Physics 318: Problem Set 1

Due Wednesday, Jan 30, 2008

1. A particle is projected vertically upward in a constant gravitational field with an initial velocity v_0. Show that if there is a retarding force proportional to the square of the instantaneous velocity, then the velocity of the particle when it returns to the initial position is

$$\frac{v_0 v_{\rm t}}{\sqrt{v_0^2 + v_{\rm t}^2}},$$

where $v_{\rm t}$ is the terminal velocity which would be attained by a freely falling particle. [Hint: change the dependent variable in the differential equation from time t to distance z using $dv/dt = (dv/dz)(dz/dt) = v dv/dz$.]

2. Spherical polar coordinates (r, θ, φ) are defined by

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

a. Using the chain rule, show that the vector $d\mathbf{r} = dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z$ can be written as $dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\varphi\mathbf{e}_\varphi$, where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit vectors. Find the explicit form of these vectors on the basis $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$.

b. Express the velocity $\dot{\mathbf{r}}$, the kinetic energy $m\dot{\mathbf{r}}^2/2$ and acceleration $\ddot{\mathbf{r}}$ of a particle in spherical coordinates, by expressing the time derivatives $\dot{\mathbf{e}}_r$, $\dot{\mathbf{e}}_\theta$ and $\dot{\mathbf{e}}_\varphi$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$.

3. A point mass m is moving in the potential $U(x) = kx^2/2 + \alpha x^4$ with $k > 0$, a harmonic oscillator with an anharmonic perturbation.

a. Find an integral expression for the period T of oscillation as a function of the total energy E, using the relation

$$T = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \frac{U(x)}{E}}},$$

where x_1 and x_2 are the turning points of the motion.

b. Evaluate the integral in the limit $\alpha E \ll k^2$, using the substitution $\sin^2\varphi = U(x)/E$ and expressing x and dx as functions of φ to first order in α. Discuss the result. Does the period depend on the amplitude of oscillation?

4. Consider a central force of the form $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = \mathbf{r} f(|\mathbf{r}|, |\dot{\mathbf{r}}|, t)$.

a. Show that a particle of mass m moving under the influence of this force has a angular momentum $\mathbf{L}$ which is constant.

b. We can choose the Cartesian coordinates (x, y, z) so that the angular momentum is in the z direction, $\mathbf{L} = L_z \mathbf{e}_z$. Show that $\mathbf{r}$ and $\dot{\mathbf{r}}$ lie in the xy plane.

c. Let $A(t)$ be the area swept out by the line joining to the location of the particle from time $t = 0$ until time t. Show that $dA = r^2 d\theta/2$, where (r, θ) are polar coordinates in the xy plane.

d. Deduce that $dA/dt = L_z/(2m)$, and hence that dA/dt is constant in time. [This is Kepler's second law for planetary motion.]

5. Suppose a force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ satisfies $\nabla \times \mathbf{F} = 0$. Argue as follows that there exists a potential $U(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$. Fix a point $\mathbf{r}_0$ in space. For any curve $\mathcal{C}$ joining $\mathbf{r}_0$ and $\mathbf{r}$, define

$$U_{\mathcal{C}}(\mathbf{r}) = -\int_{\mathcal{C}} d\mathbf{r}' \cdot \mathbf{F}(\mathbf{r}'),$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different curves joining $\mathbf{r}_0$ and $\mathbf{r}$, show using Stokes theorem that

$$U_{\mathcal{C}_1}(\mathbf{r}) - U_{\mathcal{C}_2}(\mathbf{r}) = \int_A d\mathbf{A} \cdot (\nabla \times \mathbf{F}),$$

where A is the area enclosed between $\mathcal{C}_1$ and $\mathcal{C}_2$. Deduce that $U_{\mathcal{C}}(\mathbf{r})$ is independent of the choice of curve $\mathcal{C}$.