Physics 318: Problem Set 2

Due Wednesday, Feb 6, 2008

1. Consider a system of N particles with masses m_n, positions $\mathbf{r}_n(t)$, and velocities $\dot{\mathbf{r}}_n(t)$. The total mass M and center of mass $\mathbf{R}(t)$ of the system are defined by

$$M = \sum_{n=1}^{N} m_n, \qquad \mathbf{R}(t) = \frac{1}{M}\sum_{n=1}^{N} m_n \mathbf{r}_n(t).$$

We define $\mathbf{r}'_n(t) = \mathbf{r}_n(t) - \mathbf{R}(t)$ to be the position of the nth particle relative to the center of mass.

a. Show that the kinetic energy T of the system can be written as the kinetic energy of a point particle with mass M and position $\mathbf{R}(t)$, plus the kinetic energy of the system relative to the center of mass, that is

$$T = \frac{1}{2}M\dot{\mathbf{R}}(t)^2 + \frac{1}{2}\sum_{n=1}^{N} m_n \dot{\mathbf{r}}'_n(t)^2. \tag{1}$$

b. Show that the angular momentum $\mathbf{L}$ of the system can be written as the angular momentum of a point particle with mass M and position $\mathbf{R}(t)$, plus the angular momentum of the system relative to the center of mass, that is

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_{n=1}^{N} m_n \mathbf{r}'_n \times \dot{\mathbf{r}}'_n. \tag{2}$$

c. Show that if the system is isolated (so that the external forces $\mathbf{F}_n^{(e)}$ vanish), and if the inter-particle forces $\mathbf{F}_{nm}$ satisfy the strong form of Newton's third law, then (i) the first term in the kinetic energy (1) is conserved, and (ii) both of the terms in the angular momentum (2) are individually conserved.

2. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail of the rocket. Since these gases are generated a from chemical reaction of the fuels carried in the rocket, the mass of the rocket is not constant, but decreases as the fuel is expended.

a. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric friction, is

$$m(t)\frac{dv(t)}{dt} = -v_{\text{ex}}\frac{dm(t)}{dt} - m(t)g,$$

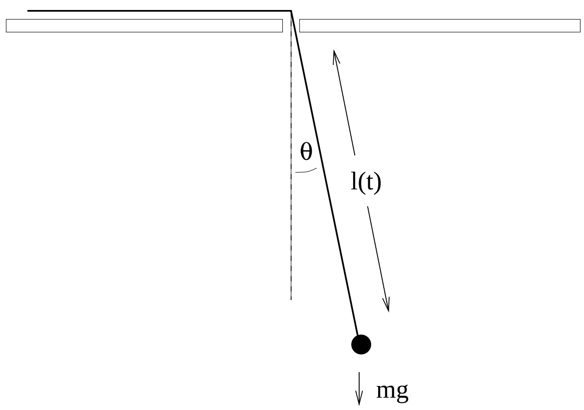
where g is the gravitational acceleration, $m(t)$ and $v(t)$ are the mass and velocity of the rocket at time t, and v_{ex} is the (constant) speed of the exhaust gases relative to the rocket.

b. By dividing across by $m(t)$ and integrating with respect to time, assuming that $v = 0$ at $t = 0$, show that

$$v(t) = -gt + v_{\text{ex}} \ln\left[\frac{m(0)}{m(t)}\right].$$

c. Suppose that we want the final velocity to exceed the escape velocity of the Earth's gravitational field, $v_{\text{esc}} = 1.1 \times 10^4\,\text{m}\,\text{s}^{-1}$. The maximum exhaust velocity that chemical reactions can achieve is roughly $v_{\text{ex}} = 3000\,\text{m}\,\text{s}^{-1}$. Deduce that at least 97.5% of the initial mass of the rocket must be fuel. [This constraint is the reason for the use of multi-stage rockets such as the Saturn V of the Apollo program.]

3. A point mass m is attached to the end of a thin, light string. The string is hanging vertically in a uniform gravitational field and at its top passes through a small hole in a table, so the length $l(t)$ of the hanging portion of the string can be changed as time passes. Treat $l(t)$ as a specified function of time. Assume that the mass and string move in the two-dimensional plane xz plane, where the z axis is vertical and the x axis is horizontal. Let $\theta(t)$ be the angle between the string and the vertical.



a. Write down the constraints and the Lagrange equations of the first kind for this system, in polar coordinates.

b. Solve for the Lagrange multiplier and determine the constraint force. Show that the equation of motion for the angle θ is

$$\ddot{\theta} = -2\frac{\dot{l}}{l}\dot{\theta} - \frac{g}{l}\sin\theta.$$

c. Show that the total mechanical energy of the system is

$$\frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) - mgl\cos\theta.$$

Is this energy conserved? Why?

4. A thin light string of length l is attached to two points at $(x, y) = (-a, 0)$ and $(x, y) = (a, 0)$, where $a < l/2$, and hangs freely in a uniform gravitational field g. A particle of mass m moves without friction along the string.

a. Show that the constraint can be written as

$$\sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2} = l.$$

b. The equilibrium location of the particle is $x = 0$, $y = -y_0$, where $y_0 = \sqrt{l^2/4 - a^2}$. For small deviations from the equilibrium, show that the constraint simplifies to

$$y = -y_0 + \frac{2y_0}{l^2}x^2 + O(x^4).$$

c. Compute the force of constraint and show that the equation of motion for $x(t)$ for small x is

$$\ddot{x} + \frac{4gy_0}{l^2}x = 0.$$

What is the period of the motion?