Structure Functions and Allan Variance

Recall (for the WSS case)

\[ R_x(\tau) = \langle x(t)x(t + \tau) \rangle \]  \text{autocorrelation}

\[ C_x(\tau) = \langle [x(t) - \langle x(t) \rangle][x(t + \tau) - \langle x(t + \tau) \rangle] \rangle \]  \text{autocovariance}

which we have said are useful in time series analysis in several ways.

But there are processes for which \( R_x(\tau) \) is not a function only of lag (i.e. non-WSS) and there are data sets for which the data interval \([0, T]\) does not satisfy \( W_x \ll T \) where \( W_x = \text{correlation time of the process (if it exists)} \).

E.g.

1. random walks
2. \( 1/f \) noise and other red-noise processes with power law spectra \( \propto f^{-\alpha} \).

In these cases the sample mean \( \bar{X}(t) \) may not be estimatable (it will vary wildly over a realization and between realizations) and the nonstationarity needs to be contended with in other ways (whether the process is ‘signal’ or ‘noise’).
Sometimes the cure is the **structure function**. Define the *first increment*

\[ \Delta X(t, \tau) = x(t) - x(t + \tau). \]

Being a difference, this quantity acts in some ways as a *derivative*. The second moment of \( \Delta X(t, \tau) \) is the first order structure function:

\[ D_x(\tau) \equiv \langle [x(t) - x(t + \tau)]^2 \rangle \]

Advantages:

1) No estimate of the sample mean is needed

2) Sometimes the SF is a function of only \( \tau \) when \( R_x \) is not (stationary increments). A first-order random walk can be generated by a running integral of white noise. The random walk is nonstationary but the white noise is not (by assumption). Therefore the structure function will depend only on the lag \( \tau \) and not on the absolute time.

For a WSS process this becomes

\[
D_x(\tau) = \langle x^2(t) \rangle + \langle x^2(t + \tau) \rangle - 2 \langle x(t)x(t + \tau) \rangle \]

\[
2\sigma_x^2 \equiv 2R_x(0) \quad R_x(\tau)
\]

\[
D_x(\tau) = 2[R_x(0) - R_x(\tau)]
\]

which clearly yields nothing more than the ACF does.
Structure Function for a Random Walk

Consider a random walk \( [= \text{shot noise with } h(t) = U(t)] \):

\[
S(t) = \sum_i a_i h(t - t_i)
\]

where \( h(t) = U(t) = \text{unit step function} \) and the \( t_i \) are Poisson distributed. One can show that if \( \langle a_i \rangle = 0 \) and \( t_1 < t_2 \) then

\[
R_s(t_1, t_2) = \lambda \langle a^2 \rangle \int_0^\infty d\alpha U(t_1 - \alpha)U(t_2 - \alpha)
\]

\[
= \lambda \langle a^2 \rangle \int_0^{t_1} d\alpha = \lambda \langle a^2 \rangle t_1
\]

So generally

\[
R_s(t_1, t_2) = \lambda \langle a^2 \rangle t_\prec, \quad t_\prec = \min(t_1, t_2)
\]

Now compute the structure function:

\[
D_x(\tau) = \langle [x(t) - x(t + \tau)]^2 \rangle
\]

\[
= \langle x^2(t) \rangle + \langle x^2(t + \tau) \rangle - 2\langle x(t)x(t + \tau) \rangle
\]

\[
= \langle a^2 \rangle [\lambda t + \lambda(t + \tau) - 2\lambda t]
\]

\[
= \lambda \langle a^2 \rangle t \quad \text{for } \tau > 0
\]

\[
D_x(\tau) = \lambda \langle a^2 \rangle \tau \quad \Longrightarrow \text{Dependence only on } \tau; \text{ the RW has \textbf{stationary increments}.}
\]
**Allan variance**

Structure functions are most useful in analyses of nonstationary random processes. Why? Because they can remove trends in the data before calculating a second moment.

SFs provide standardized statistics for phase and frequency instabilities in precision frequency sources (flicker noise = $1/f$ noise).

Let $\nu(t)$ be the time-dependent frequency of a clock or synthesizer and define the normalized frequency,

$$y(t) = \frac{\nu(t)}{\langle \nu \rangle}$$

Define a running average frequency as

$$\bar{y}(t) = \frac{1}{\tau} \int_t^{t+\tau} dt \ y(t)$$

Then the Allan variance is

$$\sigma_y^2(\tau) = \frac{1}{2} \langle [\bar{y}(t+\tau) - \bar{y}(t)]^2 \rangle$$

This proportional to the first order structure function of $y$:

$$\sigma_y^2(\tau) \equiv \frac{1}{2} D_y(\tau)$$
Figure 1: Plot of the square root of the Allan variance for a few frequency standards (left) and contributions to the Allan variance in a typical frequency standard (right).
Higher order structure functions

$D_x(\tau)$ removes any mean component and thus acts like a 1st derivative or difference:

$$\Delta_x^{(1)}(t, \tau) = x(t + \tau) - x(t).$$

The second order increment is

$$\Delta_x^{(2)}(t, \tau) = x(t + 2\tau) - 2x(t + \tau) + x(t)$$

whose second moment is the second order structure function,

$$D_x^{(2)}(\tau) \equiv \langle [\Delta_x^{(2)}(t, \tau)]^2 \rangle.$$

It removes a ramp function in a time series just as the first order structure function removes the mean.
Continuing, the mth increment of the phase (following Rutman 1978)

\[ \Delta^{(m)}_x(t, \tau) \equiv \sum_{\ell=0}^{m} (-1)^{\ell} \binom{m}{\ell} x\left[t + (m - \ell)\tau\right] \]

has variance

\[ D^{(m)}_x(t, \tau) \equiv \langle [\Delta^{(m)}_x(t, \tau)]^2 \rangle \]

that is the m\textsuperscript{th} order structure function.

The m\textsuperscript{th} increment is useful for identifying a deterministic \( t^m \) power-law term in a time series and for identifying step functions in the (m-1)\textsuperscript{th} derivative: \( \Delta^{(m)}_x = 0 \) if \( x \) is a polynomial of order \( p < m \) and \( \Delta^{(m)}_x \) is independent of \( t \) for \( p = m \). Step functions in the k\textsuperscript{th} derivative of \( x \) have increments \( \Delta^{(k+1)}_x(t, \tau) \) that are pulses in time \( t \) comprising piecewise polynomials (of order \( k \)) with an amplitude \( \propto a_k \tau^{k-1} \) (where \( a_k \) is the amplitude of the step function). For example, a step function in \( \dot{x} \) at \( t = -\tau \) yields a triangular pulse \( \Delta^{(2)}_x = \Delta \dot{x} \tau (1 - |t/\tau|) \) for \( |t| \leq \tau \). Correspondingly, the m\textsuperscript{th} order structure function may be time invariant whereas \( x \) itself may be nonstationary.