at large solar elongations (but small enough so that \( \theta_s \gg \frac{1}{2\pi \sqrt{\frac{\lambda}{B}} } \)), we may have \( P_{nm} \ll 1 \), and the frequency spectrum of the intensity is then roughly as shown in Figure (29). It consists of three rectangles of unit area, one due only to the interstellar scintillations (ISS) corresponding to the first term of equation (407), another due to only the interplanetary scintillations (IPS) corresponding to the second term of equation (407), and the third a convolution term characteristic mainly of interplanetary scintillations (denoted IPS(+ISS)). It is interesting to note that with a high pass filter which eliminates the interstellar scintillation component, the remaining interplanetary scintillations appear to be enhanced twofold, and have a modulation index \( m = \left( \langle I^2 \rangle - 1 \right)^{\frac{1}{2}} = 2^{\frac{1}{2}} \).

At small enough solar elongations (which is estimated later) it is possible to have \( P_{nm} \gg 1 \), and for this case the frequency spectrum is expected to be roughly as shown in Figure (29). The interstellar component of the intensity variations is completely quenched, and the interplanetary component is partially quenched. Note that if the received intensity is observed with a low pass filter which eliminates the interplanetary component, then no intensity variations will be observed if \( P_{nm} \gg 1 \), whereas for \( P_{nm} \ll 1 \) only the interstellar component is observed, which has a modulation index \( m = 1 \).
We can estimate the solar elongation at which \( P_{nm} \approx k \frac{\theta_y}{\theta_5} \theta_5 = 1 \) only with rather large uncertainty because the parameters for the interstellar scintillations are poorly known. However, for reference assume the 'standard' ISS parameters \( \lambda = 50 \text{ km/sec} \) and \( \zeta_z = 30 \text{ minutes at a radio frequency of 300 MHz for a pulsar at a given distance.} \) Then in order to have \( P_{nm} > 1 \), the IPS time scale \( \zeta_y \) must obey the inequality,

\[
\zeta_y < \zeta_{y*} \equiv \frac{2\pi \lambda \theta_y}{\zeta_z \mu |\mu|} \approx 0.03 \left( \frac{\lambda}{100 \text{ cm}} \right)^2 \left( \frac{30 \text{ min}}{\zeta_z (300)} \right) \sec (41.1)
\]

\( \zeta_y \) is a function of solar elongation of the source, or of the radial distance from the sun to the point of closest approach of the radio ray, \( R \). Assume that the radio wavelength is such that for a range of elongations, down to the minimum at a distance \( R_0 \), exists so that \( \zeta_y(R_0) \ll \frac{\lambda \theta_y}{\mu |\mu|} \) which is equivalent to \( \theta_5 \gg \frac{1}{2\pi} \sqrt{\frac{\lambda}{\theta_y}} \) at closest approach to the sun. Over a not too large range of radio wavelengths \( \zeta_y(R) \propto \lambda^{-1} \) (Salpeter, 1967) so that the range of elongations for which the IPS are in the Rayleigh limit increases with increasing wavelength. However, the value of \( \zeta_{y*} \) may still not fall within the limits of \( \zeta_y(R_0) = \sqrt{\frac{\lambda \theta_y}{\mu |\mu|}} \), which is necessary to get the \( P_{nm} > 1 \) behaviour discussed previously. These lower and upper limits on \( \zeta_{y*} \) are:
\[ \tau_z(R_o) \sim 10 \left( \frac{100 \text{ cm}}{\lambda} \right) \left( \frac{R_o}{1 \text{ a.u.}} \right)^2 \text{ sec.} \]
\[ \sqrt{\frac{\lambda \gamma}{|u|}} \sim 1 \left( \frac{\lambda}{100 \text{ cm}} \right)^{1/2} \text{ sec.} \]

Figure (29) shows the different possible cases as a function of wavelength, minimum elongation, and \( \tau_z(300) \). It is evident from the Figure that for very large \( \tau_z(300) \) or small \( \theta_5 \), that \( P_{nm} \) may be small compared with unity for the entire range of elongations, and therefore no information about \( \theta_5 \) may be obtained. However, observation in this limit would be of interest for the possibility of observing very strong interplanetary scintillations which most continuous sources do not display. On the other hand for very small \( \tau_z(300) \), or large \( \theta_5 \), we may have \( P_{nm} \gg 1 \) for the full range of solar elongation. With decreasing solar elongation the scintillations would then change from only ISS variations to only IPS variations at an elongation roughly where \( \theta_5 \simeq \frac{1}{2\pi/\gamma} \).
E. Cross Wavelength Correlation:

Up to this point we have discussed only the behaviour of the wave field on the ground at a single radio wavelength. However, the alteration of the diffraction pattern with wavelength can provide useful information about the scintillations not accessible to study at just one wavelength. The pattern on the ground at one wavelength will differ from that at another because (i) the scattering amplitudes $a_n$ and $b_m$ depend on wavelength and (ii) the path length phases $\frac{1}{2} k z \Theta_n^2$ and $\frac{1}{2} k z \Theta_m^2$ depend on wavelength in that $k = 2\pi / \lambda$. In the following we will be interested in the correlation between wave quantities at two different wavelengths $\lambda$ and $\lambda' \sim \lambda$ where $\frac{\lambda' - \lambda}{\lambda} = \frac{\Delta \lambda}{\lambda} \ll 1$.

