1. This problem is to help develop your intuition on the relationship of discrete sampling in one domain and periodicity in the other. Consider a CT function that is periodic with period $P$.

$$f(t) = \sum_j g(t - jP),$$

where $g(t)$ is an arbitrary pulse shape but could be a gaussian function, for example. The FT of $g(t)$ is $\tilde{G}(f)$. In each of the steps below, sketch the results in both domains ($t$ and $f$), assuming a gaussian function. Label the results appropriately to indicate separations, widths, etc.

(a) Show that the FT of $f(t)$ can be written in the form

$$\tilde{F}(f) = \tilde{G}(f) \sum_m \delta(f - m/P)$$

where $m$ is an integer. You may think of FT as being a sampled version of the FT of the pulse shape, $\tilde{G}(f)$ with samples in frequency spaced by $1/P$.

(b) Now sample the time-domain function by multiplying by the “bed of nails” function with samples spaced by the interval $\Delta t$, where $k$ is an integer:

$$f_s(t) = f(t) \sum_k \delta(t - k\Delta t)$$

Show that the FT of this periodic, DT function is periodic in frequency with spacing $\ell/\Delta t$, where $\ell$ is an integer.

(c) Interpret your results.

2. In Lecture 3 we considered a sampled complex sinusoid with noise added. Here you will analyze just the sinusoidal part of the signal.

(a) Show that the $N$-point DFT of a complex, DT sinusoid $x_n = A e^{i\omega_0 n\delta t}$ is

$$\tilde{X}_k = N^{-1} \sum_{n=0}^{N-1} x_n e^{-2\pi i nk/N} = A N^{-1} e^{i\phi_n} \frac{\sin \frac{N}{2} (\omega_0\delta t - 2\pi k/N)}{\sin \frac{\pi}{2} (\omega_0\delta t - 2\pi k/N)}. \quad (1)$$

Hint: consider a finite sum geometric series.

(b) The integer index $k$ implies that the DFT is sampled at a discrete set of frequencies; however, the true signal frequency, $f_0 = \omega_0/2\pi$, generally will not coincide with the nearest discrete frequency. By considering values of $k$ that yield frequencies nearest $f_0$, investigate the dependence of the maximum of the DFT on $f_0$ to find the range of amplitudes that
you will see as a function of \( f_0 \). You will thus quantify the so-called “scalloping effect”. Hint: plot the magnitude of \( \hat{X}_k \) vs. \( f_0 \) for the case where \( f_0 \) equals one of the discrete frequencies. Then plot the magnitude when \( f_0 \) falls exactly in between two of the discrete frequencies.

3. This problem explores the effects of aliasing that arise in numerical integration. Estimate the value of the integral

\[
I = \int_{-\infty}^{\infty} dx \, e^{-x^2} \cos 2\pi x
\]

by using a summation

\[
\hat{I} = \sum_{k=-\infty}^{\infty} e^{-x^2_k} \cos 2\pi x_k
\]

where \( x_k = k\Delta x \) and the “hat” denotes ”estimate”.

i. Without explicitly doing any integral (i.e. by using Fourier transform theorems), determine the value of \( I \). Hint: multiply the integrand \( e^{-x^2_k} \cos 2\pi x_k \) by \( e^{2\pi i ft} \) and calculate the FT, then set \( f = 0 \) to get the value of the original integral.

ii. Estimate the value of \( I \) using the summation, \( \hat{I} \), by considering the Fourier Transform of the integrand, by using the convolution theorem, and by considering aliasing. Consider \( \Delta x = 0.5 \) in the summation for \( \hat{I} \). Sketch the FT of the integrand. By what percentage will the estimate \( \hat{I} \) for \( I \) be in error? You can obtain this answer without explicitly doing any integrals and certainly without writing code!

iii. Same as ii except for \( \Delta x = 0.4 \).

iv. To see the effects of aliasing further, you could plot the value of \( \hat{I} \) vs. \( \Delta x \).

v. Explain your results.

4. Numerically investigate a sinusoid + noise, \( x_n = A e^{i2\pi f_0 n \delta t} + n_n, n = 0, \ldots, N - 1 \), sampled at times \( t_n = n\Delta t \). For specificity, consider \( f_0 = 50.1367 \) Hz, \( \Delta t = 0.005 \) sec and a total number of samples, \( N = 1024 \). For the noise you can use for each of the real and imaginary parts of \( n_n \) the pseudo noise, \( \sqrt{6}(u_n - 1/2) \), where \( u_n \) is a random number between 0 and 1 generated with a standard random number generator; the normalization is such that \( n_n \) has zero mean and unit variance, \( \sigma_n^2 = \langle |n_n|^2 \rangle = 1 \). Investigate the spectral analysis of this signal by doing the following:

(a) Generate a realization of \( x_n \) for the choice \( (S/N)_t \equiv A/\sigma_n = 0.25 \).

(b) Calculate \( \hat{X}_k \), the N-point DFT of \( x_n \), using an FFT algorithm. Note that \( x_n \) is complex.

(c) Calculate the squared magnitude of the DFT, \( |\hat{X}_k|^2 \) as an estimator for the spectrum, \( S_k \).

(d) Consider the detection scheme consisting of finding the peak of the spectrum, \( S_k \), and identifying the k value and the amplitude. Calculate S/N for the peak as \( (S/N)_{DFT} = \)
\[ S_{k_{\text{max}}}/\sigma_S \], where \( \sigma_S \) is the rms noise of the spectrum, which you can compute from the spectrum while excluding the spectral line.

(e) Repeat steps (a)-(d) for a large number of realizations (10 or 100, say) and investigate the statistics of the S/N and of \( k_{\text{max}} \). How much do they vary? What are their means and standard deviations?

(f) Now develop a new detection scheme that takes into account that the true frequency of the sinusoid does not fall neatly upon one of the DFT’s frequency samples. For example, you might consider combining 2 or more samples that include the peak. Develop a scheme without customizing it to the particular frequency of this example. I.e. develop a scheme that could be applied if you do not know \( f_0 \) and you don’t know a priori whether the line is split between one or more bins of the DFT.

(g) Apply your scheme to a large number of realizations and evaluate its performance as you did the detection scheme in (d).

(h) You can consider other values of \( (S/N)_t \) also, e.g. a factor of two larger or smaller.