1. Introduction

A clear picture on how galaxies form is not yet available. It is even necessary to specify what we mean by the term “galaxy formation”: since galaxies are gravitationally bound objects, by “formation” do we mean the process that leads to the gravitational assembly of most of the mass? Or, since a galaxy is generally identified observationally by its starlight, is the time of “formation” that by which most of its stellar mass is actually shining? It may well be that neither of this concepts is correct, as the process of galaxy formation may be a continuous process of accretion, dotted by episodes of merger.

What is well agreed on is that the process is driven by gravitational instability, which amplifies density fluctuations seeded very early on after the Big Bang. Let’s review briefly the historical background of the main ideas that play a role in this field.

1.1. Historical Background

• In 1664, Issac Newton first derived the law of Universal Gravitation. he also proposed that the Universe had to be infinite, for a finite universe would collapse to its center due to the mutual gravitational attraction of its constituents. He also realized that the mass distribution had to be homogeneous, for any deviation from homogeneity would lead to gravitational collapse: in other words, if the Universe is in equilibrium, that equilibrium is unstable.

• In 1917, Albert Einstein obtained a self-consistent set of equations of the gravitational field, within the framework of the Theory of General relativity. As Newton before him, he found that his equations described an unstable Universe, unless a cosmological constant term Λ was introduced.

• In 1922 and 1924, Aleksandr Fridman reported a new set of solutions to the gravitational field equations; these included expanding solutions as well as collapsing solutions after an
earlier expansion to a maximum radius. This set of solutions was independently found by Georges Lemaître in 1927.

• In 1929, Edwin Hubble discovered the velocity–distance relation and measured an expansion rate $H_0 = 500 \text{ km s}^{-1} \text{ Mpc}^{-1}$, showing that we live in an expanding Universe, as in one of Fridman’s solution. However, the age of a Fridman expanding Universe with $\Lambda = 0$ is $< H_0^{-1}$; given the value of $H_0$ measured by Hubble, $H_0^{-1} \approx 2 \times 10^9 \text{ yrs}$, is less than the age of the Solar System. The adoption of a $\Lambda \neq 0$ term was used early on as a means to resolve the discrepancy between $H_0^{-1}$ and that of its oldest constituents.

• In the late 1940’s, George Gamow and his collaborators Ralph Alpher and Robert Herman attempted to explain the origin of chemical elements by primordial nucleosynthesis, taking place in the early hot phase of the universal expansion. This program did not work, due in part to the difficulties of overcoming the absence of stable isotopes of atomic number 5 and 8, and in part to the success of scenarios that showed that heavy elements could effectively be produced within stars (Burbidge, Burbidge, Fowler and Hoyle 1956). Gamow and co-workers however made the prediction that the Universe should be bathed in a background radiation field with the spectral characteristics of a blackbody, with $T \sim 5 \text{ K}$, representing the relic of the hot early phase.

• By the early 1960’s it was appreciated that the cosmic abundance of He appeared to be remarkably constant everywhere, at about 25% by mass. Stellar nucleosynthesis was recognized to be largely insufficient in accounting for that much He. In 1964, Hoyle and Tayler showed that primordial nucleosynthesis could account for the observed He abundance quite accurately. Wagoner, Fowler and Hoyle also showed that the observed abundances of $^3\text{He}$, $^4\text{H}$ and $^7\text{Li}$ — difficult to account in a pure stellar nucleosynthesis scenario — could be explained by primordial processes.

• In 1965, Arno Penzias and Robert Wilson reported the discovery of the CMB, the existence of which had been predicted by Alpher, Gamow and Herman.

• In 1992, the COBE satellite discovered fluctuations in the CMB (other than the dipole, which by then was known already), revealing deviations from homogeneity of a few–parts–in–a–million that were present in the early Universe, which evolved into the large amplitude fluctuations such as galaxies, clusters and superclusters which characterize large–scale structure at $z = 0$.

It is within this cosmological framework that the modern picture of galaxy formation emerges. The fundamental physical ingredient of that picture is that of gravitational instability, whereby gravity locally overcomes pressure and universal expansion, eventually driving matter towards collapse.
\begin{itemize}
\item The growth of density fluctuations in an otherwise stationary medium was first solved by James Jeans in 1902. Gravitational collapse ensues if the size of the fluctuation exceeds a threshold value referred to as \textbf{Jeans' length} $\lambda_J$. Fluctuations of scale $> \lambda_J$ grow exponentially, as gravity overcomes pressure gradients.
\item In the 1930’s and 1940’s, Tolman and Lifshitz studied gravitational instability in an expanding Universe. As Jeans did, they found that fluctuations with size $> \lambda_J$ grow, but the growth rate is slower than in the static case, and the specific form of the growth rate depends on the parameters of the Cosmology.
\end{itemize}

We will start our inspection of the galaxy formation scenario by revisiting the picture of gravitational instability in an expanding Universe. These notes make heavy use of Coles & Lucchin (1995), Padmanabhan (1993) and Longair’s (1998).

\subsection*{1.2. Epoch of Galaxy Formation: a Rough Estimate}

Note the following. If an object of mass $M$ formed by gravitational collapse from an initially small amplitude density fluctuation of radius $r$:

$$M(r) = (4\pi/3) r^3 \bar{\rho}_{\text{matter}} = (4\pi/3) r^3 (3H_0^2/8\pi G) \Omega_{\text{matter}}$$

where $\Omega_{\text{matter}} = \bar{\rho}_{\text{matter}}/\rho_{\text{crit}}$. For $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$, we can rewrite

$$M(r) = 1.16 \times 10^{12} h^2 \Omega_{\text{matter}} r_{\text{Mpc}}^3 M_{\odot}.$$  \hfill (2)

For a conventional choice of cosmological parameters, e.g. $h = 0.7$ and $\Omega_{\text{matter}} = 1/3$,

$$M(r) = 1.9 \times 10^{11} r_{\text{Mpc}}^3 M_{\odot}.$$  \hfill (3)

i.e., a roughly “$L^*$ galaxy” such as the Milky Way formed from coalescence of matter over a Mpc sized comoving volume.