First consider the $Z$ screen alone and for simplicity the correlation of the wave amplitude at $\lambda$ with the amplitude at $\lambda'$, but at the same point on the ground. At the longer wavelength $\lambda'$ it is convenient to choose the scattering angles so that $\Theta_n(\lambda') = \frac{\lambda'}{\lambda} \Theta_n(\lambda)$. The corresponding amplitudes at $\lambda'$ and $\lambda$ are $\tilde{a}_n$ and $a_n$, respectively. Let

$$\eta_Z(\Delta \lambda) \equiv \langle A_Z(x, \lambda) \tilde{A}_Z^*(x, \lambda') \rangle$$

denote the correlation. Then,

$$\eta_Z(\Delta \lambda) = \left\langle \sum_{\eta, \eta'} a_\eta \tilde{a}_{\eta'}^* \exp \left[ 2\pi i \left( \frac{1}{\lambda} \Theta_n - \frac{1}{\lambda'} \Theta_n' \right) \cdot x - i \pi Z \left( \frac{1}{\lambda} \Theta_n^2 - \frac{1}{\lambda'} \Theta_n'^2 \right) \right] \rightangle$$

$$= \sum_{\eta} a_\eta \tilde{a}_{\eta}^* \frac{i \Delta \lambda}{\lambda} \frac{kz}{\lambda^2} \Theta_n^2$$

(416)
One can simplify (416) in the Rayleigh limit because the path length phase \( \frac{1}{2} kZ \theta_s^2 \) is not only large compared with unity but also much larger than any phase differences introduced by the screen (Section A). Therefore, over a range of wavelengths \( \frac{\Delta \lambda}{\lambda} \approx (kZ \theta_s^2)^{-1} \) the corresponding change of the phase \( \phi_z \) which determines the \( a_n \) is small, \( \frac{\Delta \lambda}{\lambda} \phi_z \approx \phi_z (kZ \theta_s^2)^{-1} \ll 1 \). Thus we have \( a_n = \tilde{a}_n \) rather accurately. The wavelength decorrelation of the diffraction pattern is then due entirely to the geometrical path lengths.

From (416) one obtains,

\[
\eta_z(\Delta \lambda) \approx \sum_n |a_n|^2 e^{i \frac{\Delta \lambda}{\lambda} \frac{kZ}{2} \theta_s^2} \tag{417}
\]

\( \eta_z(\Delta \lambda) \) is unity for \( \Delta \lambda/\lambda = 0 \), and falls to zero for \( \frac{\Delta \lambda}{\lambda} \gg \frac{\Delta \lambda_z}{\lambda} = (\frac{1}{2}kZ \theta_s^2)^{-1} \ll 1 \). The fractional wavelength scale \( \Delta \lambda_z/\lambda \) is a characteristic of the \( z \) screen and its distance from the earth. It is related to the characteristic time delay \( \Delta t_z \) between beams at one wavelength by \( \omega \Delta t_z = \frac{1}{2} kZ \theta_s^2 \), where \( \omega = 2\pi c/\lambda \) is the angular radio frequency.

A formula analogous to (417) holds when only the \( y \) screen is present; the correlation \( \eta_y(\Delta \lambda) \) falls to zero from unity for \( \Delta \lambda/\lambda \) increased from zero to \( \frac{\Delta \lambda}{\lambda} \gg \frac{\Delta \lambda_y}{\lambda} = (\frac{1}{2} k_y \theta_s^2)^{-1} \).

With both screens present and both in the Rayleigh limit, the amplitude correlation is:
\[ \eta(\Delta \lambda) \equiv \left\langle A(x, \lambda) A^*(x, \lambda) \right\rangle \] 

\[ \eta = \sum_{n,m} |a_n|^2 |b_m|^2 \exp\left\{ i \frac{\Delta \lambda}{\lambda} \left( \frac{1}{2} k Z \Theta_{m}^2 + \frac{1}{2} k Z \Theta_{n}^2 \right) \right\} \] 

Thus the correlation falls from unity at \( \Delta \lambda / \lambda = 0 \) to zero for \( \Delta \lambda / \lambda \) much larger than the smaller of \( \Delta \lambda / \lambda \) or \( \Delta \lambda / \lambda \). (The larger of the two ray paths for the \( \zeta \) or \( Z \) screen alone will be larger than \( P_{nm} \) in any case.) Note that if \( P_{nm} \ll 1 \) in equation (418), then \( \eta(\Delta \lambda) = \eta_{Z}(\Delta \lambda) \cdot \eta_{\zeta}(\Delta \lambda) \). This is consistent with equation (400).

For a concrete illustration of equation (418) assume that both screens may be described by Gaussian irregularities, and thus Gaussian angle densities (equation 64). Then we may use the correspondence \( \sum_{n} |a_n|^2 \rightarrow \int_{-\infty}^{\infty} d^2 \Theta W_{Z}(\Theta) \ldots \) and \( \sum_{m} |b_m|^2 \rightarrow \int_{-\infty}^{\infty} d^2 \Theta W_{\zeta}(\Theta) \ldots \), where \( W_{Z}(\Theta) = (2\pi \Theta_{Z}^2)^{-1} \exp(-\Theta_{Z}^2/2\Theta_{Z}^2) \) and \( W_{\zeta}(\Theta) = (2\pi \Theta_{\zeta}^2)^{-1} \exp(-\Theta_{\zeta}^2/2\Theta_{\zeta}^2) \). Doing the requisite integrals in (418) one finds,

\[ \eta(\Delta \lambda) = \frac{-\mathcal{K}_+ \mathcal{K}_-}{(\frac{\Delta \lambda}{\lambda} + i \mathcal{K}_+)(\frac{\Delta \lambda}{\lambda} + i \mathcal{K}_-)} \] 

\[ \mathcal{K}_+ = \frac{(k Z \Theta_{Z}^2 + k Z \Theta_{\zeta}^2) \pm \left\{ (k Z \Theta_{Z}^2 + k Z \Theta_{\zeta}^2)^2 - 4(1 - \frac{3}{2}) k Z \Theta_{Z}^2 k Z \Theta_{\zeta}^2 \right\}^{1/2}}{2 \left(1 - \frac{3}{2} Z\right) k Z \Theta_{Z}^2 k Z \Theta_{\zeta}^2} \]

\[ \eta(\Delta \lambda = 0) \equiv 1 ; \quad \mathcal{K}_+ \mathcal{K}_- > 0 ; \quad \mathcal{K}_+ > \mathcal{K}_- > 0 \]
Because \( \frac{\gamma}{Z} < 1 \), the roots \( \kappa_\pm \) are positive definite.

