Applications of the Fourier Transform

Astro 6523, Spring 2013

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Pulsar Searching
The discovery of radio pulsars

- Hewish et al.: interplanetary scintillation at 3.7 m.
- In 1967, Jocelyn Bell discovered radio pulses from the sky.
The discovery of radio pulsars
What is a pulsar?

(The Crab pulsar is visible even at optical frequencies.)
What is a pulsar?

Crab pulse profiles at radio, IR, optical, X-ray wavelengths.
What is a pulsar?

- Pulsars are rapidly rotating neutron stars that act as radio light houses, generating periodic pulses.
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- Pulsar surveys:
  - High data rate sampling;
  - Compensation for dispersive propagation through the Galaxy;
  - Fourier analysis to identify periodic pulsar signals.
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Challenges include the high data rate, the weakness of the pulsar signals, the sparseness of neutron stars in the Galaxy, and the growing presence of terrestrial radio frequency interference.

⇒ A classic data-mining problem!
Extreme matter physics in and near the NS:

- $10 \times$ nuclear density: one teaspoon $\sim 10^8$ tons.
- High-temperature superfluid and superconductor.
- Magnetic fields: $B \sim B_q = 4.4 \times 10^{13}$ Gauss.
- Gravity $\sim 10^9 \times$ Earth’s gravity.
- EM forces $\sim 10^{11} \times$ gravity at NS surface.
Neutron Star Science

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  * Magnetospheres
  * Radiation mechanisms (coherent plasma processes)
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- Laboratories for Gravity:
  - Tests of theories of gravity (binary pulsars)
  - Gravitational wave detectors (millisecond pulsars).
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- Probes of turbulent and magnetized in the ionized gas in the Milky Way.

- End states of stellar evolution:
  Massive stars $\Rightarrow$ neutron stars or black holes.
Radio Pulsars: The Lighthouse Model

We understand why pulsars pulse ...
Radio Pulsars: The Lighthouse Model

We understand why pulsars pulse ...

... but not why they shine.
(At least, not very well!)

Radio emission is a *tiny* part of the NS energy budget, and models are still somewhat speculative.
Pulsar properties: Pulse profiles

- Individual pulses vary in shape and have jitter – “weather”.
- Overall profile is stable, and evolves very slowly – “climate”.

Consecutive single pulse profiles for B1133+16
Due to propagation through the interstellar medium, pulsar signals are delayed at lower frequencies.
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\[ \text{Delay} \propto \frac{1}{\nu^2} \int_0^L n_e \, dl = \frac{\text{DM}}{\nu^2} \]

Refractive indices for cold magnetized plasma:

\[ n_{l,r} \sim 1 - \frac{\nu_p^2}{2\nu^2} \mp \frac{\nu_p^2 \nu_B ||}{2\nu^3} \]

\[ \nu \gg \nu_p \sim 2\text{kHz} \]

\[ \nu \gg \nu_B || \sim 3\text{Hz} \]
Due to propagation through the interstellar medium, pulsar signals are delayed at lower frequencies.

Propagation velocities are frequency dependent:

Phase velocity \( v_p = \frac{\omega}{k} = \frac{c}{n_{l,r}} \)

Group velocity \( v_g = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k} \left( \frac{kc}{n_{l,r}} \right) \)

Group delay = \( \Delta \) (Time of Arrival)
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Group delay = \( \Delta (\text{Time of Arrival}) \)

\[
\begin{align*}
t &= t_{DM} \pm t_{RM} \\
t_{DM} &= 4.15 \text{ ms} \times DM \, \nu^{-2} \\
t_{RM} &= 0.18 \text{ ns} \times RM \, \nu^{-3}
\end{align*}
\]

Dispersion Measure \( DM = \int ds \, n_e \)
Units: pc cm\(^{-3}\).
Rotation Measure \( RM = 0.81 \int ds \, n_e B_{||} \)
Units: rad m\(^{-2}\).
Basic data unit = a dynamic spectrum

10^6 – 10^8 samples x 64 µs

Fast-dump spectrometers:
- Analog filter banks
- Correlators
- FFT (hardware)
- FFT (software)
- Polyphase filter bank

