Universal Expansion: Classical Analogy

While not formally correct, a Newtonian treatment of an expanding Universe yields an advantageously intuitive picture of the dynamics. We shall thus use it, and later point out the inadequacies of the analogy.

Assume the Universe is filled by a mass distribution of density $\rho$, and is homogeneous, isotropic and Euclidean. The Universe expands. If the expansion preserves homogeneity and isotropy, the length $l(t)$ of a segment connecting two galaxies G1, G2, that partake of the expansion ("comoving observers") can be written as

$$l(t) = a(t)l(t_0)$$

(1)

where $l(t_0)$ is the length of the segment at some arbitrary time $t_0$ (e.g. now) and $a(t)$ is a scale factor. Preservation of homogeneity and isotropy demands that $a(t)$ be independent of location in space and orientation, so it is only a function of time. Taking the derivative

$$\dot{l}(t) = \dot{a}(t)l(t_0) = \frac{\dot{a}(t)}{a(t)}l(t) = H(t)l(t)$$

(2)

where $H(t)$ is referred to as the Hubble parameter. The rightmost equality above is valid if $a(t)$ is multiplied by any arbitrary constant.

Suppose $l(t)$ is small, so that $\dot{l}(t) \ll c$. G1 sends a light signal of wavelength $\lambda_{em}$ at time $t$, which is received by G2 at time $t + \Delta t = t + l(t)/c$; the wavelength of the signal received by G2 is $\lambda_{obs} = \lambda_{em} + \Delta \lambda$ where (using the non-relativistic Doppler formula)

$$\frac{\Delta \lambda}{\lambda_{em}} \approx \frac{\dot{l}(t)/c}{\lambda_{em}} = \frac{\dot{a}(t)}{a(t)} \Delta t = \frac{\Delta a(t)}{a(t)}$$

(3)

If the observer’s time is $t_0 = t + \Delta t$, we get

$$\frac{\lambda_{obs}}{\lambda_{em}} - 1 = \frac{a(t_0)}{a(t)} - 1$$

(4)

where $z$ is the redshift.
In Newtonian mechanics, the gravitational field outside a spherically symmetric mass distribution is the same as if the mass were concentrated at the center of the sphere. This results from the application of Gauss’ theorem to the gravitational field. It can also be shown that the gravitational field at any point within an empty cavity, produced by a mass distribution spherically symmetric surrounding it, is null. Thus, we can study the gravitational effect of a homogeneous and isotropic Universe upon a test particle of mass \( m \) at any point within it as if the particle were on the surface of a sphere of radius \( l \); the field produced by the matter outside the sphere is null, so the only relevant gravitational force will be that produced by the mass within the sphere. The equation of motion of the particle is
\[
ml = -\frac{GMm}{l^2} = -\frac{4\pi}{3} Gm \bar{\rho} l
\]
(5)

As the Universe expands, \( \bar{\rho}(t) = \bar{\rho}(t_0) [a(t_0)/a(t)]^3 \); note also that we can set \( a(t_0) \equiv 1 \). Defining \( \bar{\rho}(t_0) \equiv \rho_0 \) and applying eqn. (1):
\[
\ddot{a}(t) = -\frac{4\pi}{3} \frac{G \rho_0}{a^2(t)}
\]
(6)

Multiplying both sides by \( \dot{a} \):
\[
\ddot{a} \dot{a} = \frac{1}{2} \frac{d}{dt} \dot{a}^2 = -\frac{4\pi}{3} \frac{G \rho_0}{a^2(t)} \dot{a}(t) = \frac{4\pi}{3} G \rho_0 \frac{d}{dt} \left( \frac{1}{a} \right)
\]
(7)

which reduces to
\[
\frac{d}{dt} \left( \dot{a}^2 - \frac{8\pi}{3} \frac{G \rho_0}{a} \right) = 0
\]
(8)

and integrates to
\[
\dot{a}^2 + k c^2 - \frac{8\pi}{3} \frac{G \rho_0}{a} = 0
\]
(9)

where \( k \) is an integration constant, referred to as the curvature parameter, because in the general relativistic derivation it is related to the curvature of space. Let’s now consider the consequences of this derivation.
• From Eqn. (6), we can see that if $\rho_0 > 0$, then $\dot{a}$ can never be zero, i.e. \textit{a Universe filled with matter cannot be static}.

