Lecture 7

Applications:
- Fresnel-Fraunhofer diffraction simulations = convolution problem

Notes on web site:
- short, on particular topics mentioned in lecture
- look under lectures 6 and 7
- more will be added as we go on

Assignment 2
Utility of Structure Functions in Wave Propagation

Modeling and simulations of propagation through random phase screens are commonly done for studies of propagation through the atmosphere, the ionosphere, the interplanetary medium, and the interstellar medium. Though these media are quite differently physically, the underlying mathematics of wave propagation is the same.

Consider the simple case of a plane wave propagating through a thin screen that changes the electromagnetic (EM) phase randomly:
Thin Screen Scattering and Diffraction Pattern

Pulsar

Phase screen

Diffraction pattern
LOCATING PULSAR EMISSION REGIONS USING INTERSTELLAR SCINTILLATION

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ABSTRACT

Radio emission regions in pulsar magnetospheres appear to reside in the relativistic flow above the magnetic polar cap at altitudes of 1 to 100 stellar radii. However, published constraints on the pulse-phase resolved locations and sizes of emission regions are few in number and inconsistent. We present several methods that use interstellar scattering-induced structure in dynamic spectra to map emission regions as a function of spin phase and apply them to observations of two pulsars, PSRs B0834+06 and B1133+16. We use a two-dimensional correlation analysis of dynamic spectra to identify spatial offsets between emission regions, and parabolic arcs in the secondary spectra (the power spectra of the dynamic spectra) to place further limits on the shapes and locations of emission regions. For PSR B1133+16, the resolving power is low and no differential effects are seen across pulse phase, placing both upper bounds on emission region sizes and also on instrumental effects that could decorrelate dynamic spectra. For PSR B0834+06, the resolving power is much higher. Upper bounds on the shift of the cross-correlation function from zero time delay limit any spatial offset parallel to the pulsar’s space velocity. We identify a decorrelation of the phase-resolved dynamic spectrum that can be attributed to the extension of the emission region in the direction transverse to the pulsar’s space velocity. Structure in the secondary spectrum also supports this interpretation. Relative to the pulsar, the emission regions are extended in spin longitude but unresolved in latitude. If the magnetic field is dipolar, we estimate the emission altitude to be $\sim 15\%$ of the light cylinder radius.

Subject headings: stars: neutron, pulsars: general, pulsars: specific (PSR B0834+06, PSR 1133+16), ISM: structure, techniques: spectroscopic

![Diffraction pattern sampled in time and $\rightarrow$ 2D Spectrum frequency](image)

Figure 1. Emission and Scattering Geometries. Panel a shows a schematic pulse profile, labeling fiducial components 1 and 2, separated by time $w$ and phase $\Delta \theta = 2 \pi w / P$, where $P$ is the pulse period. Panels b and c show the magnetosphere geometry when components 1 and 2, respectively, are beamed in the direction of the observer. The dashed lines show the magnetic field lines along which components 1 and 2 are beamed. If the magnetic field is dipolar, the emitting regions are offset by a separation $\Delta s = \frac{r_{\text{cm}} \Delta \theta}{3}$, where $r_{\text{cm}}$ is the emission altitude, as the components are beamed along field lines with different curvature. Panel d shows the scattering geometry between two points in a scattering screen. The pulsar is at transverse position $r_1 + v_\perp t$, the scattering points $r_1$ and $r_2$ are co-moving at velocity $v_\perp$, and the observer is at position $r_0$, moving at velocity $v_o$. Radio waves from the two points in the screen scatter toward the observer and interfere at the observer’s position. The observed intensity is determined by the phase difference between the two paths, which depends on observing frequency and the path length difference. The path length difference changes with time, thus intensity modulations are observed in both time and frequency. If the emission regions for different pulse components are spatially offset, the relative phases difference for the pulse components will be different, and both will show intensity modulations. The observed structures in the dynamic spectrum (DS, see main text) are caused by scattering through more complicated screen geometries that follow the same principle.
Kolmogorov Turbulence

- Kolmogorov derived the spatial spectrum of velocity fluctuations for an incompressible fluid that is stirred at large scales. Kinetic energy cascades from large to small scales and is dissipated by collisions on small scales.