The typical radius of a visible $L^*$ galaxy at $z = 0$ is $\sim 10 - 20$ kpc; however, we know that much of the mass of the galaxy resides in an invisible dark matter (DM) halo, which most likely extends much farther out than the visible stars; let’s conservatively assume that the halo radius is 50 kpc; then such a galaxy represents a density enhancement $\delta \rho/\bar{\rho}_{\text{matter}} \sim (1000/50)^3 \sim 10^4$ (similarly, a cluster of galaxies represents a density enhancement of $\sim 10^2$ to $10^3$, while a supercluster filament has $\delta \rho/\bar{\rho}_{\text{matter}} \sim$ several). Consider the density fluctuation that eventually led to an $L^*$ galaxy at $z = 0$. Since $\bar{\rho}_{\text{matter}}$ evolves like $(1 + z)^3$, $\delta \rho/\bar{\rho}_{\text{matter}}$ for that fluctuation was $\leq 1$ at $z \simeq 20$. Galaxies could then not have “separated” from the Hubble flow earlier than $z \simeq 20$; if they had, they’d be much denser objects now (similarly, the “formation epoch” for clusters must be $z \leq 5$ to 10).
2. Linear Regime: Jeans’ Instability

A century ago, J. Jeans first investigated the process of gravitational instability that now constitutes the cornerstone of our understanding of star and galaxy formation. He showed that density fluctuations superimposed on a smooth (homogenous and isotropic) gas can evolve and lead to gravitational collapse if self–gravity can overcome the gas pressure. The criterion for a fluctuation of scale $\lambda$ to be gravitationally unstable is simply that $\lambda > \lambda_J$, where $\lambda_J$ is a threshold value known as Jeans length, which is defined in terms of the properties of the gas.

A detailed treatment of gravitational instability in a static medium can be found in MB (Chapter 5) and in CL (Chapter 10). We are interested in solving that problem in the case of an expanding Universe (see Longair Chapter 11). The static case will of course be a subset of that solution. In the formal development, we will keep with the most general case, and consider the static case in passing.

**Step 1** The fundamental equations of gas dynamics are:

- The **equation of Continuity**, which is equivalent to a statement of conservation of mass
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{4}
  \]
  where $\vec{v}$ is the velocity

- **Euler’s equation**, which describes the motion of a fluid element
  \[
  \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p - \nabla \phi \tag{5}
  \]
  where $p$ is the pressure

- **Poisson’s equation**, which relates the matter density distribution $\rho$ to the gravitational potential $\phi$
  \[
  \nabla^2 \phi = 4\pi G \rho \tag{6}
  \]
  where $G$ is the gravitational constant

The parameters $\rho$, $\vec{v}$, $p$ and $\phi$ are defined for a given point in space and time, $(\vec{x}, t)$; the partial derivatives thus describe variations of those parameters at that fixed point in space and $\vec{v} = d\vec{x}/dt$. The description given by equations 4–6 is referred to as the Eulerian description of gas dynamics. When the coordinate system is chosen in such a way that derivatives refer to changes in a gas parcel as it moves along with speed $\vec{v}$, the description is
referred to as a *Lagrangian* one. It can be shown that the derivatives $d/dt$ in the Lagrangian description are related to those in the Eulerian description in the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

(7)

so that the equations of gas dynamics can be written in Lagrangian form:

| \frac{dp}{dt} = -\rho \nabla \cdot \vec{v} \quad (8) |
| \frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p - \nabla \phi \quad (9) |
| \nabla^2 \phi = 4\pi G \rho \quad (10) |

The Lagrangian form is preferred in cosmological applications, for the behavior of the fluid is better analyzed in a comoving reference frame, i.e. one where the velocity $\vec{v}$ is that of the universal expansion.

**Step 2** Now consider a small perturbation superimposed on the “background” fluid, i.e.

$$\rho = \rho_0 + \delta \rho, \quad \vec{v} = \vec{v}_0 + \delta \vec{v}, \quad p = p_0 + \delta p, \quad \phi = \phi_0 + \delta \phi$$

(11)

where we refer to $\rho_0, \vec{v}_0, p_0, \phi_0$ as the *unperturbed* or *background* solution and to $\delta \rho, \delta \vec{v}, \delta p, \delta \phi$ as the *perturbation*. The unperturbed gas satisfies

$$\frac{d\rho_0}{dt} = -\rho_0 \nabla \cdot \vec{v}_0$$

(12)

$$\frac{d\vec{v}_0}{dt} = -\frac{1}{\rho_0} \nabla p_0 - \nabla \phi_0$$

(13)

$$\nabla^2 \phi_0 = 4\pi G \rho_0$$

(14)

while for the perturbation we have

$$\frac{d(\rho_0 + \delta \rho)}{dt} = -(\rho_0 + \delta \rho) \nabla \cdot (\vec{v}_0 + \delta \vec{v})$$

(15)

$$\frac{d(\vec{v}_0 + \delta \vec{v})}{dt} = -\frac{1}{(\rho_0 + \delta \rho)} \nabla (p_0 + \delta p) - \nabla (\phi_0 + \delta \phi)$$

(16)

$$\nabla^2 (\phi_0 + \delta \phi) = 4\pi G (\rho_0 + \delta \phi)$$

(17)

Expanding eqn. 15, dropping terms with products of perturbations and subtracting eqn. 12, we obtain
\[
\frac{d}{dt}(\delta \rho) = -\nabla \cdot \delta \vec{v} \quad (18)
\]

Expanding eqn. 16, dropping terms with products of perturbations, assuming that the unperturbed solution is homogeneous and isotropic (so that \nabla p_o = 0 and \nabla \rho_o = 0) and subtracting eqn. 13, we can get

\[
\frac{d(\delta \vec{v})}{dt} + (\delta \vec{v} \cdot \nabla)\vec{v}_o = -\frac{1}{\rho_o} \nabla (\delta p) - \nabla (\delta \phi) \quad (19)
\]

Finally, subtracting eqn. 14 from eqn. 17, we get

\[
\nabla^2 (\delta \phi) = 4\pi G \delta \rho \quad (20)
\]

**Step 3** In an expanding Universe, it is convenient to use comoving coordinates \( \vec{r} \), so that \( \vec{x} = a(t) \vec{r} \), where \( a(t) \) is the scale factor. Then the velocity is

\[
\vec{v} = \frac{d\vec{x}}{dt} = \frac{d[a(t)\vec{r}]}{dt} = \vec{r} \left[ \frac{da(t)}{dt} \right] + a(t) \frac{d\vec{r}}{dt} \quad (21)
\]

