Linear, Shift-invariant Systems and Fourier Transforms

• Linear systems underly much of what happens in nature and are used in instrumentation to make measurements of various kinds.

• We will define linear systems formally and derive some properties.

• We will show that exponentials are natural basis functions for describing linear systems.

• Fourier transforms (CT/CA), Fourier Series (CT/CA + periodic in time), and Discrete Fourier Transforms (DT/CA + periodic in time and in frequency) will be defined.

• We will look at an application that demonstrates:
  1. Definition of a power spectrum from the DFT.
  2. Statistics of the power spectrum and how we generally can derive statistics for any estimator or test statistic.
  3. The notion of an ensemble or parent population from which a given set of measurements is drawn (a realization of the process).
  4. Investigate a “detection” problem (finding a weak signal in noise) and assess the false-alarm probability.
Types of Signals

By “signal” we simply mean a quantity that is a function of some independent variable. For simplicity, we will often consider a single independent variable (time) e.g. $x(t)$. Later we will consider 2 or more dimensions of general variables.

A signal is characterized by an amplitude as a function of time and 4 kinds of signals can be defined depending on whether the time and amplitude are discrete or continuous.

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<th>AMPLITUDE</th>
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<td>($m$ bits per sample)</td>
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<td>DT, CA</td>
<td>($\infty$ bits per sample)</td>
<td>Analog Signals ($\infty$ bits per sample)</td>
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Quantum mechanics says there are only DT, DA signals but much of what we will do is in the classical regime.
Examples

CT/CA  Light intensity from a star
       (ignore photons and counting statistics)

CT/DA  Earth’s human population

DT/CA  Intensity of the moon at times of the full moon
       \[|t_{j+1} - t_j| \sim 28 \text{ days}\]

DT/DA  Earth’s population at times of the full moon
Approach taken in the course

Theoretical treatments (analytical results) will generally be applied to DT/CA signals, for simplicity.

For the most part, we will consider analog signals and DT/CA signals, the latter as an approximation to digital signals. For most analyses, the discreteness in time is a strong influence on what we can infer from the data. Discreteness in amplitude is not so important, except insofar as it represents a source of error (quantization noise). However, we will consider the case of extreme quantization into one bit of information and derive estimators of the autocovariance.

Generically, we refer to a DT signal as a time series and the set of all possible analyses as “time series analysis”. However, most of what we do is applicable to any sequence of data, regardless of what the independent variable is.

Often, but not always, we can consider a DT signal to be a sampled version of a CT signal (counter examples: occurrence times of discrete events such as clock ticks, heartbeats, photon impacts, etc.).

Nonuniform sampling often occurs and has a major impact on the structure of an algorithm.

We will consider the effects of quantization in digital signals.
Linear Systems

Consider a linear differential equation in \( y \)

\[
f(y, y', y'', \ldots) = x(t), \quad y' \equiv \frac{dy}{dt}, \text{etc.}
\]

whose solutions include a complete set of orthogonal functions. We can represent the relationship of \( x(t) \) (the driving function) and \( y(t) \) (the output) in transformational form:

\[
x(t) \rightarrow \text{system} \quad h(t) \rightarrow y(t)
\]

where \( h(t) \) describes the action of the system on the input \( x \) to produce the output \( y \). We define \( h(t) \) to be the response of the system to a \( \delta \)-function input. Thus, \( h(t) \) is the “impulse response” or Green’s function of the system.

We wish to impose linearity and shift invariance on the systems we wish to consider:

**Linearity:**

If \( x_1 \rightarrow y_1 \) and \( x_2 \rightarrow y_2 \) then \( ax_2 + bx_2 \rightarrow ay_1 + by_2 \), for any \( a, b \)

E.g. \( y = x^2 \) is not a linear operation.