- Compressible fluids can show a similar or identical spectrum because the velocity fluctuations can drive pressure variations.

- Magnetized plasmas, including the ionosphere, the interplanetary medium (solar wind), the interstellar medium show Kolmogorov-like turbulence in a number of tracers. Presumably the intergalactic medium, which is almost completely ionized, does too.

- Measurements show that turbulence is three dimensional in some cases and two dimensional in cases where an ambient magnetic field strongly influences particle motions.

An example wavenumber spectrum for the gas density is (for isotropic turbulence)

$$P_n(q) = C_n^2 q^{-11/3} \quad \text{for} \quad 2\pi/\ell_o \leq q \leq 2\pi/\ell_i,$$

where

$$\ell_i = \text{inner scale, i.e. the smallest scale}$$

$$\ell_o = \text{outer scale, i.e. the largest scale}$$

More generally we would have a spectrum that depends on the vector wavenumber $q$ and the correlation function of the density fluctuation $\delta n(x)$ would be

$$\langle \delta n(x) \delta n(x + \delta x) \rangle = \int dq \ P_n(q) e^{iq\cdot\delta x}. $$
Examples of Kolmogorov-like turbulence

• Motions in interstellar gas clouds (velocity field; spectroscopy)
  • Inner scale = dissipation length
  • Outer scale = cloud size (sub-pc to 10s of pc)

• Density fluctuations in the interstellar medium (probed with pulsar measurements and VLBI of active galactic nuclei)
  • Inner scale = dissipation length (ion gyro radius in ISM)
  Evidence for 1000 km sized turbules
  • Outer scale = sizes of HII regions and large scale structures in the Galaxy (up to pc \(\rightarrow\) kpc)
Power Law Distributions

• Common in nature
  – Earthquake amplitudes
  – Galaxy correlation function
  – Burst amplitude distribution from the Crab pulsar
  – Etc.

• Necessarily have cutoffs in order to avoid divergences

• Play a key role in modeling of astrophysical processes

• We will consider the Pareto, Levy distributions when talking later about sampling parameter spaces
Fresnel Diffraction

Huygens’ principle says that each point within an aperture $A(x)$ illuminated by a plane wave radiates spherical waves. At a position $x$ the scalar electric field is given by the Kirchoff Diffraction Integral (KDI)

$$\varepsilon(x, \lambda) = \frac{e^{ikD}}{\lambda D} \int dx' A(x') e^{i\Psi(x-x')}$$

$$\Psi(x, x) = \frac{k}{2D}|x - x'|^2$$

$$k = \frac{2\pi}{\lambda}$$

$$\frac{k}{D} = \frac{2\pi}{\lambda D} = \frac{1}{\frac{\tau^2_F}{r_F^2}} = \frac{1}{\text{(Fresnel scale)}^2}$$

Can be recast as a convolution problem:

$$\varepsilon(x, \lambda) \propto A(x) \ast K(x)$$

$$K(x) = e^{\frac{1}{2} \left( \frac{x}{\tau_F} \right)^2} = \text{Fresnel function}$$

- Locations near the aperture: Fresnel diffraction
- Far from aperture: Fraunhofer diffraction
- Transition distance = Fresnel distance = $D_F \frac{(\text{size of aperture})}{\lambda}$
Phase changing screens

\[ A(x) = e^{i \phi(x)} \]

The screen phase can be written as

\[ \phi(x) = k \int_{\Delta z} dz \, \delta n_r(x), \quad k \equiv 2\pi/\lambda \]

where \( \delta n_r(x) \) describes the variable part of the index of refraction.