In an expanding Universe, the *unperturbed* solution’s velocity \( \vec{v}_o \) can be identified with the Hubble expansion velocity \([\frac{d a(t)}{dt}] \vec{r}\) and the *perturbation* \( \delta \vec{v}_o \), also referred to as the *peculiar velocity*, with \( a(t) (\frac{d \vec{r}}{dt}) = a(t) \vec{u} \). Equation 19 then becomes

\[
\frac{d(a \vec{u})}{dt} + (a \vec{u} \cdot \nabla) \vec{u}_o = -\frac{1}{\rho_o} \nabla (\delta p) - \nabla (\delta \phi) \quad (22)
\]

Note that, since \( a \) is the same throughout the Universe, \( \frac{da}{d\vec{x}} = 0 \), hence \( \frac{d \vec{x}}{dr} = ad \vec{r} \) and derivatives with respect to the comoving space coordinate relate to those with respect to \( \vec{x} \) as \( d/\text{d}r = (1/a)d/\text{d}x \). The operator \( \nabla \), which involves a derivative with respect to \( \vec{x} \), relates to its analog \( \nabla_c \) which is obtained when space derivatives are taken with respect to the comoving coordinates, in the form \( \nabla = a^\text{\text{-}1} \nabla_c \). Now consider the second term on the left-hand side of Eqn. 22; \( \nabla \dot{a} = 0 \), and \( \nabla \vec{r} = \frac{d \vec{r}}{d \vec{x}} = a \frac{d \vec{r}}{dt} = 1/a \), hence \( (a \vec{u} \cdot \nabla) \dot{a} \vec{r} = \vec{u} \dot{a} \). and eqn. 22 becomes

\[
\frac{d\vec{u}}{dt} + 2\left(\frac{\dot{a}}{a}\right) \vec{u} = -\frac{1}{\rho_o a^2} \nabla_c (\delta p) - \frac{1}{a^2} \nabla_c (\delta \phi) \quad (23)
\]
If we next

- take the divergence in comoving coordinates of eqn. 23;
- assume that the perturbations are adiabatic in character, so that \( \delta p = c_s^2 \delta \rho \), where \( c_s \)
  is the adiabatic sound speed; then the term \((1/ \rho_c a^2) \nabla_c^2 (\delta p)\) becomes \((c_s^2/ \rho_c a^2) \nabla_c^2 (\delta \rho)\);
- by virtue of eqn. 20, which is equivalent to \( \nabla_c^2 \delta \phi = 4 \pi G a^2 \delta \rho \), the term \( a^{-2} \nabla_c^2 (\delta \phi) \) can be written as \( 4 \pi G \delta \rho \);
- convert eqn. 18 to its “comoving” analog to write

\[
- \nabla_c \cdot \mathbf{\bar{u}} = \frac{d}{dt} \left( \frac{\delta \rho}{\rho_o} \right)
\]

and if we take the time derivative

\[
- \nabla_c \cdot \frac{d \mathbf{\bar{u}}}{dt} = \frac{d^2}{dt^2} \left( \frac{\delta \rho}{\rho_o} \right)
\]

equation 23 then yields

\[
\frac{d^2}{dt^2} \left( \frac{\delta \rho}{\rho_o} \right) + 2 \left( \frac{\dot{a}}{a} \right) \frac{d}{dt} \left( \frac{\delta \rho}{\rho_o} \right) = \frac{c_s^2}{a^2} \nabla_c^2 (\delta \rho) + 4 \pi G \rho_o \delta
\]

For simplicity in writing, we define \( \delta = \delta \rho/\rho_o \), so that

\[
\frac{d^2 \delta}{dt^2} + 2 \left( \frac{\dot{a}}{a} \right) \frac{d \delta}{dt} = \frac{c_s^2}{a^2} \nabla_c^2 \delta + 4 \pi G \rho_o \delta
\]

**Step 4** We now have a differential equation for the density perturbation \( \delta \). We assume its solution can be described as the superposition of plane waves, i.e.

\[
\delta = c(k_c) e^{i[\vec{k}_c \cdot \vec{r} - \omega(k_c)t]}
\]

Substituting \( \delta \) with a generic plane wave \( \delta \propto e^{i[\vec{k} \cdot \vec{r} - \omega(k_c)t]} \) in the right-hand side of Eqn. 27, we obtain

\[
\frac{d^2 \delta}{dt^2} + 2 \left( \frac{\dot{a}}{a} \right) \frac{d \delta}{dt} = \delta (4 \pi G \rho_o - k_c^2 c_s^2)
\]

which describes the evolution of the perturbation. The wavevector in proper coordinates is \( \vec{\tilde{k}} = a^{-1} \vec{k}_c \) and \( k_c = |\vec{k}_c| \). Eqn. 29 is a fundamental tool of Cosmology: box it.
2.1. Linear Regime: Gravitational Instability in a Static Medium

In a static medium, $\dot{a} = 0$ and we can take $a \equiv 1$, $k = k_c$, so that
\[
\frac{d^2\delta}{dt^2} = \delta(4\pi G \rho_o - k^2 c_s^2)
\]
We are interested in solutions which evolve in time, i.e. with $\omega \neq 0$. Eqn. 30 admits such solutions only provided that the dispersion relation
\[
\omega^2 = c_s^2 k^2 - 4\pi G \rho_o
\]
is satisfied. The value of $k$ for which $\omega = 0$ is referred to as Jeans’ wavenumber and the corresponding wavelength
\[
\lambda_J = \frac{2\pi}{k_J} = c_s \left( \frac{\pi}{G \rho_o} \right)^{1/2}
\]
is referred to as Jeans’ length and, correspondingly, we define the Jeans mass as
\[
M_J = \frac{4\pi \rho_o \lambda_J^3}{3}
\]

- For wavelengths $\lambda < \lambda_J$, $\omega^2 > 0$, $\omega$ is real and the solution is oscillatory, i.e. a sound wave. In this case, pressure gradients provide support against gravitational collapse.

- For wavelengths $\lambda > \lambda_J$, $\omega^2 < 0$, $\omega$ is imaginary and the solution grows exponentially with time, with a characteristic timescale
\[
\tau = |\omega|^{-1} = (4\pi G \rho_o)^{-1/2} \left[ 1 - \left( \frac{\lambda_J}{\lambda} \right)^2 \right]^{-1/2}
\]
For $\lambda \gg \lambda_J$, $\tau$ is a relative of the free-fall collapse time $\tau_{ff} \sim (G \rho_o)^{-1/2}$.