• If $k < 0$ then
  \[ \dot{a} = \frac{8\pi G \rho_0}{3a} - kc^2 > 0 \] 
always. The Universe expands forever; the expansion rate diminishes with time, but never becomes zero. The Universe is said to be \textit{open}.

• If $k > 0$ then there is a time for which $\dot{a}(t) = 0$; the Universe reaches a maximum size
  \[ a_{\text{max}} = \frac{8\pi G \rho_0}{3kc^2} \] 
then starts to collapse. The Universe is said to be \textit{closed}.

• If $k = 0$
  \[ \dot{a}^2 = \frac{8\pi G \rho_0}{3a} \Rightarrow \dot{a} = \sqrt{\frac{8\pi G \rho_0}{3}} a^{-1/2} \]
which can be written as
  \[ \sqrt{\dot{a}} da = \sqrt{\frac{8\pi G \rho_0}{3}} dt \] 
and integrated to
  \[ a(t) = \left[ \frac{3}{2} \left( \frac{8\pi G \rho_0}{3} \right)^{1/2} \right]^{2/3} t^{2/3} \]
The Universe then expands forever, $a(t) \propto t^{2/3}$, but the rate of expansion, by Eqn. (12), $\dot{a}(t) \propto a^{-1/2} \propto t^{-1/3}$ tends to zero as $t \to \infty$. The Universe is said to be \textit{flat}. Replacing $\rho(t) = \rho_0 a^{-3}(t)$ in Eqn. (12), applying the definition of $H(t)$ and solving for $\rho$ we find that the density of a flat Universe (referred to as the \textit{critical density}) is related to the Hubble parameter as

\[ \rho = \rho_{\text{crit}} \equiv \frac{3H^2(t)}{8\pi G} \] 

For the currently most likely value of the Hubble parameter, $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$, $\rho_{\text{crit}} = 9.2 \times 10^{-30}(H_0/70)^2$ g cm$^{-3}$.

\begin{center}
\textbf{Universal Expansion: Relativistic Extension}
\end{center}

\begin{itemize}
\item Newtonian mechanics does not account for the contribution of relativistic particles to the universal contents. A General Relativity treatment introduces that contribution, which can be obtained by replacing $\rho$, of the previous treatment, with
\[
\rho \quad \Rightarrow \quad \rho + 3p/c^2
\]
where $p$ is the pressure of the relativistic energy component. Equations (6) and (9), which describe the evolution of the scale factor, can now be written respectively as
\begin{equation}
\ddot{a}(t) = -\frac{4\pi}{3} G(\rho + 3p/c^2)a(t)
\end{equation}
and
\begin{equation}
a^2(t) + kc^2 - \frac{8\pi}{3} G(\rho + 3p/c^2)a^2(t) = 0
\end{equation}
These are generally referred to as the \textit{Friedmann Equations}. They were first derived by the Russian physicist A. Friedmann in 1924, from Einstein’s gravitational field equations.

\item As noted before for the Newtonian derivation, we see from Eqn. (17) that the Universe cannot be static unless
\[
\rho = -3p/c^2
\]
i.e. unless either the energy density or the pressure is negative.

When Einstein first derived the gravitational field equations in 1917, he assumed the Universe was static. In order to force such
a solution, he introduced a term in the field equation which would have such effect. Without such “corrective” term, the relativistic field equation — which we will neither derive nor solve in this course — looks like
\[ R_{ij} - (1/2)g_{ij}R = \frac{8\pi G}{c^4} T_{ij} \]  
(20)
where the left–hand side — which includes the Ricci tensor \( R_{ij} \), the scalar curvature \( R \) and the metric tensor \( g_{ij} \) — describes the properties of the space geometry, while the right–hand side — which includes the momentum–energy tensor \( T_{ij} \) — describes its matter–energy contents. Equations (17) and (18) can be derived from Equation (20). However, none of the solutions resulting from it are, as said above, static. The introduction of an additional term in the field equation was allowed, without violating any of the assumptions used to derive the field equation; it implied however the presence of an energy component of — as we shall see — rather peculiar properties. The field equation with Einstein’s so–called \[ \text{cosmological constant } \Lambda \] is
\[ R_{ij} - (1/2)g_{ij}R - \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij} \]  
(21)

