Lecture 16

- CLEAN algorithm
  - Spectral estimation, image restoration
  - General deconvolution tool
- Maximum entropy applications
  - General solution for PDFs (constraints, partition function)
  - Entropy expression for power spectra/images:
  - Maximum entropy spectrum for Gaussian processes
  - Relationship to autoregressive model fitting
- Other approaches (HRM, Cholesky decomposition)

Another reference for `classical' spectral analysis methods

- `Spectrum and spectral density estimation by the Discrete Fourier transform (DFT), including a comprehensive list of window functions and some new flat-top windows.'
  - http://www.rssd.esa.int/SP/LISAPATHFINDER/docs/Data_Analysis/GH_FFT.pdf
Signals embedded in the radial velocity noise

Periodic variations in the \( \tau \) Ceti velocities*

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ABSTRACT

Context. The abilities of radial velocity exoplanet surveys to detect the lowest-mass extra-solar planets are currently limited by a combination of instrument precision, lack of data, and “jitter”. Jitter is a general term for any unknown features in the noise, and reflects a lack of detailed knowledge of stellar physics (asteroseismology, starspots, magnetic cycles, granulation, and other stellar surface phenomena), as well as the possible underestimation of instrument noise.

Aims. We study an extensive set of radial velocities for the star HD 10700 (\( \tau \) Ceti) to determine the properties of the jitter arising from stellar surface inhomogeneities, activity, and telescope-instrument systems, and perform a comprehensive search for planetary signals in the radial velocities.

Methods. We performed Bayesian comparisons of statistical models describing the radial velocity data to quantify the number of significant signals and the magnitude and properties of the excess noise in the data. We reached our goal by adding artificial signals to the “flat” radial velocity data of HD 10700 and by seeing which one of our statistical noise models receives the greatest posterior probabilities while still being able to extract the artificial signals correctly from the data. We utilised various noise components to assess properties of the noise in the data and analyse the HARPS, AAPS, and HIRES data for HD 10700 to quantify these properties and search for previously unknown low-amplitude Keplerian signals.

Results. According to our analyses, moving average components with an exponential decay with a timescale from a few hours to few days, and Gaussian white noise explains the jitter the best for all three data sets. Fitting the corresponding noise parameters results in significant improvements of the statistical models and enables the detection of very weak signals with amplitudes below 1 m s\textsuperscript{-1} level in our numerical experiments. We detect significant periodicities that have no activity-induced counterparts in the combined radial velocities. Three of these signals can be seen in the HARPS data alone, and a further two can be inferred by utilising the AAPS and Keck data. These periodicities could be interpreted as corresponding to planets on dynamically stable close-circular orbits with periods of 13.9, 35.4, 94, 168, and 640 days and minimum masses of 2.0, 3.1, 3.6, 4.3, and 6.6 \( M_\oplus \), respectively.

Key words. methods: statistical – methods: numerical – techniques: radial velocities – stars: individual: HD 10700
The 978 AAPS RVs have a similar character to the HARPS data (Fig. 1, middle panel). This data set, too, can be described as being flat and no periodic signals have been reported by the AAPS group despite an extensive baseline of the time-series of 4923 days (Wittenmyer et al. 2011b). This data does not have such clear annual gaps as the HARPS data (Fig. 1) and deviates from the mean by approximately 5.0 m s$^{-1}$ on average.

The smallest set of RVs was that measured using the HIRES (Fig. 1, bottom panel). The 567 HIRES RV measurements have a baseline of 3446 days and nothing has been reported about this data set in the literature. The velocities deviate roughly 2.9 m s$^{-1}$ from the mean and do not have significant gaps apart from relatively narrow annual gaps corresponding to the visibility of HD 10700 from the Keck telescope’s northern location in Hawaii.

