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SINGLE PULSE VS. PERIODICITY DETECTION

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ABSTRACT

We compare detection of single pulses against detection of a train of pulses in a periodicity search, taking into account the distribution of pulse amplitudes. We derive the conditions that must be satisfied in order that single pulse detection be more sensitive than periodicity searches.

Consider a time interval of length T in which there are N_p pulse periods. Pulse intensity maxima are described by a continuous PDF $f_{\rm S}(S)$ and its integral, the CDF $F_{\rm S}(S)$.

Single-pulse Detection: Given system noise $S_{\rm sys}$ expressed in Janskys, the signal to noise ratio (S/N) of the pulse peak $S_{\max}$ in the sum of $n_{\rm pol}$ polarization channels when the pulse is match filtered is

$$(S/N)_{\rm pk} = \eta (n_{\rm pol}\Delta\nu W)^{1/2} S_{\rm sys}^{-1} S_{\max}, \tag{1}$$

where $\eta \sim 1$ is a pulse-shape dependent factor. For a Gaussian pulse shape $\eta = (\pi/8\ln 2)^{1/4} \approx 0.868$. Single pulse detection entails specifying a threshold S/N of at least 3 and probably much larger, depending on how many overall statistical trials are made in a survey and what level of false-alarm can be tolerated.
For PDFs with long tails, one should take into account the fact that $f_{\rm S}$ is truncated at the maximum likely intensity in the interval. Assuming that all pulses have the same width W, we define the maximum pulse intensity likely to occur in the interval from $[1 - F_{\rm S}(S_{\max})]N_p = 1$, implying

$$F_{\rm S}(S_{\max}) = 1 - N_p^{-1}. \tag{2}$$

Periodicity Search: In a search for a periodic signal in the time series, the Fourier analysis effectively measures the period-averaged mean intensity for the time interval. The period averaged intensity is defined to be $\langle S\rangle_\phi = \zeta(W/P)\langle S\rangle$, $\langle S\rangle$ is the mean *peak* intensity and ζ is a shape-dependent constant ~ 1. For a Gaussian pulse with FWHM W, $\zeta = \sqrt{\pi/\ln 2}/2 \approx 1.06$. In terms of the pulse-peak PDF, the ensemble mean of the peak intensity is

$$\langle S\rangle \equiv \int dS\, S f_{\rm S}(S). \tag{3}$$

However, recognizing that PDFs with long tails yield a maximum intensity in a finite number of pulse periods, we define a modified mean intensity,

$$\langle S\rangle' = \frac{\int_0^{S_{\max}} dS\, S f_{\rm S}(S)}{\int_0^{S_{\max}} dS\, f_{\rm S}(S)} = \left(1 - N_p^{-1}\right)^{-1} \int_0^{S_{\max}} dS\, S f_{\rm S}(S). \tag{4}$$

The S/N for a periodicity search over an interval $T \equiv N_p P$ is

$$(S/N)_{\rm HS} = (n_{\rm pol}\Delta\nu T)^{1/2} S_{\rm sys}^{-1} \langle S\rangle_\phi \; h_\Sigma, \tag{5}$$

where the harmonic sum is

$$h_\Sigma = N_h^{-1/2} \sum_{\ell=1}^{N_h} R_\ell, \tag{6}$$

with R_ℓ equal to the ratio of the ℓ-th harmonic's Fourier amplitude to that of the DC value. The leading factor $N_h^{-1/2}$ takes into account that the threshold increases as the number of harmonics summed increases. For a Gaussian pulse shape,

$$R_\ell = \exp\left[-\left(\frac{\pi W \ell}{2P\sqrt{\ln 2}}\right)^2\right], \tag{7}$$

and the number of harmonics maximizing h_Σ is $N_{h\max} \approx P/2W$. The value of the harmonic sum at the maximum is well approximated by $h_{\Sigma\max} \approx \frac{1}{2}\left(\frac{P}{W}\right)^{1/2}$. The S/N of the periodicity search using the harmonic sum is then

$$(S/N)_{\rm HS} = (\zeta/2)(n_{\rm pol}\Delta\nu W)^{1/2} S_{\rm sys}^{-1} N_P^{1/2} \langle S\rangle'. \tag{8}$$