Jeans’ analysis can also be applied to the case of a system of collisionless particles. An analogous dispersion relation and definition of $\lambda_J$ is obtained to that of a fluid as above, provided that $c_s$ is replaced with $v_s$, where
\[
v_s^2 = \frac{\int v^{-2} f d^3v}{\int f d^3v}
\]
and $f$ is the phase space distribution function (DF) of the particles in the system. For a Maxwellian DF $f(v) = \rho(2\pi \sigma^2)^{-3/2}exp(-v^2/2\sigma^2)$, $v_s = \sigma$. 
2.2. Linear Regime: Jeans Instability in an Expanding Universe

The second term on the left-hand side of Eqn. 29, which was set to zero in the previous section, needs now to be taken into account. A dispersion relation holds, which is identical to Eqn. 31, and an instability criterion, i.e. that perturbations with $\lambda > \lambda_f$ can (under special circumstances) grow in amplitude and collapse, also holds. However, an important change with respect to the static case is that the growth rate of the instability is not exponential.

The study of the evolution of small perturbations in an expanding Universe is a complex problem. It is described in detail in Padmanabhan (1993) and Coles & Lucchin (1995). Here we will see a rapid survey of the basic concepts.

Let’s briefly refresh our memory on cosmic evolution. The Universe is made up of various components characterized by different forms of the equation of state, i.e. a relationship of the kind

$$p = w\rho c^2$$

where $\rho \leq 1$. For a non relativistic gas, $w$ is very small, so that it is often approximated by $w = 0$, as if it were pressureless dust. For a fluid of relativistic particles, i.e. photons, $p = \rho c^2/3$, hence $w = 1/3$ (and dark energy has $w < 0$). Earlier on we found that the evolution of the density behaves like

$$\rho a^{3(1+w)} = \text{const.}$$

so that the density of a non–relativistic matter component evolves like

$$\rho_{\text{matter}} \propto a^{-3}(t) \propto (1 + z)^3,$$

that of a relativistic particle fluid

$$\rho_{\text{rel}} \propto a^{-4}(t) \propto (1 + z)^4$$

while that of vacuum energy with $w = -1$ is

$$\rho_\Lambda = \text{const.}$$

The evolution of the scale factor $a(t)$ is determined by the relative importance of the various components of the energy density in the Universe. However, at any given time one can consider that only one of them is dominant, and thus that it determines the behavior of $a(t)$. For a flat Universe dominated by a component characterized by a given value of $w$, the evolution of the scale factor with time can be obtained from

$$a(t)/a_0 = (t/t_0)^{2/[3(1+w)]}$$

so that
• for a matter-dominated Universe \((w \simeq 0)\) 
  \[ a(t) \propto t^{2/3} \]

• for a radiation-dominated Universe \((w \simeq 1/3)\) 
  \[ a(t) \propto t^{1/2} \]

• for a \(\Lambda\)-dominated Universe \((w \simeq -1)\) 
  \[ a(t) \propto e^{H_s t} \]

About 10 sec after the Big Bang, most electrons and positrons in the Universe annihilate, resulting in an increase in the photon density. It follows an epoch in which the dominant form of energy density in the Universe is constituted by radiation. Since the energy density of radiation decreases as \((1+z)^4\) and that of matter does so as \((1+z)^3\), by \(z_{eq}\) the two become equal. After \(z_{eq}\), the Universe enters the “matter era”. That takes place at

\[ 1 + z_{eq} \simeq 4.3 \times 10^4 \Omega_{\text{matter}} h^2 / K_o \]

where \(K_o \simeq 1.68\) if there are only three kinds of neutrinos.

Soon after that, at a redshift \(z_{\text{rec}} \simeq 1300\), 90\% of proton-electron pairs have combined into hydrogen atoms and the bulk of baryonic matter is thus neutral. Since \(\Omega_{\text{matter}}\) is thought to be \(> 1300 / (4.3 \times 10^4 h^2)\), \(z_{eq} > z_{\text{rec}}\); for \(\Omega_{\text{mass}} \simeq 1/3\), \(z_{eq} \simeq 7000\). Baryon and radiation energy density are equal at a somewhat lower \(z\): for \(\Omega_{\text{baryon}} \sim (1/7) \Omega_{\text{DM}}\), then equality of baryon and radiation energies occur at \(z \sim 1000\).

Consider now a flat, matter-dominated Universe. Then \(\rho = (6\pi G t^2)^{-1}\) and \(\dot{a}/a = 2/3 t\). Eqn. 29 thus becomes

\[
\frac{d^2\delta}{dt^2} + \left(\frac{4}{3t}\right)\frac{d\delta}{dt} = \frac{2\delta}{3t^2} \left(1 - \frac{k_s^2 c_s^2}{4\pi G \rho}\right) \quad (43)
\]

In this case, we define \(\lambda'_J = (\sqrt{24/5}) \sqrt{\pi c_s^2 / G \rho} \simeq \lambda_J\). For \(\lambda > \lambda'_J\), the second term in parenthesis on the right-hand side of Eqn. 43 is \(\ll 1\). In that case, we test a power-law solution \(\delta \propto t^n\) by entering it in Eqn. 43; that yields

\[ n(n-1) + 4n/3 - 2/3 = 0 \quad (44) \]

which has solutions for \(n = 2/3\) and for \(n = -1\), i.e. for large \(\lambda\) Eqn. 43 allows a growing mode \((\delta_+ \propto t^{2/3})\) and a decaying one \((\delta_- \propto t^{-1})\). In general, the solution for \(\lambda > \lambda'_J\) is

\[ \delta \propto t^{-\left[1 \pm \frac{1}{2}\sqrt{(\lambda'/\lambda)^2 + 1}\right]/6} \quad (45)\]
For $\lambda < \lambda'_J$, the solutions are oscillating, as in the static case.

\[ \text{Consider now a flat, radiation-dominated Universe.} \]

Then $\rho = (32\pi G t^2)^{-1}$ and $\dot{a}/a = 1/2t$. Eqn. 29 thus becomes

\[
\frac{d^2\delta}{dt^2} + \left(\frac{1}{t}\right) \frac{d\delta}{dt} = \frac{\delta}{t^2} \left(1 - \frac{3k^2c_s^2}{32\pi G \rho} \right) \quad (46)
\]

In this case, we define $\lambda'_J = \sqrt{3\pi c_s^2 / 8G \rho} \simeq \lambda_J$. For $\lambda > \lambda'_J$, the second term in parenthesis on the right-hand side of Eqn. 46 is $\ll 1$. In that case, we test a power-law solution $\delta \propto t^n$ by entering it in Eqn. 46 and, analogously to the matter-dominated case, we obtain a growing solution $\delta_+ \propto t$ and a decaying one $\delta_- t^{-1}$.