The (scalar) electric field emerging from the screen is

\[ E(x, z) = e^{i[k z - \omega t + \phi(x)]} \]

The autocorrelation function of the field is (letting \( x_1 = x \) and \( x_2 = x + b \))

\[ R_E(x_1, x_2) = \langle E(x, z) E^*(x + b, z) \rangle \]

\[ = \langle e^{i[\phi(x) - \phi(x+b)]} \rangle \]

\[ = \langle e^{i[\Delta \phi(x,b)]} \rangle \]

The phase difference, \( \Delta \phi(x, b) \) is a random process. By inspection, you might describe \( R_E(x_1, x_2) \) as the characteristic function of the phase difference \( \Delta \phi \) evaluated at \( \omega = 1 \).

Now assume that \( \phi(x) \) is a Gaussian process with stationary statistics. Since it is a spatially varying quantity, it is said to have *homogeneous* statistics. This means that \( \Delta \phi \) is also a Gaussian process. Why?
A role for correlation functions, structure functions, and characteristic functions!

For a general Gaussian random variable $Y$ the characteristic function for zero mean is

$$\Phi_Y(\omega) = \langle e^{i\omega Y} \rangle = e^{-\frac{1}{2}\omega^2\sigma^2}.$$ 

We can apply this result to the correlation function defined above to get

$$R_E(x_1, x_2) \rightarrow R_E(b) = e^{-\frac{1}{2}\langle [\Delta \phi(b)]^2 \rangle}.$$ 

We can rewrite this in terms of the phase structure function defined as

$$D_\phi(b) \equiv \langle [\Delta \phi(b)]^2 \rangle$$

so that

$$R_E(b) = e^{-\frac{1}{2}D_\phi(b)}.$$
**Propagation to the observation plane:** Usually one wants to know what the wave field is in a plane downstream of the screen.

A fundamental result is that the field correlation function $R_E(b)$ propagates unaltered from the screen to the observer. This can be shown using the wave equation (see literature).

Secondly, a fundamental theorem relates $R_E(b)$ to the apparent image of the source of plane waves $I(\theta)$ by the *van Cittert-Zernike theorem*:

$$R_E(b) \iff \langle I(\theta) \rangle$$

Example: Suppose the phase structure function has a “square-law” form

$$D_\phi(b) = 2\sigma_\phi^2 \left( \frac{b}{b_1} \right)^2.$$ 

When $\sigma_\phi \gg 1$, the field correlation function is Gaussian in form and then so too is the image. The wave propagation describes *scattering* and this result tells us that the scattered image has a Gaussian shape. For the atmospheric case, the scattered image $I(\theta)$ is called the “seeing” disk.

Note that all of the above involves ensemble averages. Individual realizations of scattered sources show *speckles* that average into the Gaussian shape. All of the above is also one dimensional. It is straightforward to extend the results to two dimensional screens.
Linear Shift-Invariant Systems

A workhorse concept in engineering
Often applicable to physical systems
Output = filtered version of input
\[ \varepsilon_{\text{emitted}}(t) \rightarrow \boxed{g(t)} \rightarrow \varepsilon_{\text{measured}}(t) \]
“Filtering” means convolution:
\[ \varepsilon_{\text{measured}}(t) = \int dt' \varepsilon_{\text{emitted}}(t')g(t - t') \]
Which is often written as (* = convolution):
\[ \varepsilon_{\text{measured}}(t) = \varepsilon_{\text{emitted}}(t) * g(t) \]
g(t) is called the impulse response
Linear Shift-Invariant Systems II

The convolution

\[ \varepsilon_{\text{measured}}(t) = \varepsilon_{\text{emitted}}(t) \ast g(t) \]

is easily expressed in the frequency domain by taking Fourier transforms (denoted by \( \sim \)):

\[ \tilde{\varepsilon}_{\text{measured}}(\nu) = \tilde{\varepsilon}_{\text{emitted}}(\nu) \tilde{g}(\nu) \]

Often the FT of the filter is written as \( G(\nu) \) and is called the \textbf{transfer function} of the linear system.