E.g. WAPP, GBT correlator + spigot card, new PALFA correlator
Astrophysical effects are typically buried in noise and RFI.
New Pulsars

- Periodicity Search (FFT)
- Single-pulse search (matched filtering)

Known Pulsars

- Arrival time Monitoring
- Polarization Analysis
- Scintillation Studies

\[
I(\nu, t) \Rightarrow I(\text{DM}, t)
\]

\[I(\nu, t) \Rightarrow I(\text{DM}, t) \Rightarrow I(\text{DM}, f)\]

\[
I(\nu, t) \Rightarrow Q(\nu, t) \Rightarrow U(\nu, t) \Rightarrow V(\nu, t)
\]
Dedispersion
Two methods:

Coherent:
- operates on the voltage proportional to the electric field accepted by the antenna, feed and receiver
- computationally intensive because it requires sampling at the rate of the total bandwidth
- “exact”

Post-detection:
- operates on intensity = $|\text{voltage}|^2$
- computationally less demanding
- an approximation
Dispersed Pulse

$\Delta t = 8.3 \, \mu s \, \text{DM} \, \nu^{-3} \, \Delta \nu$

Coherently dedispersed pulse
Postdetection Dedispersion:
Sum intensity over frequency after correcting for dispersion delay

Residual time smearing:
\[ \Delta t = \left[ \Delta t_{DM}^2 + \left( \frac{1}{\Delta \nu} \right)^2 \right]^{1/2} \]
\[ = \left[ (a \Delta \nu)^2 + (\Delta \nu)^{-2} \right]^{-1/2} \]
\[ \Rightarrow \text{minimum smearing time across a channel when} \]
\[ \Delta \nu = \left[ 8.3 \, \mu s \, DM \nu^{-3} \right]^{-1/2} \]
Coherent Dedispersion
pioneered by Tim Hankins (1971)

Coherent dedispersion works by explicit deconvolution:

\[ E_{\text{measured}}(t) = E_{\text{emitted}}(t) * \text{FFT}\{e^{ik(\omega)z}\} \]

\[ \Rightarrow E_{\text{emitted}}(t) \approx E_{\text{measured}}(t) * \text{FFT}\{e^{-ik(\omega)z}\} \]

Comments and Caveats:

• Software implementation with FFTs to accomplish deconvolution (Hankins 1971)
• Hardware implementations: real-time FIR filters (e.g. Backer et al. 1990s-present)
• Resulting time resolution: \( 1 / (\text{total bandwidth}) \)
• Requires sampling at Nyquist rate of 2 samples \( \times \) bandwidth
  \[ \Rightarrow \text{Computationally demanding} \]
• Actual time resolution often determined by interstellar scattering (multipath)
• Most useful for low-DM pulsars and/or high-frequency observations
Issues In Pulsar Survey Optimization

• Broad luminosity function for pulsars
  • Beam luminosity
  • Geometric beaming \( L_b = L_b(P, \dot{P}) \)

• Pulse sharpness
  • Intrinsic pulse width \( W \)
  • Smearing from propagation effects
    » Dispersion across finite bandwidth (correctable)
    » Multipath propagation (scattering in the ISM)
  • Smearing from orbital acceleration

• Intermittency of the pulsar signal
  • Nulling, giant pulses, precession, eclipsing
  • Interstellar scintillation
Issues In Pulsar Survey Optimization

• Combine the signal over time and frequency while maximizing S/N through matched filtering:
  • **Dedispersion**: sum over frequency while removing the dispersive time delays
  • **Single pulses**: match the shape and width of the pulse
  • **Periodic pulses**: match the period as well as the pulse shape and width
  • **Orbital motion**: match the change in pulse arrival times related to the changing Doppler effect

⇒ **Single-pulse searches:**
  Search vs. (DM, W)

