Physics 318: Problem Set 3

Due Wednesday, Feb 13, 2008

1. Consider a particle of mass m and charge q acted on by electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. These fields can be described in terms of the scalar and vector potentials $\Phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$, for which $\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Take the generalized coordinates to be the components $(x, y, z) = (x^1, x^2, x^3)$ of the particle's position, and take the Lagrangian to be

$$\mathcal{L}(x^i, \dot{x}^i, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\Phi(\mathbf{x}, t) + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t).$$

a. Show that the Lagrange equations of motion for this Lagrangian give back the Lorentz force law

$$m\ddot{\mathbf{x}} = q\mathbf{E} + q\dot{\mathbf{x}} \times \mathbf{B}.$$

b. Show that the Hamiltonian $\mathcal{H} = \sum_i p_i \dot{x}^i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{x}^i$, is

$$\mathcal{H} = \frac{1}{2}m\dot{\mathbf{x}}^2 + q\Phi(\mathbf{x}, t).$$

Note that this differs from the sum $T + U$ of kinetic and potential energies.

2. Consider a system of N particles with positions $\mathbf{r}_n(t)$ for $1 \leq n \leq N$. Suppose that the system is subject to the holonomic constraints $g_\alpha(\mathbf{r}_1, \ldots, \mathbf{r}_n, t) = 0$ for $1 \leq \alpha \leq f$, where f is the number of constraints. For a given motion $\mathbf{r}_n(t)$ and at a given time t, consider a set of infinitesimal displacements $\delta\mathbf{r}_n$ which have the property that they satisfy the constraints:

$$g_\alpha\big(\mathbf{r}_1(t) + \delta\mathbf{r}_1, \ldots, \mathbf{r}_N(t) + \delta\mathbf{r}_N\big) = 0 \tag{1}$$

for $1 \leq \alpha \leq f$. Such displacements are called virtual displacements.

a. By expanding Eq. (1) to linear order in the virtual displacements, deduce that

$$\sum_{n=1}^{N} \nabla_n g_\alpha\big(\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots, \mathbf{r}_N(t), t\big) \cdot \delta\mathbf{r}_n = 0$$

for $1 \leq \alpha \leq f$.

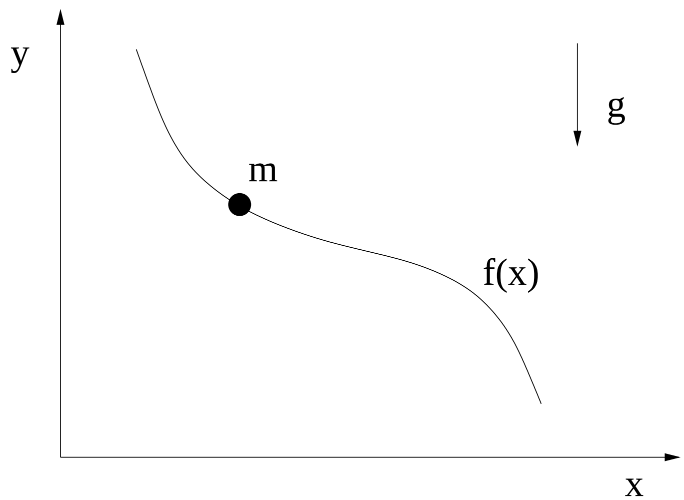
b. Deduce that the net work done by the constraint forces $\mathbf{F}_{\mathrm{C}\,n}$ in the virtual displacement is zero, ie

$$\sum_{n=1}^{N} \mathbf{F}_{\mathrm{C}\,n} \cdot \delta\mathbf{r}_n = 0.$$

This is called d'Alemberts principle of virtual work. It is sometimes used as a starting point for deriving the Lagrange equations.

3. A point mass m is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical axis with constant angular velocity ω. Obtain the Lagrange equations of motion assuming the only external forces arise from gravity. What are the constants of the motion? Show that if ω is greater than a critical value ω_0, there can be a solution in which the particle remains stationary on the hoop at a point other than at the bottom, but that if $\omega < \omega_0$, the only stationary point for the particle is at the bottom of the hoop. What is the value of ω_0?

4. Consider a point mass m moving under the influence of the gravitational force $\mathbf{F} = -mg\mathbf{e}_y$. The mass is constrained to slide along a given curve $y = f(x)$ in the $x - y$ plane. You may set $z = 0$ from the start and consider two dimensional motion.



a. Formulate the constraint and find the Lagrange equations of the first kind. Eliminate the Lagrange parameter λ, and simplify the resulting equations of motion as much as possible. Express the constraint force as a function of x and $\dot{x}$, and show that it orthogonal to the curve of the motion.

b. Explain why the total mechanical energy $E = T + U$ is conserved. Use this fact to derive a first order differential equation for $x(t)$. [Assume that $\dot{x} > 0$.] Then express the constraint force as a function of x.

c. A skier descends a slope with profile $y = -ax^n$ with $a > 0$ and $n \geq 1$. She starts at the top at $(x, y) = (0, 0)$ with zero velocity, and slides straight down without friction under the influence of gravity. If the slope steepens sufficiently, the skis will leave the ground at some point. Formulate a condition for when this happens. For what values of the parameter n, and at which point, do the skis leave the ground?