3. Statistical analysis and modelling
Table 5. MAP estimates and the corresponding 99% BCs of the parameters of the three-Keplerian model for the HARPS RVs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>HD 10700 b</th>
<th>HD 10700 c</th>
<th>HD 10700 d</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0.17 [0.041]</td>
<td>0.11 [0.028]</td>
<td>0.21 [0.042]</td>
</tr>
<tr>
<td>$K$ [m s$^{-1}$]</td>
<td>0.60 [0.30, 1.02]</td>
<td>0.89 [0.52, 1.32]</td>
<td>0.89 [0.42, 1.22]</td>
</tr>
<tr>
<td>$\omega$ [rad]</td>
<td>2.2 [0, 2\pi]</td>
<td>2.8 [0, 2\pi]</td>
<td>3.9 [0, 2\pi]</td>
</tr>
<tr>
<td>$M_b$ [rad]</td>
<td>3.4 [0, 2\pi]</td>
<td>3.9 [0, 2\pi]</td>
<td>5.1 [0, 2\pi]</td>
</tr>
<tr>
<td>$m_b \sin i$ [M$_\odot$]</td>
<td>2.0 [1.0, 3.3]</td>
<td>4.0 [2.2, 5.7]</td>
<td>5.5 [2.6, 7.5]</td>
</tr>
<tr>
<td>$a$ [AU]</td>
<td>0.106 [0.099, 0.110]</td>
<td>0.197 [0.184, 0.204]</td>
<td>0.379 [0.356, 0.393]</td>
</tr>
<tr>
<td>$\sigma_f$ [m s$^{-1}$]</td>
<td>1.05 [1.01, 1.12]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. Lomb-Scargle periodograms of the HARPS data residuals after removing moving average components from the noise (top panel) and removing the 35, 14, and 95 day signals (subsequent panels). The dotted, dashed, and dot-dashed lines indicate the analytical 0.1%, 1%, and 10% FAPs, respectively.

Fig. 6. Phase-folded signals of the three-Keplerian solution with the other two signals and the moving average component (MA(10)) of the noise removed.

contain much more maxima likely arising from noise correla...
Data Modeling

• Detection and model fitting both require:
  – Signal model
  – Noise model (not always white, Gaussian noise!)

• Optimization is always with respect to a complete model
  – Detection:
    • Minimum false alarms
    • Maximum detection probability
    • Maximum completeness
  – Model fitting:
    • Robust parameter estimation (no bias, minimum errors)
  – Model comparison / hypothesis testing
Methods of Tuomi et al.

- Data = radial velocity measurements (RVs) of the star Tau Ceti = HD10700
- Nonuniform sampling → LS periodogram
- Noise in data:
  - Measurement noise
  - Jitter noise in HD10700 from asteroseismic activity, starspots, granulation, etc
- Signal model: k sinusoidal signals from planets
- Noise model:
  - Gaussian noise from measurement errors (instrumental)
  - Gaussian noise from effects not instrumental
  - Autoregressive, moving average model (with parameters TBD) to account for possible correlations
- Bayesian modeling to choose between noise models and parameters
- Used ~Markov Chain Monte Carlo method (Metropolis algorithm, specifically) to determine best model parameters and 99% confidence interval from the posterior PDF.
- Priors for signal parameters and noise parameters
  - E.g. a Gaussian PDF for eccentricities that disfavored highly eccentric orbits.
D. Autoregressive (AR) process: depends on past values + white noise:

\[ x_t = n_t - \sum_{j=1}^{M} \alpha_j x_{t-j}, \]

where

\( n_t = \) discrete white noise.

\( M = \) order of AR model

\( \alpha = \) coefficients of AR model.

AR processes play a role in **maximum entropy spectral estimators**.

By taking the Fourier transform of the expression for \( x_t \) we can solve for

\[ \tilde{X}_f = \frac{\tilde{N}_f}{1 + \sum_{j} \alpha_j e^{-2\pi i j f}} \]
E. **Moving average (MA) process:** is a moving average of white noise:

\[ x_t = \sum_{j=0}^{N} \beta_j n_{t-j}. \]

F. **ARMA process:** AR and MA combined.

G. **ARIMA process:** An integrated ARMA process.
Autoregressive processes (M=2)

\[ x_t = n_t + \sum_{i=1}^{M} a_i x_{t-i} \]

\[ n_t = \text{white noise} \]

\[ a_i = \text{AR coefficient} \]

From Scargle 1981
Habitable-zone super-Earth candidate in a six-planet system around the K2.5V star HD 40307*

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S. S. Vogt6, and R. P. Butler6

ABSTRACT

Context. The K2.5 dwarf HD 40307 has been reported to host three super-Earths. The system lacks massive planets and is therefore a potential candidate for having additional low-mass planetary companions.