⇒ **Periodicity searches:**
  Search vs. (DM, W, P, [orbital parameters])
Dedispersion at a single known DM

\[ I(t) \rightarrow \sum_{v} \]
Dedispersion over a set of DMs

Frequency

<table>
<thead>
<tr>
<th></th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td></td>
</tr>
</tbody>
</table>

time
Pulsar Periodicity Search

```
|FFT(f)|
```

FFT each DM’s time series
Example Periodicity Search Algorithm
PULSAR SEARCHING

MATCHED FILTERS

THRESHOLD TESTS

DEDISPERSE

FFT

HARMONIC SUM

THRESHOLD TESTS

FOLD

CANDIDATE LISTS & PROFILES

\[ \mathcal{I}(t,v) \]

\( \{ \text{DM}_j, j=1, N_{\text{DM}} \} \)

\[ \mathcal{I}(t, \text{DM}) \]

(Remap t for acceleration search)

\[ \mathcal{I}(f, \text{DM}) \]

(Interbinning, sideband search)

\[ \mathcal{I}(f_0, \text{DM}, N_h) \]

\( \{ f_0, \text{DM}, N_h : 1 > l_{\text{min}} \} \)
Harmonic Sum

The FFT of periodic pulses is a series of spikes (harmonics) separated by 1/P.

To improve S/N, sum harmonics. This procedure is an approximation to true matched filtering, which would give optimal S/N.

Sum how many harmonics?

The answer depends on the pulse “duty cycle” = (pulse width / P) (unknown \textit{a priori}) ⇒ need to use trial values of $N_h$:

$$h(N_h) = N_h^{-1/2} \sum_{j=1}^{N_h} \left| \frac{\text{FFT}(j)}{\text{FFT}(0)} \right|$$

Sum over harmonics

Maximize $h()$ with respect to $N_h$ to identify candidate pulsars.

Noise and RFI conspire to yield spurious candidates.

∴ Need a high threshold. How high?

Minimum detectable flux density for a single harmonic: $S_{\text{min}_1} = m \times \sigma_{\text{radiometer}}$

$m \approx 10$

Minimum detectable flux density for harmonic sum:

$$S_{\text{min}} = \frac{S_{\text{min}_1}}{h(N_h)}$$
Example Time Series and Power Spectrum for a recent PALFA discovery
(follow-up data set shown)

DM = 0 pc cm$^{-3}$

Time Series

DM = 217 pc cm$^{-3}$

Where is the pulsar?
Example Time Series and Power Spectrum for a recent PALFA discovery
(follow-up data set shown)

DM = 0 pc cm$^{-3}$

Time Series

DM = 217 pc cm$^{-3}$

Here is the pulsar
Pulsar Detection: a strong example

Candidate: PSR_1906+0749
Telescope: Arecibo
Epoch$_{topo}$ = 53534.21017361111
Epoch$_{bary}$ = N/A
T$_{sample}$ = 0.000128
Data Folded = 62503680
Data Avg = 4.008e+04
Data StdDev = 322.5
Profile Bins = 4096
Profile Avg = 6.116e+08
Profile StdDev = 3.984e+04

RA$_{2000}$ = 19:06:49.1292
DEC$_{2000}$ = 07:46:23.0592
Reduced $\chi^2$ = 529.872
P(Noise) $\sim$ 0
Dispersion Measure (DM) = 217.777

$P_{topo}$ (ms) = 143.98130841(27)
$P_{bary}$ (ms) = N/A
$P_{topo}$ (s/s) = 0.0(2.6)$\times$10$^{-13}$
$P_{bary}$ (s/s) = N/A
$P_{topo}$ (s/s$^2$) = 0.0(2.1)$\times$10$^{-16}$
$P_{bary}$ (s/s$^2$) = N/A

$P_{orb}$ (s) = N/A
$\alpha$$_1$sin($i$)/c (s) = N/A
$T_{peri}$ = N/A
$e$ = N/A
$\omega$ (rad) = N/A
Pulsar Detection: a weaker example

Candidate: 877.07ms_Cand
Telescope: Arecibo
Epoch_{topo} = 55878.84741898148
Epoch_{bary} = 55878.84535467539
T_{sample} = 6.5476e-05
Data Folded = 3909120
Data Avg = 6112
Data StdDev = 83.53
Profile Bins = 200
Profile Avg = 1.191e+08
Profile StdDev = 1.168e+04