$\Rightarrow$ In summary, for a flat Universe:

- $\delta_+ \propto t^{2/3}, \delta_- \propto t^{-1}$ if matter-dominated
- $\delta_+ \propto t, \delta_- \propto t^{-1}$ if radiation-dominated

In an expanding Universe, there are two processes that counteract gravity, in the evolution of density fluctuations. The first one is, as in the static case, pressure support. In a perturbation of size equal to the Jeans length in a static fluid, the crossing time of $\lambda_J$ at the speed $c_s$ (or $v$) is about the same as the collapse time $(G\rho)^{-1/2}$. If the latter is shorter than the former, the fluctuation will collapse. The second one is the universal expansion itself: if the characteristic expansion time is shorter than the collapse time, a fluctuation will not grow in amplitude, even if the crossing time is longer than the collapse time. Let’s consider this second effect in more detail. If the Universe is composed of several components, each with its own value of $w$, a perturbation in the density of only one of the components will evolve in the gravitational field determined by the most dominant of the components, and not just that of the perturbed component. In other words, in the expression of the Jeans length (Eqn. 32), $c_s$ (or $v$) represents the sound speed (or the velocity dispersion) of the perturbed component, but $\rho$ is the density of the component which is gravitationally dominant. Thus, if the perturbed fluctuation is present in the matter density but the gravitationally dominant component is a homogeneous radiation field, the growth of the fluctuation is stopped by the cosmic expansion, as $t_{\text{exp}} \simeq (G\rho_{\text{rad}})^{-1/2}$ is smaller than $t_{\text{coll}} \simeq (G\rho_{\text{matter}})^{-1/2}$.

If the size of the perturbed region exceeds the Hubble radius, pressure cannot play a role. Before the epoch of recombination matter and radiation will be coupled, and super-horizon perturbations will have coupled amplitude in both the matter and the radiation
energy density. In this phase of a radiation-dominated Universe: \( \delta \propto t \propto a^2 \). As the perturbation “enters” the horizon, i.e. \( \lambda \) becomes smaller than the Hubble radius at some redshift \( z_{\text{enter}} \), photons “free-stream” over the Hubble radius and cancel density fluctuations over that scale, if \( z_{\text{enter}} > z_{\text{eq}} \). We are thus in the case discussed at the end of the previous paragraph: a perturbation in the matter density, embedded in a smooth but gravitationally dominant background of radiation: the cosmic expansion regulated by radiation prevents the collapse of the perturbation: during this phase, \( \delta \sim \text{const} \). At \( z_{\text{eq}} \), matter becomes the dominant component, thus \( \delta \propto t^{2/3} \propto a \).

\( \Rightarrow \) In summary, a perturbation with \( \lambda > \lambda_J \) in a flat Universe grows like \( a^2 \), \( a^0 \) and \( a \) respectively for \( z > z_{\text{enter}} \), \( z_{\text{enter}} < z < z_{\text{eq}} \) and \( z < z_{\text{eq}} \).

\( \bigstar \) It is also useful to have in mind how \( \lambda_J \) (and the corresponding Jeans’ mass) vary with \( a \propto (1 + z) \). Remember that \( \lambda_J \propto v \rho_{\text{dominant}}^{-1/2} \).

(i) Consider first a density fluctuation in the **dark matter** component. While DM is relativistic, say for \( z > z_{\text{rel}} \), \( v \simeq c \); as the Universe expands, the velocity dispersion gets “redshifted”, i.e. it changes as \( a^{-1} \propto (1 + z) \). For \( z > z_{\text{eq}} \), \( \rho_{\text{dominant}} \simeq \rho_{\text{radiation}} \propto a^{-4} \), while for \( z < z_{\text{eq}} \), \( \rho_{\text{dominant}} \simeq \rho_{\text{DM}} \propto a^{-3} \). The associated Jeans’ mass is, of course, \( M_J \propto \rho_{\text{DM}} \lambda_J^3 \). Thus:

- \( \lambda_J \propto a^2 \) and \( M_J \propto a^2 \) for \( z > z_{\text{rel}} \)
- \( \lambda_J \propto a \) and \( M_J \simeq \text{const.} \) for \( z_{\text{rel}} > z > z_{\text{eq}} \)
- \( \lambda_J \propto a^{1/2} \) and \( M_J \propto a^{-3/2} \) for \( z < z_{\text{eq}} \)

Carrying out the estimate of \( M_J \) numerically, for \( z < z_{\text{eq}} \) Padmanabhan gives

\[
\frac{M_J}{M_\odot} = 3.2 \times 10^{14} (\Omega_{\text{mass}} h^2)^{-2} (a/a_{\text{eq}})^{-3/2}
\]

(ii) Consider now a density fluctuation which is only present in the **baryonic matter**. The evolution of \( \lambda_J \) is somewhat different than in the DM perturbation. For \( z > z_{\text{eq}} \), baryons and photons are tightly coupled, so that the pressure of the fluid is mainly provided by the relativistic particles, so that \( v^2 \simeq p/\rho \simeq \rho_{\text{radiation}}/\rho_{\text{radiation}} = c^2/3 \propto a^0 \); radiation also constitutes the dominant form of energy density, so \( \rho_{\text{radiation}} \propto a^{-4} \). Thus \( \lambda_J \propto a^2 \) for \( z > z_{\text{eq}} \). Between \( z_{\text{eq}} \) and \( z_{\text{rec}} \), baryons are still coupled to photons, so that the pressure of the fluid is mainly provided by the relativistic particles; the density, however, is dominated
by matter, so that \( v^2 \simeq p/\rho \simeq \rho_{\text{radiation}}/\rho_{\text{matter}} \propto a^{-4}/a^{-3} = a^{-1} \). Thus, for \( z_{eq} < z < z_{\text{rec}} \), \( \lambda_J \propto a^{-1}/[a^{-3}]^{1/2} = a^{3/2} \). For \( z < z_{\text{rec}} \), the analysis is similar to the DM perturbation case, so \( \lambda_J \propto a^{1/2} \). Summarizing,

- \( \lambda_J \propto a^2 \) and \( M_J \propto a^3 \) for \( z > z_{eq} \)
- \( \lambda_J \propto a^{3/2} \) and \( M_J \propto a^{3/2} \) for \( z_{eq} > z > z_{\text{rec}} \)
- \( \lambda_J \propto a^{1/2} \) and \( M_J \propto a^{-3/2} \) for \( z < z_{\text{rec}} \)

