Physics 318: Problem Set 4

Due Wednesday, February 20, 2008

1. A particle is constrained to move without friction along the curve $y = y(x)$ in two dimensions in a uniform gravitational field. It starts from rest at (x_1, y_1) and ends at (x_2, y_2), where $y_1 = y(x_1)$ and $y_2 = y(x_2)$.

a. Show that the time taken to reach the second point is

$$T[y] = \int_{x_1}^{x_2} dx \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2g[y_1 - y(x)]}}.$$

b. Compute the choice of function $y(x)$ which minimizes $T[y]$ as follows. Use the fact that the integrand in $T[y]$ does not depend on x to obtain a first integral of the Euler equation (ie to find a relation between y' and y). Integrate this differential equation using a substitution of the form $y = -k^2 \sin^2(\varphi/2) - h$ with suitable constants k and h. Show that the resulting curve is a cycloid [a curve given parametrically by $x - x_1 = a(\varphi - \sin\varphi)$ and $y_1 - y = a(1 - \cos\varphi)$ for some constant a] with a cusp at the point (x_1, y_1).

2. Generalize the derivation of the Euler equation for the function $y(x)$ which minimizes the functional $J[y] = \int_{x_1}^{x_2} dx F(y, y', x)$ for the case where the function $y(x)$ is not restricted to take particular values a the endpoints x_1 and x_2. Show that in this case, the solution $y(x)$ must satisfy, in addition to the Euler equation, the boundary condition

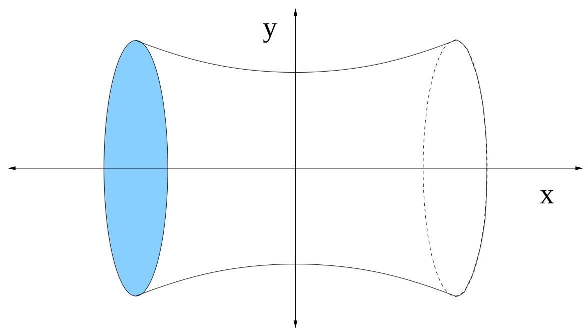
$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = \left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0.$$

3. A car is driven a distance D over a time interval T. The amount V of fuel consumed per unit time is given by $dV/dt = \alpha v + \beta \dot{v}^2$, where α and β are constants, v is the velocity and $\dot{v}$ is the acceleration. Your goal is to find the time dependence of the velocity $v(t)$ such that the total amount of fuel consumed is minimized.

a. Write down a functional $V[v]$ for the amount of fuel consumed during the ride. Next, find a functional expressing the constraint that the distance traveled during the time T is D. Derive the Euler equation for the combined functional obtained with a Lagrange multiplier. Use the fact that the integrands do not explicitly depend on time to derive a first order differential equation for $v(t)$.

b. Solve this equation and determine the constants of integration using the initial condition $v(0) = 0$ and a second boundary condition following from problem 2. Then eliminate the Lagrange multiplier λ using the constraint. Find explicit expressions for the velocity $v(t)$, acceleration $\dot{v}(t)$ and the traveled distance $d(t)$. Also compute the final velocity $v(T)$ and the total amount of fuel consumed.

4. A soap film is suspended between two parallel circular rings of radius R separated by a distance D. Neglect the effects of gravity. In equilibrium the soap film adjusts itself to minimize the surface area. In this problem you will find the shape of the soap film.



a. Take the rings to be located at $x = D/2$ and $x = -D/2$ as in the figure. The resulting soap film has a rotational symmetry about the x-axis and can be described by a function $y(x)$ with $y(\pm D/2) = R$. Derive a formula for the area $A[y]$ of the film, which is a functional of $y(x)$. Derive the associated Euler equation. Use the fact that the integrand of $A[y]$ does not depend explicitly on x to obtain a first integral of the Euler equation. Show that the solution to this equation is $y(x) = a\cosh(x/a)$, and derive the transcendental equation that determines the constant of integration $a = y(0)$.

b. In order to solve this equation for a, consider first the special case where $D \ll a$, when the two hoops are very close together. Expand the equation in a power series in D/a to quadratic order, and deduce that there are two solutions for a. Make a sketch of the resulting shapes of the soap film, and argue on physical grounds that one of these shapes corresponds to a local minimum of area and the other solution to be inconsistent with the earlier approximation. Confirm your answer by computing its area.

c. We can recast our hyperbolic cosine solution in terms of dimensionless lengths $\eta = y/R$, $\xi = x/R$ and dimensionless constants $\alpha = a/R$, $\delta = D/R$. Now consider the behavior of the film as the ratio of distance to ring radius δ is slowly increased. Graph the transcendental equation for α for various values of δ. Show graphically that the two solutions coalesce into a single solution at a critical distance δ_*, and that for $\delta > \delta_*$ there are no solutions. [At this point a continuous reduction of the surface to a configuration with two separate films inside the rings occurs.] Show that the critical distance is $\delta_* = 1.325...$