Aims. We re-derive Doppler measurements from public HARPS spectra of HD 40307 to confirm the significance of the reported signals using independent data analysis methods. We also investigate these measurements for additional low-amplitude signals.

Methods. We used Bayesian analysis of our radial velocities to estimate the probability densities of different model parameters. We also estimated the relative probabilities of models with differing numbers of Keplerian signals and verified their significance using periodogram analyses. We investigated the relation of the detected signals with the chromospheric emission of the star. As previously reported for other objects, we found that radial velocity signals correlated with the S-index are strongly wavelength dependent.

Results. We identify two additional clear signals with periods of 34 and 51 days, both corresponding to planet candidates with minimum masses a few times that of the Earth. An additional sixth candidate is initially found at a period of 320 days. However, this signal correlates strongly with the chromospheric emission from the star and is also strongly wavelength dependent. When analysing the red half of the spectra only, the five putative planetary signals are recovered together with a very significant periodicity at about 200 days. This signal has a similar amplitude as the other new signals reported in the current work and corresponds to a planet candidate with $M \sin i \sim 7 \ M_\oplus$ (HD 40307 g).

Conclusions. We show that Doppler measurements can be filtered for activity-induced signals if enough photons and a sufficient wavelength interval are available. If the signal corresponding to HD 40307 g is a genuine Doppler signal of planetary origin, this candidate planet might be capable of supporting liquid water on its surface according to the current definition of the liquid water habitable zone around a star and is not likely to suffer from tidal locking. Also, at an angular separation of $\sim 46$ mas, HD 40307 g would be a primary target for a future space-based direct-imaging mission.

Key words. methods: statistical – methods: numerical – techniques: radial velocities – stars: individual: HD 40307
Binning vs Moving Average

• Tuomi et al. argue that a moving average of their time series is better than binning because it includes more information.
• We can investigate this assertion by looking at examples in the next two slides.
• One of the issues is whether the result from binning is Nyquist sampled.
Binning vs. Moving Averages

$x(t)$  Time Series  $N = 1024$ samples

$x_s(t)$  Boxcar-smoothed time series  Filter length = 64

$x_s(t_{k \times N/M}, k = 0, \ldots, 15)$  Decimated, smoothed time series

Averaging Bins ($M=16$)  $N/M=64$ samples/bin

Bin Averaged Time Series

0  Time (sec)  51.2
Spectrum of $x(t)$ \quad FFT Length = N=1024
$\text{Var}(x) = 0.986 \quad \sum_f S_x(f) = 0.986$

Spectrum of window function $w(t)$ \quad FFT Length: N=1024

Spectrum of smoothed $x(t)$ \quad FFT Length: N=1024
$\text{Var}(x_s) = 0.012 \quad \sum_f S_{x_s}(f) = 0.012$

Spectrum of binned/decimated $x(t)$ \quad FFT Length: M=16
$\text{Var}(x_{\text{binned}}) = 0.014 \quad \sum_f S_{x_{\text{binned}}}(f) = 0.014$

Spectrum of smoothed/Nyquist $x(t)$ \quad FFT Length: $2M=32$
$\text{Var}(x_{\text{sNyquist}}) = 0.014 \quad \sum_f S_{x_{\text{sNyquist}}}(f) = 0.014$

0 \quad \text{Frequency (Hz)} \quad 10.0
$x(t)$  Time Series  $N = 1024$ samples

$\tilde{x}_s(t)$  Boxcar smoothed time series  Filter length = 16

$\tilde{x}_s(t_{k \times N/M}, k = 0, \ldots, 63)$  Decimated, smoothed time series

Averaging Bins ($M=64$)  $N/M=16$ samples/bin

Bin Averaged Time Series

0  Time (sec)  51.2
Spectrum of $x(t)$ \hspace{1cm} FFT Length = N=1024

\[ \text{Var}(x) = 1.111 \quad \sum_f S_x(f) = 1.111 \]

Spectrum of window function $w(t)$ \hspace{1cm} FFT Length: $N=1024$

Spectrum of smoothed $x(t)$ \hspace{1cm} FFT Length: $N=1024$

\[ \text{Var}(x_s) = 0.069 \quad \sum_f S_{x_s}(f) = 0.069 \]