There is an extremely important point that remains to be made, however: at \( z_{\text{rec}} \), the decoupling of matter and radiation results in a dramatic drop in the pressure of the fluid: from that of the radiation to that of the few \( 10^3 \) K baryon gas: this translates in a sudden drop in the Jeans’ mass by a factor of about \( 10^{-12} \)–\( 10^{-13} \) (depending on the baryonic mass fraction of the Universe). Numerically, Padmanabhan gives

- \( M_J/M_\odot \simeq 3.2 \times 10^{14}(\Omega_{\text{baryon}}/\Omega_{\text{mass}})(\Omega_{\text{mass}}h^2)^{-1/2}a/a_{eq}^3 \) for \( z > z_{eq} \)
- \( M_J/M_\odot \simeq 3.2 \times 10^{14}(\Omega_{\text{baryon}}/\Omega_{\text{mass}})(\Omega_{\text{mass}}h^2)^{-1/2}a/a_{eq}^{3/2} \) for \( z_{eq} > z > z_{\text{rec}} \)
- \( M_J/M_\odot \simeq 10^4(\Omega_{\text{baryon}}/\Omega_{\text{mass}})(\Omega_{\text{mass}}h^2)^{-1/2}(a/a_{\text{rec}})^{-3/2} \) for \( z < z_{\text{rec}} \)

Before entering the horizon, all perturbations grow like \( a^2 \), as they exceed the Jeans’ length which is \( \sim \) to the Hubble radius, as can easily be shown. After entering the horizon, for \( z > z_{eq} \), the growth of a \( \lambda > \lambda_J \) perturbation is stopped by the cosmic expansion, which is regulated by the dominant form of energy density, radiation. As the radiation era ends at \( z_{eq} \), and DM becomes the dominant energy density form, the DM component of the perturbation can resume growth, its amplitude increasing as \( \delta_{DM} \propto a \). However, the growth of the baryonic component of the perturbation is prevented by the tight coupling between baryons and photons via Thomson scattering. This situation changes at \( z_{\text{rec}} \), when baryonic matter becomes largely neutral and the tight coupling with radiation ceases. After recombination, baryonic perturbations can start growing again. In the meantime, DM perturbations have grown by a factor \( a_{\text{rec}}/a_{eq} \simeq 21\Omega_{\text{mass}}h^2 \). When baryons decouple from photons, they literally “fall” into the potential wells created by DM, so that there will be a rapid growth of \( \delta_{\text{baryon}} \) right after \( z_{\text{rec}} \), until \( \delta_{\text{baryon}} \simeq \delta_{DM} \). After that, they will both grow as \( a \).
Consider a perturbation containing a mass comparable with that of a normal galaxy, say $10^{11} \text{M}_\odot$. In the early stages, when the Universe is radiation-dominated and that mass exceeds that comprised within the horizon, the perturbation grows in amplitude as $a^2$. At this epoch, perturbation amplitudes are extremely small. At a $z \sim 10^6$, the perturbation enters the horizon. Since the Jeans’ length is comparable with the Hubble radius, at this time the perturbation ceases to grow, since it is smaller than the Jeans’ length. It becomes stable against gravitational collapse and it becomes an acoustic wave which oscillates at roughly constant mean amplitude. Eventually, past $z_{\text{rec}}$, $\lambda_J$ shrinks below the size of the perturbation, which starts growing again. The relative energy densities of DM, baryons and radiation play an important role in determining the details of the process.

The description of the growth of perturbations given so far only applies to very small amplitude fluctuations, i.e. to the \textit{linear regime}, for which $\delta < 1$. In order to follow the growth of perturbations into the nonlinear regime, gravitational collapse and virialization, we will have to change gears. That comes next, as we deal with the topic referred in most textbooks as the \textit{spherical collapse model}.

3. \textbf{Nonlinear Regime: The Spherical Infall Model}

After $z_{\text{rec}}$, mass perturbations with $\lambda > \lambda_J$ can grow. Self-gravity will work against the cosmic expansion, i.e. different parts of the perturbation will separate at a slower rate than that imposed by the expansion of the Universe. This will increase the density contrast. Eventually, self–gravity will locally stop the expansion, and the overdense cloud will start to contract. The collapse will eventually be arrested as the system becomes a virialized, bound structure. The treatment of non–linear collapse is mathematically difficult, and analytical solutions have been obtained only in cases of very simplified geometry, such as that in which the perturbation is spherical. This case was first described by Gunn \& Gott in 1972. This section is a summary of the treatment by Padmanabhan (1993).

Consider a spherical overdense region after $z_{\text{rec}}$, well within the Hubble radius. At some initial time $t_i$, let it have a density distribution

$$\rho(r, t_i) = \rho_{bg}(t_i)[1 + \delta_i(r)]$$

(48)

where $\rho_{bg}(t_i)$ is the background density of the Universe at $t_i$. When $\delta_i(r)$ is the same at all radii, the distribution is often referred to as “top hat” in the literature. Let $r_i$ be the initial radius of a shell within the shell, which will enclose a mass

$$M(r_i) = \rho_{bg}\left(\frac{4\pi}{3}r_i^3\right)(1 + \tilde{\delta}_i)$$

(49)
where \( \delta_i \) is the average density enhancement within the sphere. The initial time \( t_i \) is chosen so that the overdensity \( \delta_i \) is very small; the region is expanding pretty much along with the rest of the Universe.

The equation of motion of the shell is given by

\[
\frac{d^2r}{dt^2} = -\frac{GM}{r^2}
\]

Integrating the equation of motion, we get

\[
\frac{1}{2} \left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = E
\]

where the value of \( E \) determines the fate of the shell; familiarly, for \( E > 0 \), the shell will eventually expand forever, while for \( E < 0 \), for \( E = GM/r \dot{r} = 0 \), so that at that maximum radius the shell’s expansion will reverse into a collapse. We now will rewrite the equation of motion using the terminology adopted to describe cosmological models.