**Time or shift invariance** (stationarity)

If \( x(t) \rightarrow y(t) \), then \( x(t + t_0) \rightarrow y(t + t_0) \) for any \( t_0 \)

The output “shape” depends on the “shape” of the input, not on the time of occurrence.
Singularity Functions

We need some useful singularity “functions”:

1. \( \delta(t) \) defined as a functional

\[
z(t) \equiv \int_{t'} dt' \delta(t' - t) \, z(t') \quad \text{and} \quad \int_a^b dt' \delta(t' - t) = \begin{cases} 1 & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}
\]  

2. Loosely speaking, \( \delta(0) \to \infty, \delta(t \neq 0) \to 0 \); So \( \delta(t) \) has finite (unit) area.

3. \( U(t) \) unit step function (or Heaviside function)

\[
U(t) = \int_0^\infty dt' \delta(t' - t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{and} \quad \frac{dU(t)}{dt} = \delta(t)
\]

\[
\Rightarrow U(t - t_0) = \int_{t_0}^\infty dt' \delta(t' - t) = \begin{cases} 1 & t \geq t_0 \\ 0 & \text{otherwise} \end{cases}
\]
Convolutions theorem

By definition
\[ \delta(t) \longrightarrow h(t) \]

Using linearity we have
\[ a \delta(t) \longrightarrow a \ h(t) \]

Let \( a = x(t') \) then
\[ x(t') \delta(t) \longrightarrow x(t') \ h(t) \]

By shift invariance we have
\[ \delta(t - t') \longrightarrow h(t - t') \]

Combining L + SI,
\[ x(t') \delta(t - t') \longrightarrow x(t') \ h(t - t') \]

But, again by linearity, we can sum many terms of this kind. So, integrating over all \( t' \):
\[
\int_{-\infty}^{\infty} dt' \ x(t') \delta(t - t') \longrightarrow \int_{-\infty}^{\infty} dt' \ x(t') \ h(t - t')
\]

But by definition of \( \delta(t) \), LHS = \( x(t) \), so
\[ x(t) \longrightarrow \int_{-\infty}^{\infty} dt' \ x(t') \ h(t - t') = y(t) \]

By a change of variable on the RHS to \( \tilde{t} = t - t' \) we also have
\[ x(t) \longrightarrow \int_{-\infty}^{\infty} dt' \ x(t - t') \ h(t') = y(t) \]
Any linear, shift invariant system can be described as the convolution of its impulse response with an arbitrary input.

Using the notation $*$ to represent the integration, we therefore have

$$y(t) = x * h = h * x$$

Properties:

1. Convolution commutes:

$$\int dt' h(t') x(t - t') = \int dt' h(t - t') x(t')$$

2. Graphically, convolution is “invert, slide, and sum”

3. The general integral form of $*$ implies that, usually, information about the input is lost since $h(t)$ can “smear out” or otherwise preferentially weight portions of the input.

4. Theoretically, if the system response $h(t)$ is known, the output can be ‘deconvolved’ to obtain the input. But this is unsuccessful in many practical cases because: a) the system $h(t)$ is not known to arbitrary precision or, b) the output is not known to arbitrary precision.
Why are linear systems useful?

1. Filtering (real time, offline, analog, digital, causal, acausal)

2. Much *signal processing and data analysis* consists of the application of a linear operator (smoothing, running means, Fourier transforms, generalized channelization, ...)

3. Natural processes can often be described as linear systems:
   - Response of the Earth to an earthquake (propagation of seismic waves)
   - Response of an active galactic nucleus swallowing a star (models for quasar light curves)
   - Calculating the radiation pattern from an ensemble of particles
   - Propagation of electromagnetic pulses through plasmas
   - Radiation from gravitational wave sources (in weak-field regime)
We want to be able to attack the following kinds of problems:

1. *Algorithm development:* Given $h(t)$, how do we get $y(t)$ given $x(t)$ (“how” meaning to obtain efficiently, hardware vs. software, etc.) $t$ vs. $f$ domain?

2. *Estimation:* To achieve a certain kind of output, such as parameter estimates subject to “constraints” (e.g. minimum square error), how do we design $h(t)$? (least squares estimation, prediction, interpolation)

3. *Inverse Theory:* Given the output (e.g. a measured signal) and assumptions about the input, how well can we determine $h(t)$ (parameter estimation)? How well can we determine the original input $x(t)$? Usually the output is corrupted by noise, so we have

$$y(t) = h(t) \ast x(t) + \epsilon(t).$$

The extent to which we can determine $h$ and $x$ depends on the *signal-to-noise ratio*:

$$\frac{\langle (h \ast x)^2 \rangle^{1/2}}{\langle \epsilon^2 \rangle^{1/2}}$$

where $\langle \rangle$ denotes averaging brackets.