Spectrum of binned/decimated $x(t)$ \hspace{1cm} FFT Length: $M=64$

\[ \text{Var}(x_{\text{binned}}) = 0.061 \quad \sum_f S_{x_{\text{binned}}}(f) = 0.061 \]

Spectrum of smoothed/Nyquist $x(t)$ \hspace{1cm} FFT Length: $2M=128$

\[ \text{Var}(x_{\text{sNyquist}}) = 0.061 \quad \sum_f S_{x_{\text{sNyquist}}}(f) = 0.061 \]

0 \hspace{1cm} Frequency (Hz) \hspace{1cm} 10.0
Radial velocity studies of cool stars

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Our current view of exoplanets is one derived primarily from solar-like stars with a strong focus on understanding our Solar System. Our knowledge about the properties of exoplanets around the dominant stellar population by number, the so-called low-mass stars or M dwarfs, is much more cursory. Based on radial velocity discoveries, we find that the semi-major axis distribution of M dwarf planets appears to be broadly similar to those around more massive stars and thus formation and migration processes might be similar to heavier stars. However, we find that the mass of M dwarf planets is relatively much lower than the expected mass dependency based on stellar mass and thus infer that planet formation efficiency around low-mass stars is relatively impaired. We consider techniques to overcome the practical issue of obtaining good quality radial velocity data for M dwarfs despite their faintness and sustained activity and emphasize (i) the wavelength sensitivity of radial velocity signals, (ii) the combination of radial velocity data from different experiments for robust detection of small amplitude signals, and (iii) the selection of targets and radial velocity interpretation of late-type M dwarfs should consider Hα behaviour.

1. Introduction

Over the past two decades, the field of exoplanets has made extraordinary progress. Rather than wondering...
Periodogram of the residuals to the three-planet solution as a function of the blue cut-off.

Hugh R. A. Jones et al. Phil. Trans. R. Soc. A
2014;372:20130088
The open circles represent data acquired with Keck and the filled circles represent those from HARPS-S.

Hugh R. A. Jones et al. Phil. Trans. R. Soc. A
2014;372:20130088

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Key stellar parameters plotted against radial velocity RMS. The plots are of versus radial velocity RMS (top), spectral type versus radial velocity RMS (middle) and activity ($\log_{10}(LH\alpha/L_{bol})$) versus RMS velocity (bottom).

Hugh R. A. Jones et al. Phil. Trans. R. Soc. A
2014;372:20130088
Linear/Fixed Window vs. Data Adaptive Spectral Analysis

- FT approaches (DFT, LS) require ad hoc treatment of missing data
  - → sidelobes, leakage, bias
  - The level of sidelobes, leakage and bias may be acceptable in some circumstances but not others
  - In imaging problems, there may be no choice (the optics does the Fourier transforming)
  - The estimator is linear in the autocorrelation (or cross correlation for a cross spectrum)
  - Sidelobes can be removed using the CLEAN algorithm

- Maximum entropy spectral analysis is nonlinear in the ACF and has data adaptive windowing so that sidelobes and bias are minimized.
- Application of the CLEAN algorithm is not needed.
## CLEAN Algorithm

<table>
<thead>
<tr>
<th>Spectral Analysis</th>
<th>Interferometric Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(f) = \text{spectral estimate}$</td>
<td>$I_{DM}(\theta) = \text{dirty map or image}$</td>
</tr>
<tr>
<td>$S_W(f) =</td>
<td>\text{FT}{W(t)}</td>
</tr>
<tr>
<td>(calculated as the FT of sample times)</td>
<td>(calculated as the FT of samples in the UV plane)</td>
</tr>
<tr>
<td>$S_C(f) = \sum C_j , \Delta(f - f_j)$</td>
<td>$I_C(\theta) = \sum_j \sum_k C_{jk} , r(\theta - \theta_{jk})$</td>
</tr>
<tr>
<td>CLEANed Spectrum</td>
<td>$r(\theta) = \text{restoration function that represents the}$</td>
</tr>
<tr>
<td>Sum over CLEAN components</td>
<td>$\text{inherent angular resolution of a sensor array}$</td>
</tr>
<tr>
<td>$\Delta(f) = \text{restoration function that represents the}$</td>
<td></td>
</tr>
<tr>
<td>$\text{inherent frequency resolution}$</td>
<td></td>
</tr>
</tbody>
</table>