Comparing Single-pulse Detection and Periodicity Detection: A comparison of the two techniques is best done by considering the ratio of the two signal-to-noise ratios:

$$\begin{aligned} r &\equiv \frac{(S/N)_{\rm pk}}{(S/N)_{\rm HS}} \\ &= \left(\frac{2\eta}{\zeta N_p^{1/2}}\right)\frac{S_{\rm max}}{\langle S\rangle'} \\ &= \left(\frac{1}{N_p}\right)^{1/2}\frac{2\eta S_{\rm max}}{\zeta\left(1-N_p^{-1}\right)^{-1}\int_0^{S_{\rm max}} dS\, S f_{\rm S}(S)}. \end{aligned} \tag{9}$$

Unimodal Distributions: Inspection of Eq. 9 indicates that in order for $r > 1$, $S_{\rm max}/\langle S\rangle' > \zeta N_p^{1/2}/2\eta$, which is a large number for typical observation times T and pulse periods P. Some PDFs satisfy this criterion for large N_p while many do not.

Exponential PDF: Consider an exponential PDF,

$$f_{\rm S}(S) = \langle S\rangle^{-1} e^{-S/\langle S\rangle} U(S), \tag{10}$$

where U is the unit step function. For this case the maximum pulse intensity encountered in N_p trials is

$$S_{\rm max} = \langle S\rangle \ln N_p \tag{11}$$

and the renormalized average is

$$\langle S\rangle' = \langle S\rangle \left[1 - \frac{\ln N_p}{N_p - 1}\right]. \tag{12}$$

The ratio of S/N for the two search methods becomes

$$r = \frac{2\eta \ln N_p}{\zeta N_p^{1/2}\left[1 - \ln N_P/(N_p-1)\right]} \tag{13}$$

For this case, $r > 1$ only for $N_p \le 47$.

Power Law Distributions: A truncated power law PDF can be written in the form

$$f_{\rm S}(S) = \mathcal{N} S^{-\alpha}, \quad S_1 \le S \le S_2 \tag{14}$$

$$\mathcal{N} = \begin{cases} (\ln S_2/S_1)^{-1} & \alpha = 1 \\ \dfrac{(1-\alpha)}{(S_2^{1-\alpha} - S_1^{1-\alpha})} & \alpha \ne 1. \end{cases} \tag{15}$$

For a flat PDF ($\alpha = 0$) all pulse intensities are equiprobable, so we expect that as N_p gets large, the maximum and minimum intensities will equal the cutoff values. We will not consider power laws with $\alpha < 0$. For $\alpha > 0$, larger amplitudes are less probable, so we expect that the maximum encountered out of N_p pulses generally will be less than S_2. We find that

$$S_{\rm max} = \begin{cases} S_1 (S_2/S_1)^{1-N_p^{-1}}, & \alpha = 1 \\ \dfrac{S_2}{\left[\left(\dfrac{N_p-1}{N_p}\right)\left(1 + \left(\dfrac{S_2}{S_1}\right)^{\alpha-1}\dfrac{1}{N_p-1}\right)\right]^{\dfrac{1}{\alpha-1}}} & \alpha \ne 1 \end{cases} \tag{16}$$

Using $S_{\rm max}$ we evaluate[1] $\langle S\rangle'$

$$\langle S\rangle' = \begin{cases} \dfrac{S_{\rm max} - S_1}{\ln S_{\rm max}/S_1} & \alpha = 1 \\ \dfrac{S_1 S_{\rm max} \ln S_{\rm max}/S_1}{S_{\rm max} - S_1} & \alpha = 2 \\ S_{\rm max}\left(\dfrac{\alpha-1}{\alpha-2}\right)\left[\dfrac{\left(\dfrac{S_{\rm max}}{S_1}\right)^{\alpha-2} - 1}{\left(\dfrac{S_{\rm max}}{S_1}\right)^{\alpha-1} - 1}\right] & \alpha \ne 1, 2. \end{cases} \tag{17}$$

[1] Note that we evaluate $\langle S\rangle'$ by integrating from the actual lower cutoff S_1 to the effective upper cutoff $S_{\rm max}$. This is valid only for monotonically decreasing PDFs, i.e. $\alpha > 0$. For a flat PDF, moreover, we should let $S_{\rm max} \to S_2$.

Flat PDF ($\alpha = 0$): First consider a flat distribution, $\alpha = 0$, for which we let $S_{\max} \to S_2$ because all intensities are equiprobable. For this case we find

$$r = \frac{4\eta}{\zeta N_p^{1/2}} \frac{1}{(1 + S_1/S_2)}. \tag{18}$$

Asymptotically, the maximum r results for $S_2/S_1 \gg 1$,

$$r_{\max} \to \frac{4\eta}{\zeta N_p^{1/2}}, \tag{19}$$

which yields $r > 1$ only for $N_p < 11$.