We also need to consider *deterministic, chaotic* and *stochastic* systems:

- Deterministic $\Rightarrow$ predictable, precise (noiseless) functions
- Chaotic $\Rightarrow$ deterministic but apparently stochastic processes
- Stochastic $\Rightarrow$ not predictable (random)
- Can have systems with stochastic input and/or stochastic system response $h(t) \longrightarrow$ stochastic output.

Not all processes arise from linear systems but linear concepts can still be applied, along with others.
In analyzing LTI systems we will find certain basis functions, **exponentials**, to be specially useful. Why is this so?

Again consider an LTI system $y = h \ast x$. Are there input functions that are unaltered by the system, apart from a multiplicative constant? Yes, these correspond to the **eigenfunctions** of the associated differential equation.

We want those functions $\phi(t)$ for which

$$ y(t) = \phi \ast h = H\phi \quad \text{where } H \text{ is just a number} $$

That is, we want

$$ y(t) = \int dt' h(t') \phi(t - t') = H \phi(t) $$

This can be true if $\phi(t - t')$ is **factorable**:

$$ \phi(t - t') = \phi(t) \psi(t') $$

where $\psi(t')$ is a constant in $t$ but can depend on $t'$. 

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We constrain $\psi(t')$ with:

$$
\begin{align*}
    i) \quad & \phi(t - t')_{t'=0} \equiv \phi(t) = \phi(t)\psi(0) \Rightarrow \psi(0) = 1 \\
    ii) \quad & \phi(t - t')_{t=t'} \equiv \phi(0) = \phi(t)\psi(t) \Rightarrow \psi(t) = \frac{\phi(0)}{\phi(t)} \\
    iii) \quad & \phi(t - t')_{t=0} \equiv \phi(-t') = \phi(0)\psi(t') \Rightarrow \psi(t') = \frac{\phi(-t')}{\phi(0)}
\end{align*}
$$

Now ii) and iii) automatically satisfy i). With no loss of generality we can set

$$
\phi(0) = 1 \Rightarrow \psi(t) = \frac{1}{\phi(t)} = \phi(-t)
$$

We want functions whose time reverses are their reciprocals. These are exponentials (or $2^st$, $a^st$, etc):

$$
\phi(t) = e^{st}
$$
Check that $e^{st}$ behaves as required:

\[ y = \phi \ast h = \int dt' \phi(t - t')h(t') \]
\[ = \int dt' e^{s(t-t')}h(t') \]
\[ = e^{st} \int dt' e^{-st'}h(t') \]
\[ = e^{st} H(s) \]

So $\phi \rightarrow \phi H(s)$

$\phi = \text{eigenvector} \quad H = \text{eigenvalue}$

Note $H(s)$ depends on $s$ and $h.$
Two kinds of systems

Causal

\[ h(t) = 0 \text{ for } t < 0 \]

output now depends only on past values of input

\[ H(s) = \int_0^\infty dt' e^{-st'} h(t') \quad \text{Laplace transform} \]

Acausal

\[ h(t) \text{ not necessarily } 0 \text{ for } t < 0 \]

\[ H(s) = \int_{-\infty}^{\infty} dt' e^{-st'} h(t') \mid_{s=i\omega} \quad \text{Fourier transform} \]

Exponentials are useful for describing the action of a linear system because they “slide through” the system. If we can describe the actual input function in terms of exponential functions, then determining the resultant output becomes trivial. This is, of course, the essence of Fourier transform treatments of linear systems and their underlying differential equations.
Convolution Theorem in the Transform Domain

Consider input $\rightarrow$ output

$$a e^{i\omega t} \rightarrow a H(i\omega) e^{i\omega t} \quad \text{linearity}$$

