# Physics 318: Problem Set 11

*Due: Wednesday, April 23, 2008*

**1.** *Perturbation theory:* Consider the perturbed harmonic oscillator

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{3}\varepsilon m x^3$$

where $\varepsilon$ is small. Starting from the zeroth order motion

$$x^{(0)}(t) = a\cos(\omega t)$$

where

$$\omega = \omega_0 + \varepsilon\omega^{(1)} + \varepsilon^2\omega^{(2)} + O(\varepsilon^3),$$

compute the motion $x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t) + O(\varepsilon^3)$ up to second order in $\varepsilon$. Eliminate any secular terms that occur by choosing the frequencies $\omega^{(1)}$ and $\omega^{(2)}$ suitably.

**2.** *Action-angle variables for a harmonic oscillator:*

For the simple harmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$

compute the action variable $P$ and the angle variable $Q$ in terms of $q$ and $p$. Show that the action variable $P$ is proportional to energy divided by frequency.

**3.** *Adiabatic Invariant:*

a. Consider a particle of mass $m$ and charge $q$ moving in a plane, perpendicular to which there is a uniform magnetic field $B$. Show that this system is described by the Lagrangian

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{1}{2}qr^2B\dot{\varphi},$$

where $(r, \varphi)$ are suitable polar coordinates. Show that the Hamiltonian for this system can be written as

$$H = \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\varphi - \frac{1}{2}qBr^2\right)^2. \tag{1}$$

Note that $p_\varphi < 0$ when $qB > 0$.

b. Suppose now that the magnitude of the magnetic field varies slowly with time, $B = B(t)$. Then the angular momentum $p_\varphi$ is exactly conserved, since the rotational symmetry about the $z$ axis is preserved. We can therefore regard the Hamiltonian (1) as the Hamiltonian for a system with one degree of freedom, where $p_\varphi$ is taken to be a constant. Deduce from this Hamiltonian that if the system starts off in circular motion, then after a gradual change of the $B$ field the final state of the system is again circular motion such that the magnetic flux $\pi r^2 B$ is conserved. [Hint: for motion close to circular motion the system can be treated as a harmonic oscillator by expanding the potential in the Hamiltonian about its local minimum at $r = \sqrt{-2p_\varphi/(qB)}$.]