- The restoring function is needed to fairly represent the resolution imposed by physics and Fourier transform properties (uncertainty principle)
CLEAN Algorithm

- The premise of CLEAN is that our data consist not only of the data amplitudes but also the detailed sampling in time or in the UV plane.
- We therefore know that the quantity of interest (true spectrum or image) is convolved with a known window function.*
- The actual algorithm is based on the fact that any function can be represented as a sum or integral over delta functions:

\[ y(x) = \int dx' \: y(x')\delta(x - x') \]

\[ y(x_j) = \sum_k y(x_k)\delta_{jk} \]

* There are instances where the sampling may not be known exactly (jitter in an analog to digital converter, errors in the locations of sensors, clock errors, etc.)
CLEAN Algorithm

Data yield \( \hat{S}(f) \) and \( S_W(f) = |\tilde{W}(f)|^2 \)

\[ \hat{S}(f) \]

Find largest amplitude in spectrum

Scale and shift \( S_W(f) \)

Subtract and calculate residual spectrum

\( \hat{S_i}(f_i)_{\text{max}} @ f_i \quad i = \text{iteration count} \)

\( \gamma_i S_W(f - f_i) \)

\( \hat{S_i}(f) \rightarrow \hat{S_{i+1}}(f) = \hat{S_i}(f) - \gamma_i S_W(f - f_i) \)

Collect CLEAN components

\( \{ \gamma_i, f_i, i = 1, N_c \} \)

Form CLEAN spectrum

\[ S_c(f) = \sum_{i=1}^{N_c} \gamma_i \Delta(f - f_i) \]
TIME SERIES ANALYSIS WITH CLEAN. I. DERIVATION OF A SPECTRUM

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ABSTRACT

We present a method of time-series spectral analysis which is especially useful for unequally spaced data. Based on a complex, one-dimensional version of the CLEAN deconvolution algorithm widely used in two-dimensional image reconstruction, this technique provides a simple way to understand and remove the artifacts introduced by missing data. We describe the method, give several examples, and point out various analogies with the conventional use of CLEAN.
Fig. 2. Analysis of the time series in Fig. 1. (a)–(e). The clean components, residual spectra, and clean spectra after one, two, three, five and one hundred iterations with gain $g = 0.5$, and (f) after one iteration with $g = 1$. Note the change to a logarithmic scale for (d)–(f).
Fig. 3. Spectral analysis of 201 equally spaced noiseless data containing harmonic components at 31 and 57 Hz; (a) time series, (b) spectral window, (c) dirty spectrum, and (d) clean spectrum after one hundred iterations of CLEAN with $g = 0.5$. 
Fig. 6. One-hundred-one randomly spaced noiseless data containing harmonic components at 31 and 57 Hz. Same parts as Fig. 3.
A CLEAN-BASED METHOD FOR DECONVOLVING INTERSTELLAR PULSE BROADENING FROM RADIO PULSES

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ABSTRACT

Multipath propagation in the interstellar medium distorts radio pulses, an effect predominant for distant pulsars observed at low frequencies. Typically, broadened pulses are analyzed to determine the amount of propagation-induced pulse broadening but with little interest in determining the undistorted pulse shapes. In this paper, we develop and apply a method that recovers both the intrinsic pulse shape and the pulse-broadening function that describes the scattering of an impulse. The method resembles the CLEAN algorithm used in synthesis imaging applications, although we search for the best pulse-broadening function and perform a true deconvolution to recover intrinsic pulse structure. As figures of merit to optimize the deconvolution, we use the positivity and symmetry of the deconvolved result along with the mean square residual and the number of points below a given threshold. Our method makes no prior assumptions about the intrinsic pulse shape and can be used for a range of scattering functions for the interstellar medium. It can therefore be applied to a wider variety of measured pulse shapes and degrees of scattering than the previous approaches. We apply the technique to both simulated data and data from Arecibo observations.

Subject headings: ISM: structure — methods: data analysis — pulsars: general — radio continuum: general — scattering
For the next iteration, and the process is repeated until the residual subtracted pulse, $D_y(t)$, is comparable to the off-pulse rms noise level.