From equation (418a) and the general relation (177a), the average intensity shape, \( \mathcal{I}(\Delta t) \), of a narrow pulse received through both \( \gamma \) and \( Z \) screens may be obtained by closing the \( \Delta k \) integration in (177a) in the lower half complex plane for \( \Delta t < 0 \) or in the upper half plane for \( \Delta t > 0 \). In the lower half plane \( \eta(\Delta k) \) (with \( \Delta k/k = -\Delta \lambda/\lambda \)) has no poles so that \( \mathcal{I}(\Delta t) = 0 \), whereas in the upper half plane \( \eta \) has two simple poles on the imaginary axis at \( \Delta k = i k \kappa_\pm \). The residues from these two poles give \( \mathcal{I}(\Delta t > 0) \).

\[
\begin{align*}
\mathcal{I}(\Delta t < 0) &= 0 \\
\mathcal{I}(\Delta t > 0) &= k_c \frac{\kappa_+ \kappa_-}{\kappa'_- - \kappa_-} \left[ e^{-\kappa_- k_c \Delta t} - e^{-\kappa_+ k_c \Delta t} \right] \\
\int_0^\infty d(\Delta t) \mathcal{I}(\Delta t) &= 1
\end{align*}
\]

(418b)

The pulse shape (418b) is to be compared with that for a single thin screen, equation (177), which is of the form \( \mathcal{I}(\Delta t > 0) \propto \exp(-\Delta t/\Delta t_Z) \). The notable difference is that \( \mathcal{I}(\Delta t) \) for the two screens has only a gradual rise from an initial value of zero at \( \Delta t = 0 \) to its later maximum, whereas \( \mathcal{I}(\Delta t) \) for a single thin screen 'jumps' discontinuously to its maximum.

The two terms of (418b) may be better understood by considering the limit where \( k_z \theta_z^2 \gg k_\gamma \theta_\gamma^2, \frac{\gamma}{Z} \). Then one finds, \( \kappa_+ = (1 - \frac{\gamma}{Z})^{-1} (k_\gamma \theta_\gamma^2)^{-1} \) and \( \kappa_- = (k_Z \theta_z^2)^{-1} \), with
\(\kappa_+ \gg \kappa_-\). Introducing the definitions \(\Delta t_Z \equiv (z/c) \Theta_s^2\) and \(\Delta t_\bar{z} \equiv (\bar{z}/c) \Theta_s^2\), one finds,

\[
\begin{align*}
\phi (\Delta t < 0) &= 0 \\
\phi (\Delta t > 0) &\approx \frac{1}{\Delta t_Z} \left\{ e^{-\frac{\Delta t}{\Delta t_Z}} - e^{-\frac{\Delta t}{\Delta t_{\bar{z}}}(1-\bar{z}/z)^{-1}} \right\} \tag{418c}
\end{align*}
\]

If the near \(\bar{z}\) screen were absent, then the second term on the right of (418c) would be absent too. Thus it is clear that the \(\bar{z}\) screen has the effect of smoothing out the 'jump' discontinuity which occurs for a single thin screen. Here the rise time of \(\phi (\Delta t)\) is \(\sim \Delta t_{\bar{z}}(1-\bar{z}/z)\) and is from the \(\bar{z}\) screen. The decay time is very much longer and is due to the \(Z\) screen. Figure (30) shows formula (418c) for \(\bar{z} \ll Z\), and for comparison the pulse shape in the absence of the \(\bar{z}\) screen.

Now consider the correlation of the intensity at wavelength \(\lambda\) with that at \(\lambda'\), but at the same point on the ground. First assume that only the \(Z\) screen is present. Then in the Rayleigh limit one obtains,

\[
\begin{align*}
\mathcal{E}_{Z}(\Delta \lambda) &\equiv \left< I_Z(x,\lambda) I_Z(x,\lambda') \right> - 1 \\
\xi_{Z}(\omega) &= \left| \sum_n |a_n|^2 \exp \left\{ i \frac{\Delta \lambda}{\lambda} k Z \Theta_n^2 \right\} \right|^2 = \left| \eta_{Z}(\Delta \lambda) \right|^2 \tag{419}
\end{align*}
\]

which is the analogue of equation (409) for the spatial correlation function in the same limit. The correlation \(\xi_{Z} = 1\) for \(\frac{\Delta \lambda}{\lambda} = 0\), and is zero for \(\frac{\Delta \lambda}{\lambda} \gg \frac{\Delta \lambda}{\lambda} z\). A formula similar to
FIGURE 30: Average Pulse Shape through the Two Screens.
equation (419) holds also for the \( y \) screen alone in the Rayleigh limit.

With both screens present we choose to simplify the formulae by treating only the cases where \( P_{nm} \) is either small or large compared with unity. For \( P_{nm} \ll 1 \), the intensity at each of the two wavelengths factors as in equation (400) and we have,

\[
A(\Delta \lambda) + 1 = \left\{ 1 + |\eta_z(\Delta \lambda)|^2 \right\} \left\{ 1 + \eta_z(\Delta \lambda) \right\}^2 ; \quad P_{nm} \ll 1 \quad (420)
\]

which is the analogue of equation (412) for the spatial correlation. For \( \Delta \lambda / \lambda = 0 \), \( \xi(0) = 3 \), and if for example \( \Delta \lambda / \lambda \ll \Delta \lambda / \lambda \) then \( \xi(\Delta \lambda) = 1 \) for \( \Delta \lambda / \lambda \gg \Delta \lambda / \lambda \) and it vanishes for \( \Delta \lambda / \lambda \gg \Delta \lambda / \lambda \).