The solution of this equation is similar to that expressed by Equations (17) and (18), provided that one replaces \( \rho \) and \( p \) with
\[ \tilde{\rho} = \rho + \frac{\Lambda c^2}{8\pi G} \quad \tilde{p} = p - \frac{\Lambda c^4}{8\pi G} \]  
(22)

Equations (17) and (18), modified according to Equations (22), have a static solution, which corresponds to
\[ \tilde{\rho} = -3\tilde{p}/c^2 \]  
(23)
in which case \( \Lambda \) must be > 0 and \( \tilde{p} \) can be < 0. A Universe with \( \Lambda > 0 \) and a relativistic component which is negligible in comparison with the non–relativistic one (i.e. \( p/c^2 \ll \rho \)) is known as the \textit{Einstein solution}, and sometimes as the \textit{dust Universe}, dust being thought of as a pressureless material. After it was proposed, this solution was shown to be unstable, and when in 1929 Hubble discovered the universal expansion, Einstein rejected it as “his greatest error ever”.

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Yet Phoenix will keep rising from the ashes. It is another interesting historical fact that, a few years earlier, Einstein had also criticized Friedmann’s solutions: first as erroneous; then, standing corrected, as “non-physical”.

Another interesting solution of Eqn. (21) is the so-called [de Sitter model], referring to a Universe which is empty \((p = 0, \; \rho = 0)\) and flat \((k = 0)\). Then

\[
\tilde{p} = \frac{\Lambda c^4}{8\pi G} \quad \text{and} \quad \tilde{\rho} = \frac{\Lambda c^2}{8\pi G}
\]

Substitution in Eqn. (18) yields

\[
\dot{a}^2 = \frac{\Lambda}{3} c^2 a^2
\]

which implies that \(\Lambda > 0\) and yields as a solution for the scale factor

\[
a(t) = A e^{\sqrt{\frac{\Lambda}{3}}} 
\]

an exponential expansion with a constant Hubble parameter given by \(H = \dot{a}/a = c\sqrt{\Lambda/3}\). We’ll encounter this model again when we discuss inflation.

\[\blackbox{Cosmological Equations of State}\]

The dynamics of the Universe is determined, via the Friedmann equations (17) and (18), with the modification expressed by Eqn. (22), when we know the equation of state of each of its components, i.e. the relationship between pressure and energy density. The quantities \(\tilde{p}\) and \(\tilde{\rho}\) can be thought as the sum of several fluid components: \(\tilde{p} = \Sigma_i p_i\) and \(\tilde{\rho} = \Sigma_i \rho_i\). The equation of state of any component \(i\) can be written in the general form

\[
p_i = w_i \rho_i c^2
\]

where the parameter \(w_i\) characterizes the component and is < 1. For example, for pressureless matter (e.g. non-relativistic matter),
$w_i \simeq 0$; for relativistic particles (e.g. photons), $w_i = 1/3$. As for so-called vacuum energy associated with the cosmological constant, as can be seen from Eqns. (22) for $p = 0$ and $\rho = 0$: $\tilde{p} = -\tilde{\rho}c^2$, i.e. $w_i = -1$.

Given a fluid component of equation of state (27), the assumption that the Universe expands adiabatically, i.e. $dU = -pdV$ implies

$$d(\rho c^2 a^3) = -pda^3 = -3\rho a^2 da$$

(28)

This can be used to describe the evolution of the fluid component with time. Note that Eqn. (28) is not independent on Eqns. (17) and (18); e.g. the second Friedmann eqn. can be easily derived from the first and Eqn. (28). The form of Eqn. (28) is however useful in this instance. Combining Eqns. (27) and (28) and integrating, we can obtain

$$\rho a^{3(1 + w_i)} = \text{const.}$$

(29)