We can choose an arbitrary $a$, so let’s use

$$\tilde{X}(\omega) e^{i\omega t} \rightarrow \tilde{X}(\omega) H(i\omega) e^{i\omega t} \quad (4)$$

By linearity we can superpose these inputs. So integrate over $\omega$ with a judicious choice of normalization $(1/2\pi)$:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{X}(\omega) e^{i\omega t} \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{X}(\omega) H(i\omega) e^{i\omega t}$$

Let’s call $LHS \ x(t)$ and the RHS $y(t)$:

$$x(t) = \frac{1}{2\pi} \int d\omega \tilde{X}(\omega) e^{i\omega t} \quad y(t) = \frac{1}{2\pi} \int d\omega \tilde{X}(\omega) H(i\omega) e^{i\omega t}$$

What is the relationship of $\tilde{X}(\omega)$ to $x(t)$?

Multiply $x(t)$ by $e^{-i\omega' t}$ and integrate to get

$$\int_{-\infty}^{\infty} dt \ x(t) \ e^{-i\omega' t} = \frac{1}{2\pi} \int dw \tilde{X}(\omega) \int_{-\infty}^{\infty} dt \ e^{i(\omega-\omega')t} \quad (5)$$
Now the integral over $t$ on the RHS gives

$$\int_{-\infty}^{\infty} dt \, e^{i(\omega-\omega')t} \rightarrow \begin{cases} 0 & \omega \neq \omega' \\ \infty & \omega = \omega' \end{cases}$$  \hspace{1cm} (6)$$

i.e. just like a delta function. So (invoking the correct weighting factor, or normalization)

$$\int_{-\infty}^{\infty} dt \, e^{i(\omega-\omega')t} = 2\pi \delta(\omega - \omega')$$  \hspace{1cm} (7)$$

Therefore the boxed RHS becomes

$$\int dw \, \tilde{X}(\omega) \, \delta(\omega - \omega') = \tilde{X}(\omega').$$  \hspace{1cm} (8)$$

Therefore we have

$$\tilde{X}(\omega') = \int_{-\infty}^{\infty} dt \, x(t) \, e^{-i\omega't}.$$

and the inverse relation

$$x(t) = \frac{1}{2\pi} \int dw \, \tilde{X}(\omega) \, e^{-i\omega t}.$$

We say that $x(t)$ and $\tilde{X}(\omega)$ are a Fourier transform pair.

Going back to equation 4 it is clear that the FT of $y(t)$ is the integrand on the RHS so

$$\tilde{Y}(\omega) = \tilde{X}(\omega) \, H(i\omega).$$

Usually we rewrite this as $\tilde{H}(\omega) \equiv H(i\omega)$ so

$$\tilde{Y}(\omega) = \tilde{X}(\omega) \, \tilde{H}(\omega)$$
Therefore, we have shown that

\[ y(t) = x(t) * h(t) \quad \text{convolution} \]

\[ \tilde{Y}(\omega) = \tilde{X}(\omega) \tilde{H}(\omega) \quad \text{multiplication} \]

This product relation is extremely useful for

1. Deriving impulse responses of composite systems.

2. In discrete form (i.e. digitally) for implementing convolutions: \( \omega \)-domain multiplications can be much faster than \( t \)-domain convolutions
Fourier Transform Relations

Here we summarize the Fourier transform relations for a variety of signals. Let $f(t)$ be a continuous, aperiodic function and $\tilde{F}(f)$ be its Fourier transform. We denote their relations

$$f(t) = \int_{-\infty}^{\infty} df \tilde{F}(f)e^{+2\pi if t}$$

$$\tilde{F}(f) = \int_{-\infty}^{\infty} dt f(t)e^{-2\pi if t},$$

as $f(t) \iff \tilde{F}(f)$.