For \( P_{nm} \gg 1 \), we have \( I(x) = 1 + I_3(x) \) and a calculation similar to that for equation (411) gives,

\[
\xi(\Delta \lambda) = \left| \sum_{n;m} |a_n|^2 |b_m|^2 \exp\left\{ i \frac{\Delta \lambda}{\lambda} \left( \frac{1}{2} k \frac{Z \Theta_n^2}{\Theta_n^2} + \frac{1}{2} k \frac{Z \Theta_m^2}{\Theta_m^2} + k \frac{Z \Theta_n \Theta_m}{\Theta_n \Theta_m} \right) \right\} \right|^2
\]

when both the \( z \) and \( Z \) screens are in the Rayleigh limit. Hence in this case the correlation is unity for \( \Delta \lambda / \lambda = 0 \), and it falls to zero for \( \Delta \lambda / \lambda \) larger than the smaller of \( \Delta \lambda / \lambda \) or \( \Delta \lambda / \lambda \).
F. More Considerations on Interplanetary-Interstellar Scintillations:

Let us introduce two dimensionless parameters $\gamma'_{\tilde{Z}}$ and $\gamma'_{Z}$ to characterize the strength of the interplanetary ($\tilde{Z}$) scattering or the interstellar ($Z$),

$$\Theta_{S} = \frac{1}{2\pi} \sqrt{\frac{\lambda}{\gamma'_{\tilde{Z}}}} \gamma'_{\tilde{Z}} ; \; \gamma'_{\tilde{Z}} \gg 1$$

$$\Theta_{S} = \frac{1}{2\pi} \sqrt{\frac{\lambda}{Z}} \gamma'_{Z} ; \; \gamma'_{Z} \gg 1$$

(422)

Then the observable parameters may be expressed as,

$$\frac{\Delta \lambda_{\tilde{Z}}}{\lambda} = \frac{4\pi}{\gamma'^2_{\tilde{Z}}}$$

$$\frac{\Delta \lambda_{Z}}{\lambda} = \frac{4\pi}{\gamma'^2_{Z}}$$

$$\tau_{\tilde{Z}} = \frac{1}{\gamma'^2_{\tilde{Z}}} \left\{ \frac{\sqrt{\lambda_{\tilde{Z}}}}{|U|} \right\}$$

$$\tau_{Z} = \frac{1}{\gamma'^2_{Z}} \left\{ \frac{\sqrt{\lambda_{Z}}}{|U|} \right\}$$

(423)

$$P_{nm} \approx k_{\tilde{Z}} \Theta_{S} \Theta_{S} = \frac{1}{2\pi} \gamma'_{\tilde{Z}} \gamma'_{Z} \left( \frac{\tilde{Z}}{Z} \right)^{1/2}$$

Figure (31) shows the possible cases as a function of $\gamma'_{\tilde{Z}}$ and $\gamma'_{Z}$. Apparently there is a necessary condition in order for $P_{nm}$ to be larger than unity which is:

$$\gamma'_{Z} \text{ or } \gamma'_{\tilde{Z}} \gg \gamma'_{*} = \sqrt{2\pi} \left( \frac{Z}{\tilde{Z}} \right)^{1/4}$$

(424)

A numerical estimate of (424) for $Z \approx 100$ pc and $\tilde{Z} = 1$ a.u. is $\gamma'_{*} \approx 170$.

For an enumeration of possible cases consider first that where $\gamma'_{Z} > \gamma'_{*}$, and different values of $\gamma'_{\tilde{Z}}$, increasing
FIGURE 31: Scintillation Regimes.
from $Y' / Y \approx 1$, as would be expected to occur for a source with decreasing solar elongation. For $Y' / Y < Y' / Y < 1$, $P_{nm} < 1$, and the correlation of intensities at two wavelengths is expected to be roughly as shown in the bottom part of Figure (32a). If $Y' / Y$ is larger, $Y' / Y > Y' / Y$, then $P_{nm} > 1$, and the wavelength correlation is shown in the middle part of Figure (32a). This is the anomalous case mentioned previously because the spatial variations are due to the interplanetary scattering whereas the wavelength correlation is determined by the interstellar scattering. For even larger $Y' / Y > Y'$ the correlation $\xi(\Delta \lambda)$ is shown on the top part of Figure (32a).

The second and only remaining case is that where $Y' / Y < Y'$. The different possibilities are shown in Figure (32b) for the three distinct ranges of $Y' / Y$.

From the prior discussion we may conclude that very small bandwidths $\Delta \lambda / \lambda$ will be needed to observe the quenching of the interplanetary interstellar scintillations which occurs as $P_{nm}$ increases through unity. The largest and therefore optimum bandwidth permitted for observing the full effect of the quenching is that where $\Delta \lambda / \lambda = \Delta \lambda / \lambda = \frac{4\pi}{\lambda} = 2 \left( \frac{3}{2} \right)^{2}$. This is expected to be quite small, $\approx (2000)^{-1}$ for $Y' \approx 170$.

If the intensity variations for arbitrary $Y' / Y$ and $Y' / Y$ are observed with a bandwidth larger than the smaller of $\Delta \lambda / \lambda$ or $\Delta \lambda / \lambda$, different components of the intensity variations will be suppressed. The components which are suppressed may be read off of Figure (32). For example, if the case shown
FIGURE 32: Cross Wavelength Correlation.
in the lower part of Figure (32a) applies with \( \frac{\Delta \lambda z}{\lambda} \ll \frac{\Delta \lambda Y}{\lambda} \), then a bandwidth such that \( \frac{\Delta \lambda z}{\lambda} \ll \frac{\Delta \lambda}{\lambda} \ll \frac{\Delta \lambda Y}{\lambda} \) will eliminate the interstellar variations and also the IPS(+ISS) variations leaving only the IPS component which has modulation index of \( m = 1 \). Previously Scheuer (1968) considered the finite bandwidth suppression of different components of the scintillations from two screens.
G. Examination of Preliminary Data on CP 0950:

The pulsar CP 0950 appears to be especially suited for a study of interplanetary-interstellar scintillations because (a) it approaches to within $\sim 5^\circ$ of the sun, (b) it is a relatively strong source, and (c) it is probably not very distant (Davies (1969) derives a distance to CP 0950 of about 40 pc or $z \sim 20$ pc). Because of (c) the bandwidth at $P_{nm} \approx 1$ should not be excessively narrow. Rickett (1969) found that the radio bandwidth $B \nu$ at which the modulation index of CP 0950 is reduced to half of the 'zero' bandwidth value is roughly $B \nu = 20$ MHz at a frequency $\nu = 408$ MHz. In our notation this corresponds to $\gamma_z' \approx 2\pi (2\nu / B \nu)^{1/2}$ or $\gamma_z' \approx 40$ at 408 MHz. For a distance of $z = 20$ pc and $\gamma = 1$ a.u., $\gamma_z' \approx 110$. Over a not too large range of wavelength $\gamma_z'$ (and $\gamma_z$) is proportional to $\lambda^{3/2}$ and hence at a frequency $\nu \approx 200$ MHz $\gamma_z' \approx \gamma_z^2$ and at this frequency the bandwidth required for detecting the full IPS-ISS variations should be less than about $4\lambda / \lambda \approx 4\pi / \gamma_z^2$, or $4\lambda / \lambda \approx 10^{-3}$, or a bandwidth of 200 kHz, which is reasonable considering the high flux density of CP 0950. For $\gamma_z' < \gamma_z^2$ the 200 kHz bandwidth should be sufficiently narrow, but for $\gamma_z' > \gamma_z^2$ and thus $P_{nm} > 1$, a slightly narrower one may be required.

We conclude that appropriate observations of CP 0950 and analysis could give useful data on both interplanetary and interstellar scintillations. It appears possible to obtain an estimate for $\gamma_z$ by observing the quenching of either or
both the interplanetary and interstellar components of the intensity variations which happens as $P_{nm}$ increases through unity. Whereas for small values of $P_{nm} \ll 1$ the observations could provide a test for the model assumed for the interplanetary medium showing the extent to which the thin screen approximation is valid.

Observations at large solar elongations or at high radio frequencies permit characterization of the ISS of CP 0950. Knowledge of the ISS component should permit its statistical removal from the combined IPS-ISS intensity variations.

Results of analysis of preliminary data on CP 0950 (taken by H. Craft, 1968) are shown in Figure (33) and (34). The observations were at 430 MHz with bandwidths of 1-3 MHz. We estimate $j'_z \approx 40 < j'_x \approx 110$; and thus $\frac{\Delta \lambda}{\lambda} \approx 0.8 \times 10^{-2}$, the radio bandwidth is probably sufficiently narrow for the ISS. From Figure (29) we probably also have $j'_y < 50 \ll j'_x j'_z \approx 300$. This implies $P_{nm} \ll 1$ even at the closest approach of CP 0950 to the sun, at 430 MHz. Indeed the observed intensity probability density of Figure (33) is of the form of equation (401) for this limit, and the observed modulation index as a function of solar elongation shown in Figure (34) has a maximum of about the predicted value of $m = 3^{\frac{1}{2}}$ for this limit.

Shown also in Figure (34) is the modulation index of CP 0950 corrected in a crude attempt to remove the interstellar component of the variations; the corrected index was
FIGURE 33: Intensity Probability Densities.

Intensity $I$ in units of the average $\langle I \rangle$.

Number of pulses in $\delta I = \frac{1}{2} \langle I \rangle \propto$ Probability Density $\rightarrow$

$W(I) = e^{-I}$

$W(I) = 2K_0(2\sqrt{I})$

equation (401)

CP 0450
August 24, 1968
3072 pulses
obtained via \( m_{\text{corrected}} = \sqrt{\frac{1}{2}} \left( m_{\text{observed}}^2 - 1 \right)^{\frac{1}{2}} \), which involves the ambiguous assumption that the interstellar variations of CP 0950 have a stable modulation index of unity. Shown in addition in Figure (34) is the index of CTA 21 at 430 MHz for data taken in 1967. The corrected index of CP-0950 appears to be significantly larger than that of CTA-21. However, definite conclusions on this point must await thorough observations of CP 0950 variations over a wide range of elongations. Measurements of CP 0950 variations in 1969 are presently being analyzed by G. Zeissig at the Arecibo Observatory.

Data at radio frequencies lower than 430 MHz appear to be necessary to obtain \( P_{nm} > 1 \), that is, quenching of the interplanetary variations by the finite size \( r_s \) of CP 0950.

However, even the meagre data of Figure (34) gives evidence for a strong contribution by the interplanetary medium to the CP 0950 intensity variations at an elongation smaller than that at which the finite size of continuous sources quench IPS at 430 MHz.
FIGURE 341: Observed Modulation Indices of CP 0950

- Modulation index
- 24 August 1968
- 6 Sept. 1968
- 24 Sept. 1968
- 16 May 1968
- Corrected CP 0950 indices
- Observed CP 0950 indices

\[ p = \sin(\varepsilon) \; ; \; \varepsilon = \text{solar elongation} \]
H. Finite Diameter Incoherent Sources

It is of interest to determine the effect of interstellar scattering has on the interplanetary scintillations of distant finite diameter incoherent sources. Assume a brightness distribution of the source characterized by the intensity \( B(\theta) \) as a function of angle \( \theta \), normalized such that \( \int d^2 \theta \, B(\theta) = 1 \). Furthermore, assume that the radio bandwidth is narrower than \( \Delta \lambda z / \lambda \) or \( \Delta \lambda / \lambda \). Then if \( B(\theta) \) were a delta function at \( \theta = 0 \) we would have,

\[
\mathcal{R}_I(x) = \left[ \mathcal{R}_{IZ}(x) \right]_{\theta} + \left[ \mathcal{R}_{IZ}(x) \right]_{\theta} + \mathcal{R}_{IZ}(x) \mathcal{R}_{IZ}(x) \tag{4.25}
\]

where the square brackets denote the smearing defined in equation (4.07) due to angle scattering only. However, if \( B(\theta) \) has a finite range then there is an additional angle smearing due to the intrinsic source size. The finite intrinsic size may be taken into account by finding the intensity \( I(x, \theta) \) on the ground due to an infinitesimal element of the source of strength \( B(\theta) \, d^2 \theta \), and then summing,

\[
\langle I(x) \rangle_\theta = \int d^2 \theta \, B(\theta) \, I(x, \theta) \tag{4.26}
\]

From (4.26) one can find the intensity correlation of \( \langle I(x) \rangle_\theta \) for the finite source.