A time-shifted and scaled input produces a proportionately scaled and shifted output.
Simulating power-law noise

• Applications: many phenomena in nature are processes with spectra that are power-law in form (temporally or spatially)

\[ S(f) \propto f^{-\alpha} \quad f_0 \leq f \leq f_1 \]

• Can use a linear filter with impulse response \( h(t) \):

\[
\begin{align*}
  x(t) & \rightarrow [h(t)] \rightarrow y(t) \\
  y(t) & = x(t) \ast h(t) \\
  \tilde{Y}(f) & = \tilde{X}(f) \times \tilde{H}(f) \\
  \langle |\tilde{Y}(f)|^2 \rangle & = \langle |\tilde{X}(f)|^2 \rangle |\tilde{H}(f)|^2
\end{align*}
\]

• Let \( x(t) = \) realization of white noise and \( H(f) = \sqrt{\text{shape}} \) of spectrum that is wanted
Procedure

• Generate spectrum in frequency domain
• Generate white noise realizations for real and imaginary parts of H(f)
  – random number generator: gaussian, uniform
• Multiply by $f^{-\alpha/2}$ for $f_0 \leq f \leq f_1$
• Fill vector to force Hermiticity
• Do inverse DFT to get real time-domain realization of the noise
• Can be applied to any shape of spectrum of course
• What will the statistics be in the time domain?
slop = 0

slop = 1
slope = 2

slope = 3
slope = 4

slope = 5
slope = 6
Simulating 2D Turbulence

Specify spectral index, inner and outer scales and define the corresponding wavenumbers:

\[ q_o, q_i = \text{wavenumber cutoffs (rolloffs)}, \beta = 11/3 = \text{spectral index}. \]

Generate real white noise in x domain:

\[ n(x) = \text{randn}(\text{nx, ny}) + 1.j \times \text{zeros((nx, ny))} \quad \text{nx, ny = array sizes} \]

2DFFT to wavenumber domain:

\[ N(q) = \text{fft2}(n) \]

Define a shaping function:

\[ S(q) = (q_o^2 + q_x^2 + q_y^2)^{-\beta/4} e^{-(q/q_i)^2} \]

Multiply in the q domain and inverse FFT back to x domain:

\[ \Phi_x(x) = \text{ifft2}(N(q) S(q)) \]

\[ \Phi_x \text{ will have a power spectrum proportional to } [S(q)]^2 \]
PSF for a medium with Kolmogorov turbulence

LSST = Large Synoptic Survey Telescope

Repetitive scanning of 70% of the sky over 5 years

Data mining = mission

LSST optical design. The telescope design is a modified Paul-Baker three mirror system, which produces a uniform image quality across a large field of view. The relative positions of the primary and the tertiary mirrors were adjusted during the design process so that their surfaces meet with no axial discontinuity at a cusp, allowing the Primary and the Tertiary to be fabricated from a single substrate. The 3.4-m convex secondary mirror has a 1.8 m inner opening, through which the LSST camera is inserted at the focal plane.
Newton 1717, *Optiks*:
“... the air through which we look upon the stars is in perpetual tremor; as may be seen by the tremulous motion of shadows cast from high towers, and by the twinkling of the fixed stars.... The only remedy is a most serene and quiet air, such as may perhaps be found on the tops of high mountains above grosser clouds.”
Implications for Weak Lensing Systematics:

The ellipticity of the point spread function (PSF) of any ground based telescope depends both on the properties of the atmosphere and the design and operation of the telescope and detector. Understanding the ellipticity of the PSF and its correlation across the field is critical to the success of weak lensing measurements. In particular, any residual uncorrectable ellipticity represents a floor that prevents the detailed measurements of arbitrarily low shear values.