Upon termination of the iteration, $n_c$ CLEAN components are identified:

$$C_j, t_j, j = 1, \ldots, n_c$$

denote the amplitudes and times of these CLEAN components, i.e., the collection of delta functions at the end of the CLEAN process.

The restored pulse shape is built from the ensemble of CLEAN components, $C_j$, convolved with the restoring function $C(t)$, details of which we will discuss later in this section:

$$y_r(t) = \sum_{j=1}^{n_c} \frac{C_j}{C(t/t_j)}$$

At the end of the CLEAN process, the residual noise, $D_y(t)$, is added to the convolved CCs to obtain the final restored pulse profile.

Examples of the CLEAN process are shown in Figure 1 for simulated data (left-hand panels) and real data (right-hand panels). The figure shows the measured pulse shapes, the chosen PBF (a one-sided exponential as in eq. [3]), the CCs, and the deconvolved profile. Compared to the traditional least-squares or frequency-extrapolation methods (2.1), the approach to the pulse-broadening problem outlined here is much easier to apply because it is nonparametric. Further, it does not make any assumption about the intrinsic pulse shape or its evolution with frequency, and, thus, can potentially yield useful information about the intrinsic pulse shape. Below, we elaborate on some aspects concerning the practical realization of the method.

### 2.3.1. Loop Gain

Small values of the loop gain ($\gamma$) are employed in imaging applications. The performance of the method is expected to change only marginally for sufficiently small values (say, $\gamma = 0.1$). Larger values tend to cause oversubtraction and a consequent lack of convergence. We adopted $\gamma = 0.05$ as a reasonable choice for our application. Thus, in each iteration, a delta function with an amplitude of 5% of the profile

The termination criterion is not well defined for the CLEAN technique.

**Fig. 1.** Sample plots that illustrate the CLEAN process for deconvolving the interstellar pulse broadening. *Left*: Simulated data. *Right*: Data for PSR J1852+0031 at 1475 MHz. (a) Scattered pulse shapes. (b) Model PBFs. (c) CLEAN components. (d) Final restored (CLEANed) pulse shapes.
Fig. 3.—Similar to Fig. 2, except that the PBF employed by the CLEAN deconvolution method has a more rounded shape (PBF$_2$, due to a uniform scattering medium between the pulsar and the Earth). [See the electronic edition of the Journal for a color version of this figure.]
Gravitational wave tests of general relativity with the parameterized post-Einsteinian framework

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Gravitational wave astronomy has tremendous potential for studying extreme astrophysical phenomena and exploring fundamental physics. The waves produced by binary black hole mergers will provide a pristine environment in which to study strong-field dynamical gravity. Extracting detailed information about these systems requires accurate theoretical models of the gravitational wave signals. If gravity is not described by general relativity, analyses that are based on waveforms derived from Einstein’s field equations could result in parameter biases and a loss of detection efficiency. A new class of “parameterized post-Einsteinian” waveforms has been proposed to cover this eventuality. Here, we apply the parameterized post-Einsteinian approach to simulated data from a network of advanced ground-based interferometers and from a future space-based interferometer. Bayesian inference and model selection are used to investigate parameter biases, and to determine the level at which departures from general relativity can be detected. We find that in some cases the parameter biases from assuming the wrong theory can be severe. We also find that gravitational wave observations will beat the existing bounds on deviations from general relativity derived from the orbital decay of binary pulsars by a large margin across a wide swath of parameter space.

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PACS numbers: 04.80.Cc, 04.30.-w, 04.50.Kd, 04.80.Nn

1. INTRODUCTION

Einstein’s theory of gravity has been subject to a wide array of experimental tests and has passed them all with flying colors [1]. None of these tests, however, has probed the strong-field dynamical regime that pertains to the final inspiral and merger of compact objects. The Hulse-Taylor binary pulsar PSR B1913 + 16 [2] and the double binary pulsar PSR J0737-3039A [3,4] have provided convincing evidence for the existence of gravitational waves, and have served as unique laboratories to test general relativity (GR), but these objects have relatively small orbital velocities, \( v/c \approx 10^3 \), a mere factor of 10 faster than the Earth’s orbit around the Sun. The parameter space covered by black hole mergers, where orbital velocities \( v/c \gg 10^3 \) and can approach \( v/c \approx 0.7 \), is currently terra incognita—dragons may yet lurk there.