Search Information
RA_{2000} = 19:01:22.2354
Best Fit Parameters
Reduced \chi^2 = 3.605
P(\text{Noise}) < 1.04e-59 (\approx 16.3\sigma)
Dispersion Measure (DM) = 1103.561

P_{topo} (ms) = 877.138(25)
P_{bary} (ms) = 877.073(25)
P_{topo} (s/s) = 0.0(7.6)e^{-7}
P_{bary} (s/s) = 0.0(7.6)e^{-7}
P_{topo} (s/s^2) = 0.0(1.9)e^{-8}
P_{bary} (s/s^2) = 0.0(1.9)e^{-8}

Binary Parameters
P_{orb} (s) = N/A
\alpha\sin(i)/c (s) = N/A
\omega (rad) = N/A

0 0.2 0.4 0.6 0.8 1
0 20 40 60 80
Fraction of Observation
Sub-band
1300 1400 1500
Frequency (MHz)

0 0.2 0.4 0.6 0.8 1
0 20 40 60
Fraction of Observation
Sub-band
1300 1400 1500
Frequency (MHz)

0 0.2 0.4 0.6 0.8 1
0 20 40 60
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1300 1400 1500
Frequency (MHz)

0 0.2 0.4 0.6 0.8 1
0 20 40 60
Fraction of Observation
Sub-band
1300 1400 1500
Frequency (MHz)
Pulsar Detection: a false positive?

Candidate: PSR_1902+0723
Telescope: Arecibo
Epoch_to = 53934.15555555555
Epoch_bary = 53934.16134680091
T_sample = 6.4e-05
Data Folded = 4194304
Data Avg = 1.793e+04
Data StdDev = 205.4
Profile Bins = 64
Profile Avg = 1.174e+09
Profile StdDev = 5.257e+04

Search Information
RA_2000 = 19:02:09.7300
DEC_2000 = 07:21:42.0000
Folding Parameters
Reduced $\chi^2 = 2.317$ $P(\text{Noise}) < 1.56e-08$ ($\approx 5.5\sigma$
Dispersion Measure (DM) = 106.537
P_topo (ms) = 487.818(16)
P_topo (s/s) = 0.0(4.6)x10^{-7}
P_topo (s/s^2) = 0.0(1.1)x10^{-8}

Binary Parameters
P orb (s) = N/A
e = N/A
$\alpha_s\sin(i)/c$ (s) = N/A
$\omega$ (rad) = N/A

Period = 487.81824235 (ms)
Freq = 2.049944 (Hz)

P-dot = 3.4465e-11 (s/s)
Period = 487.81824235 (ms)
Effects that broaden pulses reduce the harmonic sum, which is bad.
Survey Selection Against Binaries

NS-NS binary

Pulse shape

Phase perturbation

FFT harmonics

Harmonic Sum
Dealing With Orbital Motion

Orbital acceleration yields a time-dependent period, potentially destroying the power of the straightforward FFT + HS.

- **Long-period binaries:** \( T = \text{data span length} \ll P_{\text{orb}} \)
  - Do nothing different

- **Intermediate-period orbits:** \( T < 0.1 P_{\text{orb}} \)
  - Acceleration search: compensate the time domain or match filter in the frequency domain according to an acceleration parameter
  - Adds another search parameter: DM, P, W, a

- **Very short period orbits:** \( T > P_{\text{orb}} \) (potentially \( \gg P_{\text{orb}} \))
  - Do conventional FFT but search for **orbital sidebands**
Effects of orbital motion on detection of a relativistic binary pulsar (recent discovery at Arecibo)

Before correcting for orbital motion

After correction
Single pulse searches
Synthesis Imaging
Arrays of Antennas
Arrays of Antennas

Pair of antennas with incident plane wave:
Phase difference due to path length difference (PLD).