Our simulations demonstrate that the ellipticity may receive similar contributions from the optics and the atmosphere (1 to 2% for each), which is similar to the shear from a massive foreground cluster of galaxies. The optics contribution to the ellipticity of the PSF, however, is highly correlated on several hundred arcsecond scales. This is due to the fact that the secondary and tertiary mirrors in the LSST optics chain are relatively close to the pupil plane. Photons emitted from all points in the field of view see a similar part of the surfaces of all the mirrors. Every perturbation, therefore, affects the PSF across the field of view in a similar way. While it is important to control the overall ellipticity induced by the optics in the design of LSST, it is anticipated from these simulations that the optics contribution the ellipticity of the PSF will be easily correctable, since it is highly correlated. Studies are continuing to identify other instrumental problems that could affect weak lensing measurements.

Figure 4 demonstrates the expected PSF function when we turn on various parts of the simulator. The upper left image shows the PSF due to the optics alone with the mirror perturbations. The second image shows the PSF after the detector simulator is included. The lower left image shows the PSF when the atmosphere with no wind is included. The lower right shows the same but with wind. Clearly, the effect of wind reduces the ellipticity due to a larger part of the atmospheric turbulence that is being averaged.

http://lsst.org/lsst/science/scientist_detf
Point Source

Spherically Radiating Light

Flat Wavefront

Long Propagation Distance
guide star

flat wavefront
corrugated wavefront
ground telescope
Six layers of Kolmogorov/von Karman phase screens used for the atmospheric turbulence model.

Lee et al.
Example of time evolution of simulated LSST PSF due to atmospheric turbulence at the zenith. The shown PSFs are for the monochromatic beam at $\lambda=6000$ Å.

We do not include either the optical aberration or the charge diffusion by CCDs in this simulation. As the exposure time increases, more speckles are stacked together, which makes the resulting PSF rounder and the surface brightness irregularities decrease. The FWHM at $t\sim15s$ is $\sim0.5$ arcsec.

Lee et al.
Simulated focal plane of LSST. We used the CCD assembly/fabrication specification in Table 1 to generate the LSST focal plane tiled by 189 CCDs.
Impact of focal plane CCD height variation on PSF. We use the ZEMAX software to obtain diffraction-limited PSFs in the absence of atmospheric turbulence. The encircled stick in the lower-left corner represents 10% ellipticity as illustrated. Note both the magnitude of ellipticity induced by the focal plane error (and aberration) and the abrupt changes across the CCD borders. The large circle shows the 3.5 m diameter field of view. The sticks outside this circle are for illustrative purpose only to demonstrate that the aberration degrades severely beyond the 1.75 degree boundary.
PCA eigenPSFs. The eigenPSFs were ranked by their eigenvalues with $n = 0$ representing the largest. The example shown here is derived by performing PCA on 4000 simulated PSFs. As $n$ increases, the eigenPSF tends to possess higher-frequency features. Interestingly, the first several eigenPSFs resemble the basis functions obtained by the shapelet formalism remarkably. However, asymmetry is also evident, which reveals the characteristic pattern of the data.

Simulated LSST image. We display only a $\sim 100\text{arcsec} \times 100\text{arcsec}$ cutout to show the detail. The left panel image is the result prior to noise addition, about 2 mag deeper than the median depth (6000 s, $\sim 27.5\text{at 5}\sigma$) shown in the middle panel. The nominal 15 second exposure case is shown in the right panel.
Ellipticity distribution in simulated LSST images. We display "whiskers" overlaid on the LSST focal plane that we use for the input PSF generation. Apart from the magnitude, the PSF pattern is similar to the one in the case when only optical aberration and focal plane height errors are considered (Figure-??). The reduction in the magnitude demonstrates that the atmospheric turbulence mostly circularizes the PSF rather than introduces additional anisotropy. Because we simulate the sky near the zenith, the ellipticity due to the atmospheric dispersion is very small in the current case. Note that the correlation of the ellipticity change with the focal plane height variation is still strong.