If not accounted for, the possibility that Einstein’s theory of gravity may not correctly describe the production and propagation of gravitational waves could have dire consequences for gravitational wave astronomy. In the case of ground-based detectors, the detection of weak signals buried below the instrument noise requires accurate models of the gravitational waveforms. Errors in the modeling of these waveforms can lead to a loss in detection efficiency. When the signals are stronger, as will often be the case with space-based observations of black hole mergers, waveform templates will no longer be needed for detection, but a waveform model will be required to infer the physical parameters of the system, such as the masses and spins of the black holes, and the distance to the system. Waveform models based on an incorrect theory of gravity will lead to fundamental bias [5] in the recovered parameters. Because these waveforms would not accurately describe nature, the parameters that maximize the fit of such a waveform to data would not correspond to the true physical values of the system. This bias is distinct from that caused by imperfect modeling of GR, as explored in [6], as it reflects a fundamental lack of knowledge about the true nature of gravity, and not simply the use of inaccurate physical assumptions—see [5] for more details.

Turning the problem around, the discovery that Einstein’s theory is flawed would be the greatest result to come out of gravitational wave astronomy [7]. This has served as the motivation for the development of a wide range of tests of GR that use gravitational wave observations. These tests can be broadly classified as “extrinsic” or “intrinsic.” Extrinsic tests are possible when there is a concrete alternative theory, such as massive gravitons [8–14], or Brans-Dicke theory [9,10,14–16]. Intrinsic tests work within the confines of GR, and take the form of internal consistency checks, such as measuring the multipolar structure of the metric [17,18], or multimodal spectroscopy of black hole (BH) inspiral and ringdown waveforms [19,20]. These tests are valuable, but they do not cover the full spectrum of possibilities. The existing extrinsic tests are limited by the lack of viable alternative models, while the intrinsic tests do not so much test GR, as “test the nature of massive compact bodies within GR” (to quote [21]).
Aspects of Spectral Analysis

The outcome of any attempt to estimate the power spectrum should be assessed through consideration of the following:

(1) confidence intervals
(2) resolution
(3) handling of gaps (missing data)
(4) bias (particularly for processes with steep power laws)

Numbers (2) and (3) are related because resolution is determined by the length of the data set $T$ and one can view data after the interval $[0, T]$ as missing data.
Linear/fixed window FT methods vs. Data adaptive methods

Spectral estimators based on the Fourier transform have the properties:

no smoothing \[
\begin{align*}
1. \text{frequency resolution: } & \delta f = T^{-1} \quad T = \text{data span} \\
2. \text{estimation error: } & \varepsilon = \frac{\mathrm{Var} \hat{S}}{\langle \hat{S}(\omega) \rangle} = 1
\end{align*}
\]

with smoothing \[
\begin{align*}
3. \text{tradeoff between } & \varepsilon \text{ and } \delta f : \\
& \text{spectral window } \tilde{W} \Rightarrow \varepsilon \text{ decreases, } \delta f \text{ increases,} \\
& \text{sidelobes altered } \varepsilon = \left(\frac{2}{N_{d.o.f.}}\right)^{1/2}
\end{align*}
\]

4. Since \( \hat{R}_w(\tau) \Leftrightarrow \hat{S}_w(\omega) \), the estimator is \textbf{linear} in the correlation function estimates, \( \hat{R}_w \).

5. Also, once chosen by the analyst, the window function is \textbf{fixed} with respect to the data.

Therefore, even though the spectral estimation may be \textbf{iterative} whereby the analyst chooses by trial an \textbf{optimum} window, this optimization is performed according to intuitive/aesthetic/prior knowledge prejudice. \textit{The form of the window is not an integral part of the estimation process.}

Hence, the characterization of finite F.T. estimators as \textit{linear, fixed window, empirical} estimators.
Different Views on Missing Data in a Time Series

1. Transform (in the general sense) data using fixed window analyses; then correct the data for the effects of gaps in data, etc. (e.g. can apply a CLEAN algorithm to the spectrum).

   vs.

2. Transform the data without making assumptions about missing data (maximum entropy techniques are “maximally noncommittal” about missing data).

