1. In general, the covariance matrix of additive measurement noise $\mathbf{n}$ in least squares problems has nonzero off-diagonal terms. However, it is possible to transform the problem to one where the transformed errors have a diagonal covariance matrix, in which case the simple form of least squares solution may be applied.

   (a) Assume the inverse of the covariance matrix $\mathbf{V}$ may be factored in terms of a new matrix $\mathbf{P}$:
   
   $$\mathbf{V}^{-1} = \frac{\mathbf{P}\mathbf{P}^t}{\sigma^2},$$

   where $\sigma^2$ is an appropriate constant in the factorization. Under what conditions may this factorization be made?

   (b) Transform the model equation $\mathbf{y} = \mathbf{X}\mathbf{\theta} + \mathbf{n}$ by multiplying through by $\mathbf{P}^t$. Show that the covariance matrix of the transformed error vector (call it $\mathbf{n}^*$) is now diagonal.

2. When is a filter matched? In this problem you are to use simulations to test how close a filter has to be to the exact matched filter in order to obtain optimal results.

   Consider a 1D object
   
   $$I(x) = A(x - x_0) + n(x)/\text{SNR},$$

   where $A(x)$ is a Gaussian function with unit amplitude and width (FWHM) $W$. The additive noise $n$ is white Gaussian noise and SNR is the signal to noise ratio of the object (peak to rms noise). Simulate $I(x)$ for a vector of $x$ 1024 value in $[0, 1023]$. Use a width $W = 20.7$ samples and a location parameter $x_0 = 423.7$. Also use SNR = 5 though you are free to experiment with lower or higher S/N. For $N_r = 100$ realizations estimate the location of the object using matched filtering. I.e. calculate the CCF between a template and the data and then use a method of your choice to find the location, $\hat{x}_0$. Calculate the mean and RMS of your set of 100 estimates. Use three different templates with widths $0.5W, W$, and $1.5W$ and compare your results.

3. Through a web search identify an application of principal component analysis (PCA) in astronomy.

4. An important example of orthogonal basis functions is the analysis of a spatial stochastic process on the sphere. For this problem you may wish to refer to http://en.wikipedia.org/wiki/Spherical_harmonics.

   Consider a real process $S(\Omega)$, where $\Omega = (\theta, \phi)$, that is homogenous and isotropic, meaning that it is the spatial analog of a wide-sense stationary process. The two moments of $S$ that we need are
   
   $$\langle S(\Omega) \rangle = S_0$$
   $$\langle S(\Omega_1)S(\Omega_2) \rangle = S_0^2 + \sigma_S^2 \rho(\theta_{12}),$$

   where $\theta_{12}$ is the 1D angle between $\Omega_1$ and $\Omega_2$. The quantity $\rho(\theta_{12})$ is a normalized correlation function with $\rho_{\text{max}} = 1$ and you can consider $\rho$ to be much narrower than 1 radian. As written, the second equation above implies that the cross correlation between $S$ in the two directions depends only on the difference angle between the two directions.

   In the following, we drop the ‘12’ subscript. The appropriate basis functions for this problem are spherical harmonics, $Y^m_\ell(\Omega)$, which satisfy the orthonormality condition
   
   $$\int d\Omega Y^m_\ell Y^{m'}_{\ell'} = \delta_{\ell\ell'}\delta_{mm'},$$

   We can therefore expand the process as
   
   $$S(\Omega) = \sum_{\ell,m} a_{\ell m} Y^m_\ell(\Omega)$$
with stochastic coefficients

$$a_{\ell m} = \int d\Omega S(\Omega) Y_{\ell m}^*(\Omega).$$

The correlation function $\rho(\theta)$ can be expanded in terms of Legendre polynomials

$$\rho(\theta) = \sum_{\ell=0}^{\infty} \beta_{\ell} P_{\ell}(\cos \theta)$$

$$\beta_{\ell} = \frac{2\ell + 1}{2} \int_{-1}^{1} d(\cos \theta) \rho(\theta) P_{\ell}(\cos \theta).$$

Of great utility is the addition theorem, which relates Legendre polynomials to products of spherical harmonics:

$$P_{\ell}(\cos \theta) = \frac{4\pi}{2\ell + 1} \sum_{m=\ell}^{\ell} Y_{\ell m}(\Omega_1) Y_{\ell m}^*(\Omega_2)$$

Using these definitions,

(a) Calculate $\langle a_{\ell m} \rangle$ in terms of the moments of the process, $S$.

(b) Calculate $\langle |a_{\ell m}|^2 \rangle$ in terms of the moments of $S$ and spherical harmonics.

(c) By using the addition theorem and the orthogonality of spherical harmonics, write $\langle |a_{\ell m}|^2 \rangle$ in terms of integrals over Legendre polynomials.

(d) What can you conclude about the statistics of $a_{\ell m}$ and of $|a_{\ell m}|^2$? I.e., what are their distributions when you make fairly general assumptions about the process $S$?

(e) Complete the analogy with time series analysis and the power spectrum, for which the Wiener-Khinchine theorem tells us that the autocorrelation function and power spectrum are a Fourier-transform pair. What quantity is the analogous partner to the angular correlation function, $\rho(\theta)$?