Physics 318: Problem Set 10

Due Wednesday, April 16, 2008

1. *Hamiltonian formulation of mechanics:*

a. Consider the system described by the Lagrangian

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2} A_{ij}(q)\dot{q}_i\dot{q}_j + b_i(q)\dot{q}_i - V(q)$$

where $q = (q_1, q_2, \ldots q_f)$ and A_{ij} is symmetric. Show that the corresponding Hamiltonian is

$$H(q, p, t) = \frac{1}{2} C_{ij}(q)[p_i - b_i(q)][p_j - b_j(q)] + V(q)$$

where $\mathbf{C} = \mathbf{A}^{-1}$, i.e., $A_{ij}C_{jk} = \delta_{ik}$. Is this Hamiltonian conserved?

b. Suppose that a Lagrangian $\mathcal{L}(q, \dot{q}, t)$ does not depend explicitly on time. Show that the same is true for the corresponding Hamiltonian $H(q, p, t)$. Show also that if $\partial\mathcal{L}/(\partial q_i) = 0$ for some coordinate q_i, then $\partial H/\partial q_i = 0$ also.

2. *Particle in a magnetic field:*

a. A particle of mass m and electric charge q moves in a plane under the influence of a central force potential $V(r)$ and a constant uniform magnetic field $\mathbf{B}$ perpendicular to the plane. The magnetic field can be represented by the static vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r}.$$

Find the Lagrangian using as generalized coordinates the particle's Cartesian coordinates, and find the corresponding Hamiltonian.

b. Repeat part a. without the magnetic field, but using coordinates that rotate with respect to the previous coordinates about an axis perpendicular to the plane with an angular velocity

$$\omega = -\frac{q}{2m}\mathbf{B}.$$

c. Now specialize to weak magnetic fields (drop all terms quadratic in $\mathbf{B}$). By comparing the results from a. and b., what can you say about the dynamics of the particle in a static weak magnetic field?

3. A uniform cylinder of radius a and mass M is mounted so as to rotate freely around its axis of symmetry, which is in the vertical direction. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track of the form $\rho = a$, $\varphi = \varphi_0 + \alpha z$, where φ_0 and α are constants and (z, ρ, φ) are cylindrical coordinates. A mass m slides without friction along the track under the influence of gravity, starting from rest at the top of the cylinder. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

4. Consider a system of particles interacting with one another through potentials that depend only on the distances beweeen them and acted on by conservative central forces from a fixed point in some inertial frame. Obtain the Hamiltonian of the particles with respect to a set of axes, with origin at the center of force, which is rotating around some axis in the inertial frame with angular velocity ω. What is the physical significance of the Hamiltonian in this case? Is it conserved?

5. Consider the system described by the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2.$$

Suppose that there is a canonical transformation to new phase space coordinates Q_1, Q_2, P_1, P_2, and that

$$Q_1 = q_1^2, \qquad Q_2 = q_1 + q_2.$$

Find the most general transformation equations for P_1 and P_2. Show that with a particular choice of P_1 and P_2, the coordinates Q_1 and Q_2 are both cyclic in the new Hamiltonian. By this means obtain expressions for q_1, q_2, p_1 and p_2 as functions of time and of their initial values.