‘Empirical methods such as Blackman-Tukey, which do not invoke even a likelihood function, are useful in the preliminary, exploratory phase of a problem where our knowledge is sufficient to permit intuitive judgments about how to organize a calculation (smoothing, decimation, windows, prewhitening, padding with zeroes, etc.) but insufficient to set up a quantitative model which would do the proper things for us automatically and optimally.’ [In abstract of ”On the Rationale of Maximum Entropy Methods”, E.T. Jaynes, 1982, IEEE Proc., 70, 939.]
I. Fourier transform + deconvolution approach: Do on blackboard.

Suppose time series has gaps:
The effective window on the data:
We can write sampled data as \( x_s(t) = \Omega(t)x(t) \)
Subtract mean and put zeroes into the gaps:
We want an estimate for the power spectrum of \( x(t) \) but we can only use \( x_s(t) \) in our estimates.

**Fourier transform:**
\[
\tilde{X}_s(f) = \tilde{\Omega}(f) \ast \tilde{X}(f)
\]
\[
|\tilde{X}_s(f)|^2 = |\tilde{\Omega}(f) \ast \tilde{X}(f)|^2
\]

**Deconvolution of the Window Function:**
Suppose \( \tilde{X}(f) \) has a spectral line, then \( |\tilde{\Omega}(f)|^2 \) is convolved with the spectral line, adding large sidelobes (because of discontinuities).
These can be removed, to some extent, by deconvolving the \( |\tilde{\Omega}(f)|^2 \) from the spectral estimate.
For “simple” spectral estimates composed of a few spectral lines, the deconvolution process becomes, essentially, a subtraction.
For example a spectral estimate may appear as

![Spectral Estimate Diagram](image)

and the window function has $|F.T.|^2$ of

![Window Function Diagram](image)

By subtracting $\gamma|\tilde{\Omega}(f - f_0)|^2$ (where $\gamma$ = scale factor < 1 and often $\ll 1$ and $f_0$ is a shift) from the spectral estimate one may be able to investigate low level structure in the estimate.

This approach was developed first in the 2-D context of radio interferometry and is the core of the algorithm ‘CLEAN’.

Roberts et al. 1987 (A.J.) have developed the CLEAN technique for 1-D spectral analysis of time series.
CLEAN spectrum

\[ \sum c_i s(t-i) \ast W_i(t) + R(t) \]

CLEAN component

residual spectrum

\[ j \leq 1 \]

\[ g = \text{loop gain} \]

\[ \Delta \text{window data} \]

\[ X_w(t) = \Delta L(t) X(t) \]

\[ |\Delta L(t) \ast \Delta W(t)|^2 \text{ spectrum of data} \]

\[ \text{Find largest amplitude in spectrum: } A(f_{\text{max}}) \text{ at } f = f_{\text{max}} \]

\[ \text{Scale } |\Delta L(t)|^2 \text{ and shift } f \rightarrow f_{\text{max}} \]

\[ \text{Subtract } g \cdot A(f_{\text{max}}) |\Delta L(f - f_{\text{max}})|^2 \text{ from data spectrum} \]

\[ R(t) = \text{residual spectrum} \]

\[ \text{Are residuals consistent with noise?} \]

\[ j \leq 1 \]
II. Data Adaptive Spectral Analysis: The effective window function depends on the data and can vary with frequency.

Spectral estimators exist which have effective spectral windows that adapt to the properties of the signal (e.g. sampling, signal-to-noise, type of signal...); the effective window is produced integrally. As such, the window may be optimized with respect to the data, but the analyst to some extent has lost control over it and its properties (e.g. the effective number of degrees of freedom).

Data adaptive spectral estimators are generally nonlinear in the correlation function estimates. We will consider the following data adaptive estimators:

1. Maximum entropy method.
2. “Maximum likelihood” (in quotes because it does not result from maximizing a likelihood function, but rather is closely related to calculating an ML estimate of the power at the output of a filter).

Better name: High resolution method
Spectral Estimates and Correlation Functions

- We will see that three of the spectral estimators can be expressed in terms of the covariance matrix.
  - The covariance matrix is closely related to the autocorrelation function.
  - The three expressions are mathematically very different.
  - How can this be, given that the Wiener-Khinchine theorem applies?