We can then see that a non-relativistic matter component of the Universe ($w_i \simeq 0$) evolves like

$$\rho_{\text{matter}} \propto a^{-3} (t) \propto (1 + z)^3$$

(30)

where we have also used Eqn. (3). In the case of the relativistic particle fluid ($w_i = 1/3$), we have

$$\rho_{\text{rel}} \propto a^{-1} (t) \propto (1 + z)^4$$

(31)

while in the case of vacuum energy ($w_i = -1$),

$$\rho_{\Lambda} = \text{const.}$$

(32)

Consider Eqn. (18); if we divide all terms by $a^2$, we see that the curvature term $kc^2/a^2$ behaves as a matter/energy term of density $\rho_{\text{curv}} \sim 3kc^2/(8\pi Ga^2)$ which, since $k$ and $c^2$ are constants, depends on the scale factor as

$$\rho_{\text{curv}} \propto a^{-2} (t) \propto (1 + z)^2$$

(33)

By virtue of Eqn. (29), the curvature term is equivalent to a component of energy density with equation of state (27) characterized by $w_i = -1/3$. 

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We also define a \textit{density parameter }$\Omega$

$$\Omega(t) = \sum_i \Omega_i = \frac{\sum_i \rho_i}{\rho_{\text{crit}}}$$ \hfill (34)

where $\rho_{\text{crit}}$ is given by Eqn. (15). A value of $\Omega = 1$ corresponds to a flat Universe.

We distinguish the value of the density parameter and the Hubble parameter, as measured at $t = t_0$ (now), by $\Omega_0$ and $H_0$.

\begin{center}
\textbf{Time Evolution of Cosmological Parameters}
\end{center}

\begin{itemize}
\item Because different components of the Universe evolve at different rates, those that dominate its dynamics at one time may not do so at a different one. For example, early enough in the history of the Universe, when its temperature was very high, relativistic particles are thought to have dominated the energy density. Since their density falls off faster than that of non–relativistic matter (compare Eqns. 30 and 31), the latter eventually evolved to exceed the former. If the values of each component at the present time $t_0$ are respectively $\rho_{\text{matter},0}$ and $\rho_{\text{rel},0}$, the transition from a radiation–dominated Universe to a matter–dominated one takes place at time $t$ such that

$$\rho_{\text{rel},0} [a(t)/a(t_0)]^{-4} = \rho_{\text{matter},0} [a(t)/a(t_0)]^{-3}$$ \hfill (35)

i.e when the scale factor is

$$\frac{a(t)}{a(t_0)} = \frac{\rho_{\text{rel},0}}{\rho_{\text{matter},0}} \quad \Rightarrow \quad z = \frac{a(t_0)}{a(t)} - 1 = \frac{\rho_{\text{matter},0}}{\rho_{\text{rel},0}} \approx 2 \times 10^4 \hfill (36)$$

where the number results from observational estimates of current matter and radiation density. A similar exercise could be carried out with other components, provided that reliable observational constraints at $t = t_0$ were available.

\item We can obtain a relation between $t$ and $a(t)$ by inverting

$$t_0 - t = \int_t^{t_0} dt = \int_a^{a_0} \frac{da}{a}$$ \hfill (37)
\end{itemize}
Substituting Eqn. (18) for $\dot{a}$ and assuming a superposition of components with different equations of state, the integral on the right can be rewritten as

$$\int_{a_i}^{a_e} a^{-1} \left( (8\pi G/3) \Sigma_i \rho_i(t_o)(a/a_o)^{-3(1+w_i)} - k c^2 \right)^{-1/2} da$$

(38)

While a full knowledge of $a(t)$ requires knowing the relative importance and the equation of state of each component $i$, we can assume that at any given time only one of them is dominant. In the simple case of a flat Universe ($k = 0$), the duration of the epoch of domination of a single component $i$ can be obtained from

$$a(t)/a_o = (t/t_o)^{2/[3(1+w_i)]}$$

(39)

(for a non flat Universe, we can use Eqn. 39 with $w_i = -1/3$, as indicated in the discussion following Eqn. 33). In the flat case, if a single component dominates most of the history of the Universe, then solving Eqn. (38) for $t = 0$ yields an estimate of the age of the Universe:

$$t_o = \frac{2}{3(1+w_i)} H_o^{-1}$$

(40)