We need to consider the following functions:

1. The Dirac delta ‘function’

$$\delta(t)$$

2. A periodic train of delta functions (‘bed of nails’) with period $\Delta$:

$$s(t, \Delta) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$

3. The periodic extension $f_p(t)$ of a function $f(t)$ defined using the bed of nails function:

$$f_p(t) = f(t) \ast s(t, \Delta) \quad \ast \text{denotes convolution}$$

4. An aperiodic function $f(t)$ sampled at intervals $\Delta t$:

$$f_s(t) = f(t) \times s(t, \Delta t)$$

5. The sampled and periodically extended signal:

$$f_{ps}(t) = f_p(t) \times s(t, \Delta t)$$
### 1D Fourier Transform Theorems

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<th>Fourier transform</th>
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<tr>
<td>$\delta(t)$</td>
<td>$\delta(t)$</td>
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<tr>
<td>$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$</td>
<td>$\tilde{S}(f) = \Delta^{-1} \sum_{-\infty}^{\infty} \delta(f - k/\Delta)$ Bed of nails function</td>
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<tr>
<td>$y(t) = x(t) * h(t)$</td>
<td>$\tilde{X}(f) \tilde{H}(f)$ Convolution</td>
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<td>$C_x(\tau) \equiv \int dt x^*(t)x(t+\tau)$</td>
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<td>$x(t - t_0)$</td>
<td>$e^{-i\omega_0 t} \tilde{X}(f)$ Shift theorem</td>
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<td>$e^{+i2\pi f_0 t} x(t)$</td>
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<td>$\frac{dx}{dt}$</td>
<td>$(2\pi f)^{-1} \tilde{X}(f)$ Integration theorem</td>
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<td>$2\pi i f \tilde{X}(f)$ Derivative theorem</td>
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<tr>
<td>$x(t) = \sum_m x_m \frac{\sin 2\pi f(t - m\delta t)}{2\pi \Delta f(t - m\delta t)}$</td>
<td>$\sum_m x_m e^{-2\pi im f \delta t} \Pi \left( \frac{f}{2\Delta f} \right)$ \text{Bandlimited } \Delta f = \text{half BW.}</td>
</tr>
<tr>
<td>$x_p(t) = x(t) * s(t)$</td>
<td>$\tilde{X}(f) \tilde{S}(f)$ Periodic in time</td>
</tr>
<tr>
<td>$x_p(t) = \sum_k a_k e^{2\pi ik t/\Delta}$</td>
<td>$\Delta^{-1} \sum_k \tilde{X}(k/\Delta) \delta(f - k/\Delta)$ Fourier series</td>
</tr>
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where $a_k \equiv \Delta^{-1} \tilde{X}(k/\Delta)$
Points

1. You can bootstrap from a few basic FT pairs by using the FT theorems.

2. Narrow functions in one domain are wide in another (Uncertainty Principle, related to the scaling theorem).

3. Functions with sharp edges in one domain are oscillatory in the other (Gibbs phenomenon).

4. **Derivative theorem:**

   \[
   f(t) \iff \tilde{F}(f) \\
   \frac{df}{dt} \iff 2\pi i f \tilde{F}(f). 
   \]  

   \hspace{1cm} \text{(10)}

5. **Integration theorem:**

   \[
   f(t) \iff \tilde{F}(f) \\
   \int_t dt' f(t') \iff (2\pi i f)^{-1} \tilde{F}(f). 
   \]  

   \hspace{1cm} \text{(11)}

6. Consider a noisy signal, like white noise (which has a constant average FT but a realization of white noise is noisy in both domains). Differentiation of the noise increases the high-frequency components and thus increases the noise relative to any signal.

7. Integration of the noise reduces the high frequency components. “Smoothing” (low-pass filtering) of data is closely related to integration and in fact reduces high-frequency components.
Gaussian Functions

Why useful and extraordinary?

1. We have the fundamental FT pair:

\[ e^{-\pi t^2} \iff e^{-\pi f^2} \]

This can be obtained using the FT definition and by completing the square. Once you know this FT pair, many situations can be analyzed without doing a single integral.