\( \checkmark \) Let’s consider the initial time \( t_i \). As we said, the “cloud” is at this time expanding along with the rest of the Universe, i.e. the “peculiar” motion impressed by its local dynamics is negligible. We can then write

\[
\dot{r}_i = (\dot{a}/a)r_i = H(t_i)r_i \equiv H_ir_i
\]

where \( a \) is the cosmic scale factor. The kinetic energy per unit mass of the shell is then

\[
K_i \equiv \dot{r}_i^2/2 = (H^2_ir_i^2)/2,
\]

while the potential energy per unit mass at \( t_i \) is

\[
U = -GM/r_i = -\frac{4\pi G}{3} \rho_{bg}(t_i) r_i^2 (1 + \delta_i)
\]

Introducing the matter density parameter at time \( t_i \), \( \Omega_i = \rho_{bg}(t_i)/\rho_{crit}(t_i) \),

\[
U = -(1/2) H^2_i r_i^2 \Omega_i (1 + \delta_i) = -H_i \Omega_i (1 + \delta_i)
\]

so that the total energy of the shell is

\[
E = K_i \Omega_i [\Omega_i^{-1} - (1 + \delta_i)]
\]

The shell will eventually collapse if \( \delta_i > \Omega_i^{-1} - 1 \), a condition which, in a flat Universe, is satisfied for any positive value of \( \delta_i \). In an open Universe, on the other hand, there would be a positive threshold value for \( \delta_i \), in order for the region to collapse.
Let’s next estimate the maximum radius \( r_{\text{turn}} \) of expansion of the shell, generally referred to as the turn-around radius. At the time of maximum expansion, \( \dot{r} = 0 \), at which

\[
E = -\frac{GM}{r_{\text{turn}}} = -(r_i/r_{\text{turn}}) \, K_i \Omega_i \left(1 + \tilde{\delta}_i\right).
\]  

EQUATION 57

Equating Eqn. 56 with Eqn. 57,

\[
\frac{r_{\text{turn}}}{r_i} = \frac{1 + \tilde{\delta}_i}{\delta_i - (\Omega_i^{-1} - 1)}
\]  

EQUATION 58

which shows how shells of small overdensity will expand to larger \( r_{\text{turn}} \) and will, as a result, take longer to collapse.

The customary way to describe the evolution of the shell adopts the parametric form, already used in the description of cosmological models:

\[
r = A(1 - \cos \theta) \quad t + T = B(\theta - \sin \theta) \quad A^3 = GMB^2
\]  

EQUATION 59

With \( \theta, t \) increases indefinitely, while \( r \) reaches a maximum value for \( \theta = \pi \) before decreasing to zero for \( \theta = 2\pi \). The constants \( A, B \) and \( T \) can be set with help of the initial conditions. For example, for \( \theta = \pi, r = r_{\text{turn}} = 2A \); with the help of Eqn. 58,

\[
A = \frac{r_i}{2} \frac{1 + \tilde{\delta}_i}{\delta_i - (\Omega_i^{-1} - 1)}
\]  

EQUATION 60

and using 59 and Eqn. 49,

\[
B = \frac{1 + \tilde{\delta}_i}{2H_i \Omega_i^{1/2} [\bar{\delta}_i - (\Omega_i^{-1} - 1)]^{3/2}}
\]  

EQUATION 61

It can be shown that the value of \( T \) is small \( (T/t_i \simeq \tilde{\delta}_i \ll 1; \) see Padmanabhan), so we’ll ignore it henceforth.

We next use these equations to describe the density evolution of the perturbation in a flat Universe, for which the density evolves with time as \( \rho_{bg}(t) = (6\pi Gt^2)^{-1} \). Using Eqns. 59 and setting \( t = 0 \), the mean density contrast within the spherical shell can be written as

\[
1 + \tilde{\delta}(t) = \frac{\bar{\rho}(t)}{\rho_{bg}(t)} = \frac{3M}{4\pi r^3 6\pi Gt^2} = \frac{9}{2} \left(\frac{\theta - \sin \theta}{1 - \cos \theta}\right)^2
\]  

EQUATION 62
This equation can be used to show that for very small \( \theta \) (i.e. early in the evolution when \( \delta \) is also small), the evolution in the linear regime \( (\delta \propto t^{2/3} \propto a(t)) \) can be recovered.

\* In order to refer the evolution of perturbations to their values at \( z = 0 \), it is convenient to introduce two new variables:

\[ x = r_i \frac{a(t)}{a(t_i)} \quad \text{and} \quad \bar{\delta}_o = \frac{3}{5} \frac{\delta_i}{a(t)} = \frac{3}{5} \delta_i (1 + z_i) \tag{63} \]

where \( x \) is the comoving radius corresponding to \( r_i \) and \( \bar{\delta}_o \) is the value of the density contrast at \( z = 0 \), predicted by the application of linear theory to a perturbation \( \delta_i \) at \( z_i \). The parametric description of a spherical, overdense shell can thus be summarized as follows:

\[
r(t) = \frac{r_i}{2\bar{\delta}_i} (1 - \cos \theta) = \frac{3x}{10\bar{\delta}_o} (1 - \cos \theta) \tag{64}
\]

\[
(1 + z) = 1.21 \frac{\bar{\delta}_i (1 + z_i)}{(\theta - \sin \theta)^{2/3}} = 1.94 \frac{\bar{\delta}_o}{(\theta - \sin \theta)^{2/3}} \tag{65}
\]

\[
\bar{\delta} = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3} - 1 \tag{66}
\]

\[
1 + z_{\text{turn}} \simeq 0.57 (1 + z_i) \bar{\delta}_i \simeq 0.57 \frac{\bar{\delta}_o}{1.062}, \quad 1 + \bar{\delta}_{\text{turn}} = \frac{9\pi^2}{16} \simeq 5.6 \tag{67}
\]

where Eqn. 65 uses the expression for \( t \) in 59 and converts it to redshift using the relation \( (t/t_i)^{2/3} = (1 + z_i)/(1 + z) \). Eqns. 67 yield the epoch of turn-around and the overdensity at turn-around, as derived from Eqns. 65 and 66 for \( \theta = \pi \). Consider, for example, an overdensity of about 1% at \( z_i = 1000 \), right after \( z_{\text{rec}} \). Eqn. 67 tells us that the overdensity will turn around at a redshift \( z_{\text{turn}} \simeq 4.7 \), by which it will be 4.6 times as dense as the background. Note that \textbf{the turn-around overdensity is always 4.6 times the background density, independent on the initial conditions.}

Note also that if we define the transition from the linear to the nonlinear regime that for which Eqn. 66 yields \( \bar{\delta} \simeq 1 \), that corresponds to \( \theta \simeq 2\pi/3 \); at that time, the linear treatment predicts an overdensity of 0.57, i.e. the disagreement between the linear and nonlinear treatments is nearly a factor 2; obviously, the linear prediction is incorrect. At turnaround, for which the nonlinear treatment yields \( \bar{\delta}_m = 4.6 \), the linear treatment yields \( \bar{\delta} = 1.063 \).
4. Virialization

The spherical collapse equations predict that at $\theta = 2\pi$ the overdense region collapses into a singularity. Long before it happens, through a process named violent relaxation the collisionless DM particles reschedule the energy distribution and reach virialization. The violent relaxation process relies on the fact that, during collapse, large fluctuations in the gravitational potential take place over a timescale which is comparable with the collapse time. Responding to this fast changing potential, particles follow orbits that do not conserve energy, and thus a redistribution of the energy of the particles results. The net effect of this process is similar to that achieved by collisions in a gas, which tends to bring energy equipartition to the system (see BT Chapter 4 for details).