Suppose a single component dominates the expansion of the Universe, characterized by an equation of state $p = \omega \rho c^2$. Equation (18) can be rewritten, after a little math, as

$$H^2(t) = H_o^2 \left[ \frac{a_o}{a(t)} \right]^2 \left[ \Omega_o \left[ \frac{a_o}{a(t)} \right]^{(1+3w)} + (1 - \Omega_o) \right]$$

(41)

and using the relationship between $z$ and $a(t)$:

$$H^2(z) = H_o^2 (1+z)^2 \left[ \Omega_o (1+z)^{(1+3w)} + (1 - \Omega_o) \right]$$

(42)

We can also obtain an equation that describes the evolution of $\Omega(t)$, again for a single component Universe, since

$$\Omega(z) = \frac{\rho(z)}{3 H^2(z) / 8\pi G}$$

(43)
which, using Eqn. (29) for the evolution of the energy density:
\[ \rho_i(z) = \rho_i(1 + z)^{3(1+w)} \], becomes
\[
\Omega(z) = \frac{\Omega_0 (1 + z)^{1+3w}}{(1 - \Omega_0) + \Omega_c (1 + z)^{1+3w}}
\] (44)

Consider now the consequences of the relations derived in the last two sections.

- Tackle first Eqn. (44). You can verify that if \( \Omega_0 > 1 \), \( \Omega(z) > 1 \) at all times. Similarly, if \( \Omega_0 < 1 \), \( \Omega(z) < 1 \) at all times. Note also that, as \( z \to \infty \), i.e. as \( t \to 0 \), \( \Omega(z) \to 1 \). Conversely, a small deviation of \( \Omega(z) \) from unity very early on (at high \( z \)), will be amplified into a very large deviation at \( t = t_c \). These considerations will be refreshed when we deal with the so-called flatness problem.

- for a **matter-dominated Universe**
  \[ a(t) \propto t^{2/3} \text{ and } t_c = (2/3)H_0^{-1} \]

- for a **radiation-dominated Universe**
  \[ a(t) \propto t^{1/2} \text{ and } t_c = (1/2)H_0^{-1} \]

- for a **curvature-dominated Universe**
  \[ a(t) \propto t \text{ and } t_c = H_0^{-1} \]

- for a **\( \Lambda \)-dominated Universe**
  \[ a(t) \propto e^{H_0 t} \text{ and } t_c = \infty \]

We also see, by combining Eqn. (17) with Eqn. (27), that components of the Universe with \( w_i > -1/3 \) cause the expansion so decelerate, while components with \( w_i < -1/3 \) cause the expansion to accelerate. The latter, as indicated by Eqn. (27), have negative pressure; the acceleration of the expansion is not due to some hydrodynamical effect associated with the negative pressure (since the “fluid” is homogeneous, there are no pressure gradients and thus no net force). The acceleration is rather due to the fact that such components have negative effective gravitational mass, which is in principle allowed by General Relativity.
• Note also that the different evolution of each component allows, for example, for an early Universe dominated by radiation (relativistic particles with $w_i = 1/3$), to evolve into a matter-dominated Universe (where the dominant component has $w_i = 0$); during these two “eras”, positive gravitational mass slows down the expansion. At a later stage, the energy densities of both matter and radiation may have dropped below that of, for example, vacuum energy (with $w_i = -1$), and the expansion starts speeding up.

• For the earlier, simpler solutions obtained from the Friedmann equations which include only matter and radiation (no components with $w_i < -1/3$), both the geometry and the fate of the Universe are determined by the total value of $\Omega$. If we allow however a component with $w_i < -1/3$, the Universe may be flat but its expansion will eventually accelerate for as long as that component dominates. We could also have a positive curvature, but ever-expanding Universe.

In the second half of the XX century, the “program” of Cosmology consisted in determining the Hubble parameter and the so-called deceleration parameter

$$q(t) = -\frac{\ddot{a}(t) a(t)}{\dot{a}^2(t)}$$

($q_o$ at $t = t_o$), due to the presumption that only stuff with positive effective gravitational mass filled the Universe. Recent observational estimates, which indicate accelerating expansion, have altered the “program”. It is now necessary to characterize accurately the equation of state of each relevant component of the Universe, in order to be able to predict its fate.