\[ \Delta \phi = 2\pi \frac{PLD}{\lambda} = k \cdot PLD = kb \sin \theta. \]
Arrays of Antennas

\[ E = E_1 + E_2 \]
\[ = E_0(e^{i\phi_1} + e^{i\phi_2}) \]
\[ = E_0(e^{i\phi_1} + e^{i(\phi_1 + \Delta \phi)}) \]
\[ = E_0e^{i\phi_1}(1 + e^{i\Delta \phi}). \]

\[ I = |E|^2 = |E_0|^2 2(1 + \cos \Delta \phi) = 2I_0(1 + \cos \Delta \phi) \]
\[ \Rightarrow I = 2I_0(1 + \cos \Delta \phi) \begin{cases} 
4I_0 & \text{if } \Delta \phi = 0, \\
2I_0 & \text{if } \Delta \phi = \pi/2, \\
0 & \text{if } \Delta \phi = \pi. 
\end{cases} \]
\[ I(\theta) = 2I_0(1 + \cos(kb \sin \theta)). \]

Fringe spacing \( \Delta \theta \approx \frac{\lambda}{b} \) for \( \Delta \theta \ll 1 \).
The Fringe Pattern

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Also need to include antenna power pattern \( P_n(\theta) \):

\[ I(\theta) \rightarrow I(\theta)P_n(\theta). \]
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Relative widths: \( \lambda/b \) vs. \( \lambda/D \)

- VLA: \( D = 25m; b \sim 1–30 \) km.
- VLBA: \( D = 25m; b \sim 6000 \) km.
For an N-element array,

\[
E = \sum_{j=1}^{N} E_j e^{i\phi_j} = \sum_{j=1}^{N} E_j e^{ik \cdot X_j}.
\]

\[
I = \left| \sum_{j=1}^{N} E_j e^{ik \cdot X_j} \right|^2
\]

⇒ N “auto” terms, N(N-1) “cross” terms, N(N-1)/2 fringe patterns.
For an N-element array,
\[ E = \sum_{j=1}^{N} E_j e^{i\phi_j} = \sum_{j=1}^{N} E_j e^{ik.X_j}. \]

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⇒ N “auto” terms, N(N-1) “cross” terms, N(N-1)/2 fringe patterns.

The array response to the sky brightness distribution is, for a given pair of antennas, the integral of the product of the fringe pattern and the sky brightness.

This is \(\approx\) the Fourier transform of the sky.
Add a (controllable) phase difference at each antenna, $\hat{\phi}_j$.

With a suitable $\hat{\phi}_j$ we can control the directionality of the array.

\[
E = \sum_{j=1}^{N} E_j e^{i(\phi_j - \hat{\phi}_j)} = \sum_{j=1}^{N} E_j e^{i(k.X_j - \hat{\phi}_j)}.
\]

Let $E_j = E_0$ for all $j$, and $\Delta \phi_j = (k.X_j - \hat{\phi}_j)$. 

Then, with \( E = E_0 \sum_{j=1}^{N} e^{i\Delta \phi_j} \),

- Case 1: Perfect phase compensation, \( \Delta \phi_j = 0 \).
  Then \( E = N E_0 \) and \( I = N^2 I_0 \).
Then, with \( E = E_0 \sum_{j=1}^{N} e^{i \Delta \phi_j} \),

- Case 1: Perfect phase compensation, \( \Delta \phi_j = 0 \).
  Then \( E = N E_0 \) and \( I = N^2 I_0 \).

- Case 2: Random \( \Delta \phi_j \) (why?).

\[
I = I_0 \left| \sum_j e^{i \Delta \phi_j} \right|^2 \\
= I_0 \sum_j \sum_k e^{i(\Delta \phi_j - \Delta \phi_k)} |^2 \\
= NI_0 + I_0 \sum_j \sum_{k \neq j} e^{i(\Delta \phi_j - \Delta \phi_k)} |^2
\]

For large \( N \), final \( N(N - 1) \) terms sum \( \approx 0 \); \( I = NI_0 \).
Coherent and Incoherent Sums

With \( E = E_0 \sum_{j=1}^{N} e^{i \Delta \phi_j} \),

- “Incoherent” case: \( I = NI_0 \).
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With \( E = E_0 \sum_{j=1}^{N} e^{i \Delta \phi_j} \),

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- "Coherent" case: \( I = N^2 I_0 \).