Violent relaxation will thus bring the dominant component of the mass, DM, towards dynamic relaxation (virialization). Virialization means that the kinetic, potential and total energy are related via $E = U + K = -K$, i.e. $|U| = 2K$. In the oversimplified case of a spherically symmetric system of constant density, we can derive some interesting results.

At turn-around, all the energy of a shell is in $U$, i.e. $E = U \simeq \tau GM^2/5r_m$. We define the virial velocity $v_{vir}$ and the virial radius $r_{vir}$ via the equations

$$K = -E = \frac{3GM^2}{5r_{turn}}, \quad |U| = \frac{3GM^2}{5r_{vir}} = 2K = Mv^2$$

i.e.

$$v_{vir} = \sqrt{6GM/5r_{turn}}, \quad r_{vir} = r_{turn}/2$$

Now, the time of collapse is essentially that given by $\theta = 2\pi$, even if singularity is not reached. Using Eqs. 65 and 67, for the redshift of collapse — at which virialization is achieved — we obtain

$$1 + z_{vir} = 0.36 \delta_i (1 + z_i) = 0.63 (1 + z_{turn})$$

We can now compute the overdensity of the collapsed object at $z_{vir}$. Since $r_{vir} = r_{turn}/2$, the density of the virialized object is $\rho_{vir} = 8\rho_{turn}$. Between $z_{turn}$ and $z_{vir}$, however, the Universe expanded by $(1 + z_{turn})/(1 + z_{vir})$, and the background density has dropped by the cube of that ratio. Hence,

$$\bar{\rho}_{vir} \simeq 2^3 \rho_{turn} \simeq 45 \rho_{bg}(z_{turn}) \simeq 180 \rho_{bg}(z_{vir}) = 180 \rho_0 (1 + z_{vir})^3$$
where \( \rho_0 \) is the cosmological density at \( z = 0 \). Once the object is virialized, its density changes little; as the Universe continues to expand, and \( \rho_{bg} \propto a^{-3} \), the density contrast represented by the object increases as \( a^3 \). This is a very important result. One often finds references in the literature to \( r_{200} \), as the virial radius of a DM halo, for example; it refers to the radius within which the mean halo density is 200 times the critical density of the Universe. The factor 200 is a rounding up of the result in Eqn. 71, and indicates the radius out to which a halo is just becoming virialized at the given \( z \).

The collapse of the baryonic component is somewhat different. While it responds to the same overall gravitational potential, largely produced by the dominant DM component, its virialization proceeds through hydrodynamic processes: particle collisions result in energy equipartition among the H and He gas particles, and pressure gradients prevent total collapse. As the cloud collapses, the kinetic energy of the particles increases, and the gas gets heated; shock waves and radiative losses accompany the process, so that, in fact, baryons fall deeper into the cloud’s potential well. We shall revisit this stage at a later stage in the course. For the moment, we will compute numerical estimates of the scaling relations resulting from the spherical infall model, and compare them with the parameters we observe in real galaxies.

From the relation between \( r_{\text{turn}} \) and \( r_{\text{vir}} \), we can derive scaling laws between mass, virial radius and virial velocity

\[
 r_{\text{vir}} = (164 \text{ kpc}) \left( \Omega_{\text{matter}} h^2 \right)^{-1/3} \left( \frac{M}{10^{12} M_{\odot}} \right)^{1/3} (1 + z_{\text{vir}})^{-1} \tag{72}
\]

\[
 v_{\text{vir}} = (125 \text{ kms}^{-1}) \left( \Omega_{\text{matter}} h^2 \right)^{1/6} \left( \frac{M}{10^{12} M_{\odot}} \right)^{1/3} (1 + z_{\text{vir}})^{1/2} \tag{73}
\]

It is also useful to derive a virial temperature of the baryonic gas, i.e. a quantity related to the virial velocity in the manner

\[
 \frac{3}{2} \mu \rho_{\text{baryon}} T_{\text{vir}} = \frac{1}{2} \rho_{\text{baryon}} v_{\text{vir}}^2 \tag{74}
\]

where \( \mu \) is the mean molecular weight of the H, He mixture. For a He abundance of \( Y \simeq 0.25 \), \( \mu \simeq 0.57 m_p \), where \( m_p \) is the proton mass. Then,

\[
 T_{\text{vir}} = (3.68 \times 10^5 \text{ K}) \left( \Omega_{\text{matter}} h^2 \right)^{1/3} \left( \frac{M}{10^{12} M_{\odot}} \right)^{2/3} (1 + z_{\text{vir}}) \tag{75}
\]

Note the high value of the virial temperature of a galaxy-sized mass: between \( 10^5 \text{ K} \) and \( 10^6 \text{ K} \), the gas can radiate very efficiently, mainly through bremsstrahlung, and lose rapidly a
large fraction of its energy. Thus, as mentioned above, the baryons can fall deep in the DM potential well, well within the virial radius determined by the DM.

The overall predictions of the spherical infall model, in spite of the unrealistic simplifications made in its derivation, are roughly consistent with the overall parameters of observed galaxies, suggesting that, albeit broadly, the ideas it contains are on the right track. For example, consider an object of $M = 10^{12} \, M_\odot$, a cosmology with $\Omega_{\text{matter}} \simeq 1/3$ and $h = 0.7$ and a redshift of virialization of $z_{\text{vir}} \simeq 2$; Eqn. 72 predicts that the DM halo of that object should have a virial radius of the order of 100 kpc and a velocity dispersion of about 160 km s$^{-1}$. These values are not very far off those we observe in nearby galaxies.