If only the \( Z \) screen were present, then for radiation incident at an angle \( \theta \), one could obtain the amplitude on the ground by displacing all of the angles \( \theta_n \rightarrow \theta_n + \theta \) in equation (3.96) for the amplitude. The amplitude for scattering
from \( \mathbb{B} \) to \( \mathbb{B} + \mathbb{B}_n \) is still equal to \( a_n \). This gives
\[
A_2(x, \mathbb{B}) = A_2(x - \mathbb{B} Z), \quad \text{and hence,}
\]
\[
\langle I_z(x) \rangle_{\mathbb{B}} = \sum_{-\infty}^{\infty} d^2 \mathbb{B} \mathcal{B}(\mathbb{B}) I_z(x - \mathbb{B} Z) \tag{4.27}
\]

From (4.27) it follows that,
\[
\langle \tilde{I}_z(x) \rangle_{\mathbb{B}} = \sum_{-\infty}^{\infty} d^2 \mathbb{B} \tilde{B}(\mathbb{B}) I_z(x - \mathbb{B} Z) \tag{4.28}
\]

\[
\tilde{B}(\mathbb{B}) = \sum_{-\infty}^{\infty} d^2 \mathbb{B}' B(\mathbb{B} - \mathbb{B}') B(\mathbb{B}') \text{ ; } \sum_{-\infty}^{\infty} d^2 \mathbb{B} \tilde{B}(\mathbb{B}) = 1
\]

This result was derived previously by Salpeter (1967). It shows in analytical detail the effect of quenching a finite source may have on the spatial intensity variations.

With both screens present one may also obtain the intensity \( I(x, \mathbb{B}) \) by the replacement \( \mathbb{B}_n \rightarrow \mathbb{B}_n + \mathbb{B} \), but with the \( \gamma \) screen angles \( \theta_m \) unchanged. The angles \( \theta_m \) must not be altered because in equation (399) these are defined relative to the angle of the incident beam (see Figure 27). The substitution gives,
\[
I(x, \mathbb{B}) = \sum a_n a^*_n b_m b^*_m \exp \{ i(k(\theta_n - \theta'_n) \cdot (x - \mathbb{B} Z) + \\
+ k (\theta_m - \theta'_m) \cdot (x - \mathbb{B} \gamma) + \psi_{nm}(0) - \psi_{nm'}(0) \} \tag{4.29}
\]

The spatial dependence due to the \( Z \) screen has been altered by \( x \rightarrow x - \mathbb{B} Z \), whereas that from the \( \gamma \) screen has been changed from \( x \rightarrow x - \mathbb{B} \gamma \). The geometrical reason for this is because a wave ray for incoming radiation with an angle \( \mathbb{B} \) is dis-
placed on the $Z$ screen by a distance $\beta Z$, whereas on the $\tilde{Z}$ screen the displacement is $\beta \tilde{Z}$, both relative to the $\beta = 0$ ray. This is illustrated in Figure (35). From equations (425) and (429), and (407) and (411) appropriate for the Rayleigh limit of both screens, one may obtain the following relation,

\[
\langle \rho_I(x) \rangle_B = \left\{ \int_0^\infty d^2 \beta \int_0^\pi d\theta \tilde{B} (\beta) \tilde{W}_Z(\theta) \rho_I Z (x - Z (\beta \theta + \theta)) + \right.
\]

\[
+ \left\{ \int_{-\infty}^0 d^2 \beta \int_0^\pi d\theta \tilde{B} (\beta) \tilde{W}_Z(\theta) \rho_I Z (x - Z \beta + \theta \theta) + \right. \]

\[
\left. + \int_0^\infty d^2 \beta \tilde{B} (\beta) \rho_I Z (x - Z \beta) \rho_I Z (x - Z \beta) \right\} (430)
\]

where we have introduced the definitions,

\[
\tilde{W}_Z(\theta) \equiv \int_0^\infty d^2 \theta' W_Z(\theta - \theta') W_z(\theta'); \quad \int_0^\infty d^2 \tilde{W}_Z(\theta) = 1
\]

\[
\tilde{W}_Z(\theta) \equiv \int_{-\infty}^0 d^2 \theta' W_z(\theta - \theta') W_z(\theta'); \quad \int_{-\infty}^\infty d^2 \tilde{W}_Z(\theta) = 1
\]

where $W_Z(\theta)$ and $W_z(\theta)$ are the probability densities for angle scattering by the $Z$ and $\tilde{Z}$ screens respectively.