If the phase offset at each antenna is noise-like, incoherent sum.
If the phase offset precisely compensates for a given direction, we have a coherent sum in that direction: *beam steering*. 
Coherent and Incoherent Sums

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Note that each antenna has a power pattern, and we end up with a net power pattern for the array:

\[ P_n^{(A)} \equiv \text{the Fourier transform of the antenna locations.} \]
Beam Steering

Control the direction of the main lobe of the array by the control phases $\hat{\phi}_j$.

- Reference direction: direction of main lobe
- Array beam can be steered to any location that is within the primary beam of each antenna.
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- How? Analog delay lines, or digital delays.
- If $\exists$ multiple sets of control phases, then multiple beams on the sky! Cost: Extra processing.
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- How? Analog delay lines, or digital delays.
- If $\exists$ multiple sets of control phases, then multiple beams on the sky! Cost: Extra processing.
- If each antenna has a wide FoV (e.g., dipoles $\theta_{FWHM} \sim 1$ radian) then much of the sky can be viewed simultaneously.
- Steerable antennas: entire sky is accessible. (cf., SKA).
Rather than summing antenna outputs in real time, process each pair of antennas separately, and do all beam forming offline in software.

⇒ Fourier transformation vs. correlation (convolution).
→ Cost: much, much more processing.
(Also, not usually real-time.)
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Schematically:

\[
\text{Array}(X_j) \xrightarrow{\text{Cross correlate pairs}} \Gamma_{ij} \xrightarrow{\mathcal{F} \text{ vs. } b} I(\theta)
\]

\[
\mathcal{F} \text{ vs. } X_j \uparrow \quad P_n^{(A)}(\theta) \xrightarrow{\text{Convolve with sky}} I(\theta)
\]
The “visibility” $\Gamma$ is function of $(b_{ij}) = X_j - X_i$.

$$\Gamma_{ij} \equiv \langle E_i E^*_j \rangle = \langle E(X_i) E^*(X_j) \rangle$$

Simpler notation: $\Gamma(b)$, where

$$b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$
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$$\Gamma_{ij} \equiv \langle E_i E_j^* \rangle = \langle E(X_i) E^*(X_j) \rangle$$

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$$b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

For small fields of view, the “$w$” coordinate can be ignored if the $z$-direction $\approx$ the reference direction.

*(Assumption breaks down for very wide-field imaging.)*
The visibility function then depends on the “$u$” and “$v$” coordinates:
Visibility function $\Gamma(u, v)$.

$$
I(\theta) = I(\theta_x, \theta_y) = \int\int du \, dv \, e^{2\pi i(u\theta_x + v\theta_y)} \Gamma(u, v)
$$

(The Van Cittert-Zernike theorem.)

*Note: The primary antenna pattern $P_{n}^{(1)}(\theta)$ is implicit.*
The visibility function then depends on the “u” and “v” coordinates:
Visibility function $\Gamma(u, v)$.

$$I(\theta) = I(\theta_x, \theta_y) = \int \int du \, dv \, e^{2\pi i (u \theta_x + v \theta_y)} \Gamma(u, v)$$

(The Van Cittert-Zernike theorem.)

Note: The primary antenna pattern $P^{(1)}_n(\theta)$ is implicit.

Interferometry:

- Measure Fourier components of $I(\theta)$
- $\equiv$ Visibility $\Gamma(b, \tau)$.
- Transform to get image estimate $\hat{I}(\theta)$. 
Interferometric Imaging

Primary beam \times \text{fringe pattern of 2-element int.}

VLA: 27 \text{ antennas} \Rightarrow 351 \text{ unique pairs}
\[ I(\theta) \equiv K_\lambda \int db \, e^{ikb \cdot \theta} \Gamma(b; \tau) \]
\[ \rightarrow \int \int du \, dv \, e^{2\pi i (u\theta_x + v\theta_y)} \Gamma(u, v; \tau) \]

