1. Thoughts on Space Travel for Spring 2010 Orion: I. Wasserman, 3/31/2010

To derive the relativistic rocket equation, consider what happens when a mass Δm is ejected in the spaceship's rest frame at a speed v_0. The changes in energy and momentum resulting from the ejection are $\Delta E_0 = \gamma_0 \Delta m$ and $\Delta P_0 = \Delta E_0 v_0$. If the rocket moves at speed v relative to Earth, then in the Earth frame

$$\Delta E = \gamma(\Delta E_0 + v\Delta P_0) = \gamma \Delta E_0 (1 + v v_0) \qquad \Delta P = \gamma(\Delta P_0 + v\Delta E_0) = \gamma \Delta E_0 (v + v_0)\ . \quad (1)$$

Since $P = Ev$ is the rocket momentum relative to Earth, and $E = \gamma m$ in the same frame when the rocket rest mass is m, we get

$$\Delta P = v\Delta E + E\Delta v = v\gamma(\Delta E_0 + v\Delta P_0) + m\gamma \Delta v = \gamma(\Delta P_0 + v\Delta E_0) \quad (2)$$

and therefore

$$\gamma^2 \Delta v = \tfrac{1}{2}\Delta\left[\ln\left(\frac{1+v}{1-v}\right)\right] = \frac{\Delta P_0}{m} = v_0 \frac{\Delta E_0}{m} = -v_0 \Delta \ln m \quad (3)$$

because the rocket rest mass changes by $-\Delta E_0$ to lowest order; integrating implies

$$\frac{1+v}{1-v} = \left[\frac{m(0)}{m}\right]^{2v_0} \quad \Rightarrow \quad \frac{m(0)}{m} = \left(\frac{1+v}{1-v}\right)^{1/2v_0} \quad (4)$$

where m is the mass of the rocket when it has attained speed v and $m(0)$ is its mass at zero speed. Note that this solution does not depend on the history of mass ejection, but only on the total mass ejected.

Now consider a rocket that accelerates uniformly. This means that the acceleration is fixed in the rocket rest frame; let the acceleration be a. Then in the Earth frame we have

$$\frac{d(\gamma v)}{d\tau} = \gamma a \quad (5)$$

and the solution to this equation is

$$\begin{aligned} \gamma v &= \sinh a\tau \\ \gamma &= \cosh a\tau \\ v &= \tanh a\tau \\ \frac{1+v}{1-v} &= \exp(2a\tau) \\ x &= \frac{\cosh a\tau - 1}{a} \\ t &= \frac{\sinh a\tau}{a}\ . \end{aligned} \quad (6)$$

Now suppose we wish to go a total distance D in the Earth frame going out halfway at a uniform accleration a and then the rest of the way at uniform acceleration $-a$ so that we arrive at our destination with zero speed. Then we go halfway in a proper time

$$a\tau_{\frac{1}{2}} = \cosh^{-1}(1 + aD/2) \tag{7}$$

and the maximum speed attained is $v_{\rm max}$ where

$$\begin{aligned} v_{\rm max} &= \frac{\sqrt{aD(1 + aD/4)}}{1 + aD/2} \\ \gamma_{\rm max} &= 1 + aD/2 \ . \end{aligned} \tag{8}$$

The entire trip requires a proper time

$$2a\tau_{\frac{1}{2}} = 2\cosh^{-1}(1 + aD/2) \ , \tag{9}$$

and using the relativistic rocket equation we see that the trip requires a ratio of final mass m to initial mass $m(0)$ given by

$$\frac{m(0)}{m} = \exp\left[\frac{2\cosh^{-1}(1 + aD/2)}{v_0}\right] = \left[1 + aD/2 + \sqrt{aD(1 + aD/4)}\right]^{2/v_0} \quad ; \tag{10}$$

this can be inverted to find

$$aD = [m/m(0)]^{v_0/2} + [m(0)/m]^{v_0/2} - 2 = \{[m/m(0)]^{v_0/4} - [m(0)/m]^{v_0/4}\}^2, \tag{11}$$

which gives the distance that can be travelled in terms of the ratio of the masses, a useful relationship if limitations on the ratio of fuel to spaceship mass is invoked. The implied distance increases at $m(0)/m > 1$.

Now a comfortable journey would be at an acceleration that equals the acceleration of gravity on Earth, $g = 980\,{\rm cm\,s^{-2}} \approx 1.03c\,{\rm yr}^{-1} \approx c/0.97\,{\rm yr}$. Then, for example, a trip to α Cen, which is at $D = 4.5\,{\rm yr}$, has $gD/2 \approx 2.25 \times 1.03 \approx 2.32$. During the trip, the astronauts age 3.62 years, whereas their counterparts on Earth age

$$t = \frac{2\sqrt{aD(1 + aD/4)}}{a} = \frac{2\sqrt{gD(1 + gD/4)}}{g} \tag{12}$$

or about 6.14 years. The maximum speed attained in this journey is $v_{\rm max} \approx 0.954$, and the mean speed is $\langle v\rangle = 4.5/6.14 \approx 0.733$. Note that the astronauts age *less than* 4.5 years, a reflection of their relativistic motion and time dilation.

The amount of fuel consumed during the journey depends critically on the value of v_0. The largest value of v_0 is one, and that is for a rocket that annihilates its fuel completely.

In this case, $m(0)/m \approx 42$. But this is a *lower bound* on this ratio. For more realistic rocket fuel, the requirements are more stringent. Nuclear energy derived from H fusion, a more practical source of fuel, would have $v_0 \approx 0.122$, and in this case we would have $m(0)/m \approx 2 \times 10^{13}$.

For nuclear fuel, a practical limitation on the maximum distance that can be travelled may be derived from assuming that at most all of the H in the solar system could be packed along as fuel. That amounts to something like 10^{30} kg of H. Even with the most Spartan design, it seems hardly possible for a spaceship to weigh less than a ton, or about 1000 kg. Thus, a practical limitation would be $m(0)/m \lesssim 10^{27}$. For such a spacecraft, the distance travelled cannot exceed about $41.06/(a/g)$ light years! Longer trips can be made at lower accelerations, since the distance travelled is inversely proportional to the acceleration given $m(0)/m$, but the biological effects of protracted living at sub-g gravity may be prohibitive. The astronauts would age about $7.36/(a/g)$ years during this longest-possible nuclear powered journey.

The only practical hope for long trips at $a = g$ is for the astronauts to scavenge along the way.