Let us assume for an examination of relation (430) that the brightness distribution $B(\beta)$ may be characterized by a single angle $\beta_I$ such that $\beta_I^2 B(\beta) \ll 1$ for $|\beta| \gg \beta_I$. Then three different cases may be distinguished,
FIGURE 35: Ray Geometry for Finite Source.
\[(i) \quad \theta_I \gg \frac{\lambda}{2\pi \theta_s \gamma} \]
\[\quad (ii) \quad \frac{\lambda}{2\pi \theta_s \gamma} > \theta_I > \frac{\lambda}{2\pi \theta_s Z} \]
\[\quad (iii) \quad \theta_I \ll \frac{\lambda}{2\pi \theta_s Z} \]

} \quad (432)

where we have assumed \( \theta_s Z \gg \theta_s \gamma \). The first case (i) represents very large sources which do not scintillate at all, that is, \( \langle \Theta_I \rangle_\theta = 0 \). The third case corresponds to a point source, which is the case considered in prior sections. The intermediate case (ii) presents some interesting possibilities: If \( \frac{\lambda}{2\pi \theta_s \gamma} \approx \theta_I \gg \frac{\lambda}{2\pi \theta_s Z} \), then the last two terms in equation (430) average to zero and the following formula applies,

$$\langle S_I (r) \rangle_\theta = \left( \iint_{-\infty}^{\infty} d^2 \theta \tilde{B}(\theta) \right) S_{I \gamma}(r-\gamma \theta)$$

$$\tilde{B}(\theta) = \left( \iint_{-\infty}^{\infty} d^2 \theta' \tilde{W}_Z(\theta-\theta') \tilde{B}(\theta') \right) ; \left( \iint_{-\infty}^{\infty} d^2 \theta \tilde{B}(\theta) = 1 \right)$$

\quad (433)

The intensity variations in this limit are due entirely to interplanetary scattering; the spatial variations have a length scale of \( r_\gamma \approx \lambda/2\pi \theta_s \) and the characteristic bandwidth is \( \Delta \lambda_\gamma / \lambda \approx \left( \frac{1}{2} \kappa_\gamma \theta_s^2 \right)^{-1} \). Here the interstellar scattering has only the effect of increasing the apparent source size, that is, the width of the brightness \( \tilde{B} \) may be larger than the actual source width depending on \( W_z(\theta) \). With no interstellar scattering, \( W_z(\theta) \) is a delta function and \( \tilde{B} = \tilde{B} \).

However, if the angle spread from interstellar scattering is large, \( \theta_s \gg \theta_I \), then the apparent source brightness function
is \( \tilde{W}_z(\Theta) \), which has a width \( \sim \Theta_s \), rather than \( B_I \).

Thus \( \Theta_s \) represents a minimum angular size which may be measured with interplanetary scintillations.

As a concrete illustration of equation (433), assume for both screens Gaussian irregularities in the Rayleigh limit, where \( \Phi_{Iz}(x) = \exp\left(-\frac{\Theta_z^2 x^2}{a_z^2}\right) \), with \( \Theta_z \) the r.m.s. phase shift and \( a_z \) the scale size of irregularities for the \( z \) screen; where \( W_z(\Theta) = (2\pi \Theta_s^2)^{-1} \exp\left(-\frac{\Theta^2}{2 \Theta_s^2}\right) \) with \( \Theta_s \) the typical scattering angle for the \( z \) screen. For the source brightness distribution assume \( B(\Theta) = (2\pi \Theta_s^2)^{-1} \cdot \exp\left(-\frac{\Theta^2}{2 \Theta_s^2}\right) \). Then one finds \( \tilde{W}_z(\Theta) = (4\pi \Theta_s^2)^{-1} \cdot \exp\left(-\frac{\Theta^2}{4 \Theta_s^2}\right) \), and \( B(\Theta) = (4\pi \Theta_s^2)^{-1} \exp\left(-\frac{\Theta^2}{4 \Theta_s^2}\right) \), and finally, \( \tilde{B}(\Theta) = (4\pi \Theta_s^2)^{-1} \left(1+\Theta_s^2/\Theta_I^2\right)^{-1} \exp\left(-\frac{\Theta^2}{4 \Theta_s^2}\right) \).

Then (433) gives,

\[
\langle \Phi_I(x) \rangle_B = \left(1 + \frac{4(\Theta_I^2 + \Theta_s^2) - 1}{(a_z/\Theta_I^2)^2}\right) \exp\left\{-\frac{r^2 \Theta_I^2}{a_z^2} \left[1 + \frac{4(\Theta_I^2 + \Theta_s^2)}{(a_z/\ Theta_s^2)^2}\right]\right\}
\]

\[
m^2 = \langle \Phi_I(r=0) \rangle_B = \left(1 + \frac{4(\Theta_I^2 + \Theta_s^2)}{(a_z/\ Theta_s^2)^2}\right)^{-1}
\]

\[B_I \gg \frac{\lambda}{2\pi \Theta_s Z}\]

Formula (433a) shows that with the assumptions made the intrinsic source size \( B_I \) combines with the size \( \Theta_s \) due to scattering by the Pythagorean rule.
I. Size of Pulsar Emitting Region:

Previously we assumed that pulsars have negligible angular size. This is not a priori evident. From the work of Pisareva (1959) and Salpeter (1967), it is well known that scintillations arising in a screen a distance say $D/2$ from the earth are quenched if the finite intrinsic size of the source $\Theta_I$ exceeds a critical value given roughly by

$$\Theta_{\text{crit.}} = \frac{\lambda^2}{(2\pi \Theta_S (D/2))},$$

where $\Theta_S$ is the typical scattering angle of the screen. $\Theta_S$ for the interstellar medium is expected to be much smaller than that for the interplanetary medium. Hence a pulsar showing ISS is effectively a point source for the discussion in prior sections.