After sampling the Fourier components, transform to get:

\[ \hat{I}(\theta) = \sum_{u, v} e^{2\pi i (u\theta_x + v\theta_y)} \Gamma(u, v; \tau) \]

where \( u, v, \theta \) are discrete.
The following are Fourier transform pairs:

\[(u-v) \text{ coverage} \Leftrightarrow \text{Dirty beam} \]
\[\mathcal{F}(\text{sky flux}) \Leftrightarrow \text{Sky flux} \]
\[\text{Visibilities} \Leftrightarrow \text{Dirty map} \]

Visibilities \(\equiv \mathcal{F}(\text{sky flux}) \times \text{the}\ (u-v)\ \text{coverage}\ (1\ \text{in}\ \text{some}\ \text{places,}\ 0\ \text{in}\ \text{others}).\)

⇒ The dirty map \(\equiv \text{the}\ \text{sky}\ \text{brightness}\ \text{distribution}\ \text{convolved}\ \text{with}\ \text{the}\ \text{dirty}\ \text{beam}.\)
Interferometric Imaging: Summary

FT
IFT

u-v coverage

Synthesized ("dirty") beam

Convolution

Sky brightness distribution

Output ("dirty") map
• Control phases are based on a reference direction (the “phase center”), as in beam forming.
Complications in Interferometric Imaging

- Control phases are based on a reference direction (the “phase center”), as in beam forming.

- Phases are corrupted by:
  - Intervening media (differential refraction, scattering, ...);
  - Imperfect knowledge of coordinate system (Earth rotation);
  - Imperfect knowledge of antenna locations (tectonics; geodesy);
  - Imperfect clocks and oscillators;
  - Variable phases in electronics (temperature).
Interferometry: Summary

- Voltage correlation function: related to mutual coherence of the radiation field.
- Cross-correlations → Stokes parameter visibilities.
- Coherence time is limited by LOs, atmosphere, ...
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- Compared to single dish observations, advantages:
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- But: Limited collecting area, computation intensive.
- Larger scales are filtered out! Zero spacing flux density.
- Smearing effects due to finite bandwidth and integration time (i.e., averaging over time and frequency).
- Consider: calibration, imaging, sky mapping schemes.
How do we measure $\Gamma$, the visibility function?

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  Closure phase: $\phi_{1,2} + \phi_{2,3} + \phi_{3,1} = 0$.
  Closure amplitude: $\Gamma(1, 2) \cdot \Gamma(3, 4) = \Gamma(1, 3) \cdot \Gamma(2, 4)$

Self calibration and hybrid mapping.
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- Self calibration and hybrid mapping.

- Field of view issues: primary beam and delay beam.
How do we go from $\Gamma$ to the sky brightness distribution $B$?

- Fourier relationship between $\Gamma \leftrightarrow B$.
- Relationship is approximate in practice.
- In some cases, better to use alternatives to go from $\Gamma \rightarrow B$. 
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**e.g., Maximum Entropy Method:**
Use an *a priori* model of the sky and maximize a pre-defined statistic.
(But where do you get the model? A blank sky works.)
Constructing an image from a set of visibilities: 
FFT + CLEAN, or MEM.

- Sampling in $uv$ plane is incomplete.
- Dealing with missing data: MEM, CLEAN take different approaches (smooth extended structure vs. collection of point sources).
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Dynamic range: Dirty maps $\rightarrow 10:1$
CLEAN $\rightarrow 100s:1$
Self-cal $\rightarrow 10^4:1$
SKA goal $\rightarrow 10^6 - 10^7$. 
Further Reading

- Synthesis Imaging in Radio Astronomy II
  The “White Book” is a complete practical reference.
- Interferometry and Synthesis in Radio Astronomy
  Offers a sound theoretical background to the subject.
- EVLA and VLBA Observational Status Summaries.
  The Novice’s Guide to using the VLBA.
  (See http://www.nrao.edu/)
- Synthesis interferometry works well at radio wavelengths but is much more
difficult at optical wavelengths because of a fundamental difference in the
nature of the signals (classical regime vs. quantum mechanical). This is
explored in the final lecture in the White Book: Noise and Interferometry, V.
Radhakrishnan, in Synthesis Imaging in Radio Astronomy II (as above).
Highly recommended!