Physics 318: Problem Set 12

Due Wednesday, April 30, 2008

1. *Poisson Brackets:*

a. Show that the Poisson bracket $\{F, G\}$ of two functions F and G on phase space can be written as

$$\{F, G\} = \frac{\partial F}{\partial \eta_i} J_{ij} \frac{\partial G}{\partial \eta_j}.$$

Here the vector η and the $2f \times 2f$ matrix $\mathbf{J}$ are the quantities defined in lecture, given by $\eta_i = q_i$,$\eta_{f+i} = p_i$, $J_{ij} = 0$, $J_{i+f\,j} = -\delta_{ij}$, $J_{i\,f+j} = \delta_{ij}$, $J_{f+i\,f+j} = 0$ for $1 \leq i, j \leq f$.

b. Show that a mapping from the phase space coordinates η_i to new phase space coordinates ζ_i is canonical if and only if it preserves Poisson brackets, i.e. if

$$\frac{\partial F}{\partial \eta_i} J_{ij} \frac{\partial G}{\partial \eta_j} = \frac{\partial F}{\partial \zeta_i} J_{ij} \frac{\partial G}{\partial \zeta_j}$$

for any functions F and G on phase space.

2. *The wave equation:*

a. Show that the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

is solved by $u(x, t) = f(x - ct) + g(x + ct)$ where f and g are arbitrary functions of a single variable.

b. For the case of the string discussed in lectures, determine the functions f and g such that the boundary conditions $u(0, t) = u(l, t) = 0$ and the initial conditions $u(x, 0) = F(x)$ and $\dot{u}(x, 0) = G(x)$ are satisfied.

3. *Action principle for electromagnetic fields:* Consider the action functional

$$S[\Phi(\mathbf{x}, t), \mathbf{A}(\mathbf{x}, t)] = \int dt \int d^3x \left[\frac{1}{2}\varepsilon_0(\nabla\Phi + \dot{\mathbf{A}})^2 - \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 - \rho\Phi + \mathbf{j} \cdot \mathbf{A}\right].$$

This is a functional of the scalar potential Φ and the vector potential $\mathbf{A}$, and depends also on the charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{j}(\mathbf{x}, t)$. Here ε_0 is the permittivity of empty space and μ_0 is the permeability of empty space.

a. By varying the action with respect to the scalar potential Φ, derive the equation

$$\nabla^2\Phi + \nabla \cdot \dot{\mathbf{A}} = -\rho/\varepsilon_0.$$

b. By varying the action with respect to the scalar potential $\mathbf{A}$, derive the equation

$$\mu_0\varepsilon_0(\nabla\dot{\Phi} + \ddot{\mathbf{A}}) + \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\mathbf{j} = 0.$$

c. Using the expressions

$$\mathbf{E} = -\nabla\Phi - \dot{A}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

for the electric and magnetic fields in terms of the potentials, deduce from a. and b. the four Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dot{\mathbf{E}} + \mu_0 \mathbf{j}.$$

d. Now suppose that the electric and magnetic fields are coupled to N point particles of masses m_n, charges q_n, and positions $\mathbf{x}_n(t)$ for $1 \leq n \leq N$. The charge density and current density are then

$$\rho(\mathbf{x}, t) = \sum_{n=1}^{N} q_n \delta^3[\mathbf{x} - \mathbf{x}_n(t)], \qquad \mathbf{j}(\mathbf{x}, t) = \sum_{n=1}^{N} q_n \dot{\mathbf{x}}_n(t) \delta^3[\mathbf{x} - \mathbf{x}_n(t)].$$

The total action for the system consisting of the electric and magnetic fields and the point particles can be obtained by adding to the above action the kinetic energy

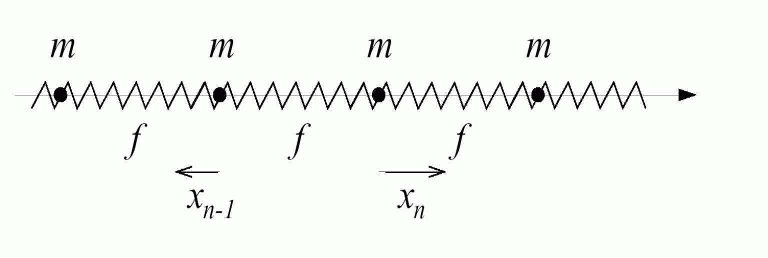
$$\sum_{n=1}^{N} \int dt \frac{1}{2} m_n \dot{\mathbf{x}}_n^2$$

for the particles. By varying this total action with respect to the positions of the particles, derive the equation of motion

$$m_n \ddot{\mathbf{x}}_n(t) = q_n \mathbf{E}[\mathbf{x}_n(t), t] + q_n \dot{\mathbf{x}}_n(t) \times \mathbf{B}[\mathbf{x}_n(t), t]$$

for $1 \leq n \leq N$.

4. Consider a simple one-dimensional model of a crystal consisting of an infinite chain of identical point masses of mass m connected by identical springs of length a and force constant f. In equilibrium, the distance between two successive is the lattice spacing a. Let the displacement of the nth point mass from its equilibrium position be $x_n(t)$.



a. Derive the Lagrangian and equations of motion for the chain.

b. Show that for an arbitrary real constant k, the Bloch wave

$$x_n(t) = \text{Re}\ [Q_k(t) \exp(ikna)]$$

defines a normal mode, i.e. that the equations of motion can be satisfied by an ansatz of the form $Q_k(t) = A_k \exp(i\omega_k t)$. Argue that without loss of generality the constant k can be restricted to the range $-\pi/a \leq k \leq \pi/a$. Make a sketch of the eigenfrequencies $\omega_k(k)$ as a function of k. This relation is called a dispersion relation.

c. Consider now a finite chain of N masses subject to the periodic boundary condition $x_n(t) = x_{n+N}(t)$. Show that this boundary condition leads to a discrete set of values k_m, $m = 1, 2, 3, \ldots$, and determine these values. How many physically different values of k_m exist, and what are the corresponding eigenfrequencies $\omega_m = \omega(k_m)$? Write down the general solution for the motion of the chain and describe its properties.