It is of interest to find if the limit $\Theta_I < \Theta_{\text{crit.}}$ for pulsars showing ISS provides any useful limits on pulsar emission models. A semi-theoretical expression for $\Theta_I$ may be made on the basis of Gold's (1969) pulsar model, which suggests an upper bound on the angular diameter of the emitting region of $r_c/D$ where $r_c = \frac{Pc}{2\pi}$ is the velocity of light radius for the pulsar of period $P$, and where $D$ is the distance to the pulsar. If we express the actual instantaneous angle diameter of the pulsar radiation as $\Theta_I = \gamma (r_c/D)$, then $\gamma < 1$. The reduction factor $\gamma$ is unknown, but one may readily imagine models in which $\gamma < w/P$ or $\gamma < \frac{1}{2} (w/P)^2$ where $w$ is the pulse width; however, we do not make any assumption beyond $\gamma < 1$. For an estimate of $\Theta_{\text{crit.}}$, we assume a normalization based on $\Theta_S \approx 0.002''$ at $\lambda = 70$ cm and $D = 40$ pc,
obtained from the formula $\Omega^2 = 8c/B_\lambda D$ which corresponds to equation (176) for a screen midway (a distance $D/2$ from the earth) between source and earth, where $B_\lambda$ is the radio bandwidth for which the modulation index is one-half. Rickett's (1969) estimate $B_\lambda = 20$ MHz and Davies (1969) independent estimate $D = 40$ pc, both for CP 0950, have been used to get $\Omega_S$. One then finds,

$$\begin{align*}
\theta_I (\text{sec of arc}) &\approx 3 \times 10^{-6} \gamma \left(\frac{P}{1 \text{ sec}}\right) \left(\frac{100 \text{ pc}}{D}\right) \\
\theta_{\text{crit}} (\text{sec of arc}) &\approx 2 \times 10^{-7} \left(\frac{70 \text{ cm}}{\lambda}\right)^{\frac{1}{1}} \left(\frac{100 \text{ pc}}{D}\right)^{\frac{3}{2}}
\end{align*}$$

(434)

The interstellar scintillations are quenched for $\theta_I > \theta_{\text{crit}}$. Hence for a pulsar of period $P = 1$ sec and $D = 100$ pc showing ISS at $\lambda = 70$ cm, there is the limit $\gamma \leq 0.06$. For the same conditions, but at $\lambda = 700$ cm, $\gamma \leq 0.006$. Clearly larger $D$ and $\lambda$ for fixed $P$ give more stringent limits on $\gamma$. Alternatively, large $D$, $\lambda$ and $P$, for fixed $\gamma$ could cause a quenching of the interstellar scintillations. (The possibility of quenching might explain the anomalous nature of the intensity variations of CP 1919, in an interpretation with ISS alone, at low frequencies reported by Rickett (1969).)
J. Source at a Finite Distance:

Previously it was assumed that the source was at a large distance beyond the far screen. But if it is at a finite distance $\mathcal{Z}_o$ on the far side of the $Z$ screen as shown in Figure (35), then prior formulae are altered. The changes are small if $\mathcal{Z}_o \gg Z$, but if the source is close to the $Z$ screen, then not insignificant numerical factors enter. It follows from the geometry of Figure (35) that the wave amplitude on the ground at the earth is:

$$A(i) = \sum_{n,m} a_n b_m \exp\left\{ ik \left( \beta_1 \theta_m \beta_2 \omega_n \right) \cdot X \right\} * \exp\left\{ - ik \left( \mathcal{Z} \beta_1 \theta_m^2 + \mathcal{Z} \beta_2 \omega_n^2 + 2 \mathcal{Z} \rho \omega_n \beta_1 \theta_m \right) \right\}$$

$$\beta_1 = 1 - \frac{\mathcal{Z}}{Z + \mathcal{Z}_o} \quad ; \quad \beta_2 = \frac{\mathcal{Z}_o}{Z + \mathcal{Z}_o}$$

(435)

where the scattering amplitudes $a_n$ and $b_m$ are the same as previously defined. In general $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Note that expression (435) may be obtained from (399), which holds for $\mathcal{Z}_o \gg Z$, by the scalings,

$$\theta_m \rightarrow \beta_1 \theta_m \quad ; \quad Z \rightarrow \mathcal{Z} / \beta_1$$

$$\omega_n \rightarrow \beta_2 \omega_n \quad ; \quad Z \rightarrow \mathcal{Z} / \beta_2$$

(436)

These replacements make the formulae of prior sections applicable to the case of finite $\mathcal{Z}_o$. 


FIGURE 36: Source at a Finite Distance.
For a concrete case where the scalings (436) are important consider \( y_o \ll y = \frac{1}{2} Z \); for example, the \( y \) screen could represent the interstellar medium and the \( Z \) screen a denser plasma in the vicinity of a pulsar. Then \( \beta_1 \approx \frac{1}{2} \) and \( \beta_2 \approx \frac{y_o}{2} \ll 1 \). One effect (a) is that the scale size of intensity variations due to the \( Z \) screen is greatly magnified,

\[
r_Z \approx \frac{Z}{y_o} \left( \frac{\lambda}{2\pi \Theta_S} \right) \]

by a factor \( Z/y_o \), compared with the case \( y_o \gg Z \). A second effect is that (b) the time delay due to the far screen is reduced,

\[
\Delta t_Z \approx \frac{1}{2} \left( \frac{Z}{c} \right) \beta_2 \Theta_S^2 \approx \frac{1}{2} \left( \frac{y_o}{c} \right) \Theta_S^2
\]

under the same comparison. And thirdly (c) \( P_{nm} \approx kY \beta_2 \Theta_S \Theta_5 \approx \frac{1}{2} kY \Theta_5 \Theta_5^2 \), and this may be greater or less than unity as before in the case \( y \ll Z \ll y_o \). Time variations of the intensity on the ground due to motion of the \( Z \) screen across the line of sight with velocity \( \bar{U} \) may be obtained by the replacement \( I_Z ( \bar{x} - \bar{U} t ) \) (for the case \( y_o \gg Z \)) \( \rightarrow \)

\( I_Z ( \beta_2 \bar{x} - \bar{U} t ) \); the velocity of the \( Z \) screen diffraction pattern, \( \bar{x} / \bar{t} = \bar{U} / \beta_2 \), is increased by a factor \( Z/y_o \).

This was previously noted by Salpeter (1969). We note that the increased velocity of the diffraction pattern and the magnified scale size conspire to give a time scale for the \( Z \) screen scintillations of

\[
\tau_Z \approx \frac{\lambda}{2\pi \Theta_S |U|}
\]

formally independent of the screen's location, \( Z \).
REFERENCES


57. Lovelace, R., 1969, unpublished result.