Steeper Power Laws: For power laws with $\alpha > 0$, the S/N ratio $r > 1$ only for small N_p for $\alpha \lesssim 1$. For intermediate cases, $1.5 \lesssim \alpha \lesssim 2.5$, r has a maximum at $N_p \gg 1$. For steep power laws, e.g. $\alpha = 3$ and larger, the ratio $r > 1$ again only for small N_p and decreases monotonically.
Figure 1 shows r for several power-law cases and for the exponential PDF. The figure illustrates the statements made in the preceding paragraph. The values for r are small for flattish PDFs because the pulses Fourier analyzed in a long-train of pulses have amplitudes that do not deviate much from the peak pulse. For very steep power laws (e.g. $\alpha = 3$ in the Figure), the probability of seeing a large pulse near the upper cutoff, S_2, is too small to outweigh the $N_p^{1/2}$ increase in S/N of the Fourier method, even for very large N_p. For intermediate cases, where r has a distinct maximum, the likelihood of seeing a large pulse outweighs the $N_p^{1/2}$ increase in S/N of the Fourier method.

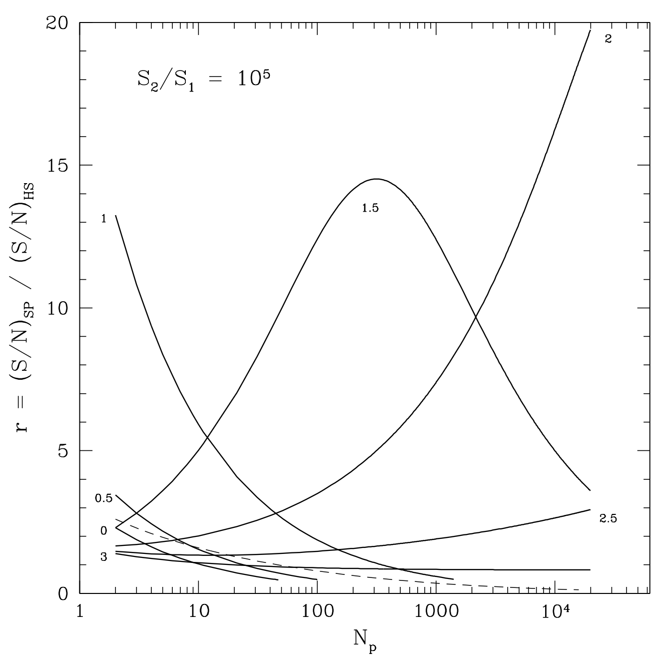

FIG. 1.— The S/N ratio, r, for power-law intensity PDFs (solid lines) and for an exponential PDF (dashed line). The exponent of the PDF (α) is shown and all power-laws are for a ratio of cutoff intensities, $S_2/S_1 = 10^5$. Single-pulse detection is superior to a Fourier harmonic-sum detection scheme if $r > 1$.

For $\alpha > 1$ and $\alpha \neq 2$ we can express r in the limit where $S_2/S_1 \gg 1$ using Eq. 16-17:

$$r \approx \left(\frac{2\eta}{\zeta}\right)\left(\frac{\alpha-2}{\alpha-1}\right) \times \begin{cases} N_p^{(3-\alpha)/2(\alpha-1)}, & N_p \ll (S_2/S_1)^{\alpha-1} \\ \left(\frac{S_2}{S_1}\right) N_p^{-1/2}, & N_p \gg (S_2/S_1)^{\alpha-1}. \end{cases} \tag{20}$$

For small N_p, r scales as $r \propto N_p^{(3-\alpha)/2(\alpha-1)}$ until $S_{\max} \to S_2$ and then r decreases as $N_p^{-1/2}$. Note that r has an increasing branch only if $1 < \alpha < 3$, consistent with Figure 1. If there is an increasing branch, then we expect a maximum at approximately where the two scaling laws in Eq. 20 are equal:

$$N_{p_{\max}} \approx \left(\frac{S_2}{S_1}\right)^{\alpha-1}. \tag{21}$$

For $S_2/S_1 = 10^5$ in the Figure, we expect the maximum for $\alpha = 3/2$ to be at $N_p \approx 10^{5/2}$, consistent with the exact expressions evaluated in the Figure. The maxima for $\alpha = 2$ and $\alpha = 5/2$ are then expected at $N_{p_{\max}} \approx 10^5$ and $10^{15/2}$ and are thus off the scale of the Figure.