2. The Gaussian is one of the few functions whose shape is the same in both domains.

3. The width in the time domain (FWHM = full width at half maximum) is

\[ \Delta t = \frac{2\sqrt{\ln 2}}{\sqrt{\pi}} = 0.94 \]

4. The width in the frequency domain \( \Delta \nu \) is the same.

5. Then

\[ \Delta t \Delta \nu = \frac{4\ln 2}{\pi} = 0.88 \sim 1. \]

6. Now consider a scaled version of the Gaussian function: Let \( t \to t/T \). The scaling theorem then says that

\[ e^{-\pi (t/T)^2} \iff T e^{-\pi (fT)^2} . \]

The time-bandwidth product is the same as before since the scale factor \( T \) cancels. After all, \( \Delta t \Delta \nu \) is dimensionless!
7. The Gaussian function has the smallest time-bandwidth product (minimum uncertainty wave packet in QM)

8. Central Limit Theorem: A quantity that is the sum of a large number of statistically independent quantities has a probability density function (PDF) that is a Gaussian function. We will state this theorem more precisely when we consider probability definitions.

9. Information: The Gaussian function, as a PDF, has maximum entropy compared to any other PDF. This plays a role in development of so-called maximum entropy estimators.
Chirped Signals

Consider the chirped signal \( e^{i\omega t} \) with \( \omega = \omega_0 + \alpha t \), (a linear sweep in frequency). We write the signal as:

\[
v(t) = e^{i\omega t} = e^{i(\omega_0 t + \alpha t^2)}.\]

The name derives from the sound that a swept audio signal would make.

1. Usage or occurrence:
   (a) wave propagation through dispersive media
   (b) objects that spiral in to an orbital companion, producing chirped gravitational waves
   (c) swept frequency spectrometers, radar systems
   (d) dedispersion applications (pulsar science)

2. We can use the convolution theorem to write

\[
\tilde{V}(f) = \text{FT} \left\{ e^{i(\omega_0 t + \alpha t^2)} \right\}
\]

\[
= \text{FT} \left\{ e^{i\omega_0 t} \right\} \ast \text{FT} \left\{ e^{i\alpha t^2} \right\}
\]

\[
= \delta(f - f_0) \ast \text{FT} \left\{ e^{i\alpha t^2} \right\}.
\]

3. The FT pair for a Gaussian function would suggest that the following is true:

\[
e^{-i\pi t^2} \iff e^{-i\pi f^2}.
\]

4. Demonstrate that this is true!

5. Within constants and scale factor, the FT of the chirped signal is therefore

\[
\tilde{V}(f) \propto e^{i(\pi(f-f_0)^2)}
\]
Three Classes of Fourier Transform

**Fourier Transform** (FT): applies to continuous, aperiodic functions:

\[
    f(t) = \int_{-\infty}^{\infty} df \ e^{2\pi i ft} \tilde{F}(f)
\]

\[
    \tilde{F}(f) = \int_{-\infty}^{\infty} dt \ e^{-2\pi i ft} f(t)
\]

Basis functions \( e^{2\pi i ft} \) are orthonormal on \([-\infty, \infty]\)

\[
    \int_{-\infty}^{\infty} dt \ e^{2\pi i ft} e^{-2\pi i ft} = \delta(t)
\]

**Fourier Series**: applies to continuous, periodic functions with period \( P \):

\[
    f(t) = \sum_{n=0}^{\infty} e^{2\pi i (n/P)t} \tilde{F}_n
\]

\[
    \tilde{F}_n = \frac{1}{P} \int_{0}^{P} dt \ e^{-2\pi i (n/P)t} f(t)
\]

\( f(t) \) periodic with period \( P \), orthonormal on \([0, P]\)

\[
    \int_{0}^{P} dt \ e^{2\pi i (n/P)t} e^{-2\pi i (n'/P)t} = \delta_{n,n'}
\]
Discrete Fourier Transform (DFT): applies to discrete time and discrete frequency functions:

\[ f_k = \sum_{n=0}^{\infty} e^{2\pi i nk/N} \tilde{F}_n \]

\[ \tilde{F}_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i nk/N} f_k \]

\[ f_k, \tilde{F}_n \text{ periodic with period } N, \quad \text{orthonormal on } [0, N] \]

\[ \sum_{n=0}^{N-1} e^{2\pi i nk/N} e^{-2\pi i k'} = \delta_{k,k'} \]

The Fourier transform is the most general because the other two can be derived from it. The DFT is not “just” a sampled version of the FT. Nontrivial consequences take place upon digitization, as we shall see.