We conclude that for certain power laws and intensity cutoffs, single-pulse searches can be superior to periodicity searches for long time series.

Crab Pulsar: At 146 MHz, Argyle & Gower (1972) find $\alpha \approx 2.5$ over at least two orders of magnitude of flux density (c.f. Figure 4-9 of Manchester & Taylor 1977). This slope is consistent with the fact that the Crab pulsar is more easily detected (and was discovered through) using its giant pulses. At 800 MHz Lundgren et al. (1995) find $\alpha \approx 3.5$ over 1.2 orders of magnitude. They also infer that the PDF for all pulses (giant and normal) is bimodal. Therefore, even though the slope found by Lundgren et al. would suggest that single-pulse detection would be inferior to a periodicity search, the apparent bimodality appears to change this conclusion.

Bimodal Distribution: A bimodal distribution provides a finite phase space for $r > 1$, even for large pulse numbers $N_p \gg 1$. Consider the bimodal distribution

$$f_S(S) = (1-g)\delta(S - S_1) + g\delta(S - S_2). \tag{22}$$

The only interesting case is when the interval contains at least one pulse with intensity S_2, requiring $gN_p \gtrsim 1$ or $g > g_{\min}$ where $g_{\min} = N_p^{-1}$. In this case we have

$$r = \frac{2\eta S_2}{\zeta N_p^{1/2}\left[(1-g)S_1 + gS_2\right]}. \tag{23}$$

To have $r > 1$ requires $g_{\min} < g < g_{\max}$ where

$$g_{\max} = \frac{(2\eta/\zeta)N_p^{-1/2} - S_1/S_2}{1 - S_1/S_2}. \tag{24}$$

Figure 2 shows regions in the two dimensional space of g and S_1/S_2 for which $r > 1$. The regions are a function of N_p.

Null Pulses: A special case of interest for the bimodal PDF is where the lowest intensity pulses are null pulses, $S_1 = 0$. Then

$$r = \frac{2\eta}{g\zeta N_p^{1/2}}. \tag{25}$$

Single pulse detection is superior to a periodicity search when

$$N_p^{-1} < g < (2\eta/\zeta)N_p^{-1/2}. \tag{26}$$

This regime corresponds to values of g between the solid and dashed lines in Figure 2 for small values of S_1/S_2.

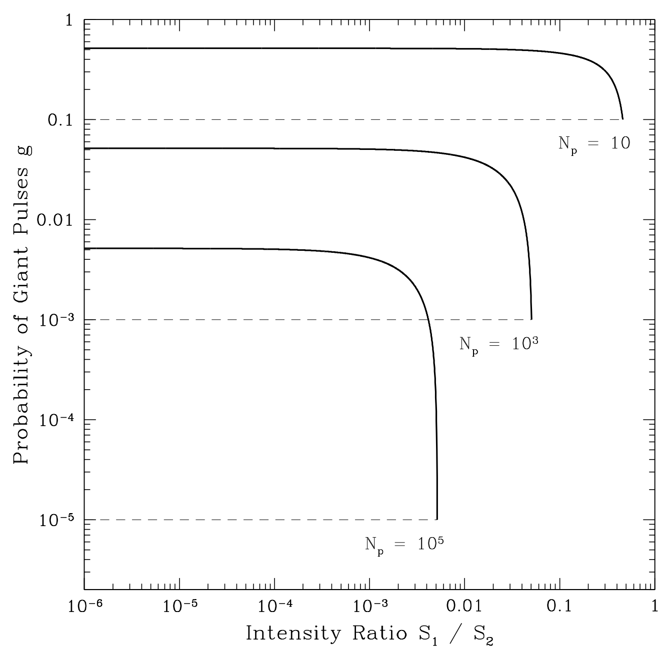

FIG. 2.— Domain for which a giant pulse search is more sensitive than a periodicity search. The pulse intensity distribution is assumed to be bimodal. S_1 is the intensity of ordinary pulses with probability $1 - g$ and S_2 is the giant-pulse intensity with probability g. The results depend on how many pulse periods, N_p, are analyzed. For each of the three cases shown, a giant pulse search is superior for values of g and S_1/S_2 between the dashed and solid lines.