Physics 318: Problem Set 5

Due Wednesday, Feb 27, 2008

1. Suppose that two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are related by

$$\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \tag{1}$$

where f is some function and q is shorthand for $(q_1, \ldots, q_f)$.

a. By computing the Euler-Lagrange equations for the Lagrangian $\mathcal{L}'$, show that the two Lagrangians give the same equations of motion.

b. Consider the case of a particle of charge q and mass m moving in electric and magnetic fields parameterized by a scalar potential Φ and a vector potential $\mathbf{A}$. The Lagrangian for this system is

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} m \dot{\mathbf{x}}^2 - q\Phi(\mathbf{x}, t) + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t).$$

An electromagnetic gauge transformation is a transformation of the form $\mathbf{A} \to \mathbf{A} + \nabla\psi$, $\Phi \to \Phi - \dot{\psi}$, where $\psi = \psi(\mathbf{x}, t)$ is an arbitrary function. Show that under such a transformation the Lagrangian transforms as in Eq. (1), and compute the form of the function f.

2. Consider the Lagrangian

$$\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} m\omega^2 (q_1^2 + q_2^2),$$

which describes two uncoupled harmonic oscillators of mass m and frequency ω. Show that this system is invariant under the symmetry operation

$$q(t) \to e^{i\alpha} q(t),$$

where $q(t) \equiv q_1(t) + iq_2(t)$ and α is an arbitrary number. Compute the corresponding conserved quantity. How does this quantity relate to the amplitudes and phases of the individual harmonic motions of q_1 and q_2?

3. The general Galilean transformation from an inertial frame $(t, \mathbf{x})$ to an inertial frame $(t', \mathbf{x}')$ can be written as

$$x_i' = \sum_{j=1}^{3} \alpha_{ij} x_j - v_i t - d_i, \qquad t' = t - t_0.$$

It is characterized by ten parameters: 3 rotation angles determining the orthogonal matrix α, 3 components of the relative velocity vector $\mathbf{v}$, 3 components of the displacement vector $\mathbf{d}$, and a time displacement t_0.

a. Consider a system of N particles with masses m_n and positions $\mathbf{r}_n(t)$ in the original frame. Show that in the case of no rotation ($\alpha = \mathbf{1}$), the total momentum $\mathbf{P}'$ in the new frame is related to the total momentum $\mathbf{P}$ in the original frame by

$$\mathbf{P}' = \mathbf{P} - M\mathbf{v},$$

where $M = \sum_n m_n$ is the total mass of the system. Show that the kinetic energy T' in the new frame is given in terms of the kinetic energy T in the original frame by

$$T' = T - \mathbf{P} \cdot \mathbf{v} + \frac{1}{2} M\mathbf{v}^2.$$

b. Find the parameters $\bar{\alpha}_{ij}$, $\bar{v}_i$, $\bar{d}_i$ and $\bar{t}_0$ of the inverse Galilean transformation

$$x_i = \sum_{j=1}^{3} \bar{\alpha}_{ij} x'_j - \bar{v}_i t' - \bar{d}_i, \qquad t = t' - \bar{t}_0.$$

c. Show that the action of two consecutive Galilean transformations can be obtained from a single Galilean transformation. Express the parameters of the combined transformation in terms of those of the two transformations. Does the order of the transformations matter?

4. *Derivation of Galilean transformations:* In this problem we derive the equations describing Galilean transformations from first principles. We start from the most general relation between two coordinate systems $(t, \mathbf{x})$ and $(t', \mathbf{x}')$:

$$t' = t'(t, \mathbf{x}), \qquad x'_i = x'_i(t, \mathbf{x}).$$

We assume that these functions are smooth.

a. We assume that the transformation preserves the time intervals between pairs of event. Given two different events $(t, \mathbf{x})$ and $(t + \Delta t, \mathbf{x} + \Delta\mathbf{x})$ in the first reference frame, show that the difference between the time coordinates in the new reference frame is $t'(t + \Delta t, \mathbf{x} + \Delta\mathbf{x}) - t'(t, \mathbf{x})$. By equating this to Δt, and using the fact that the equation is valid for all values of t, $\mathbf{x}$, Δt and $\Delta\mathbf{x}$, argue that (i) the function $t'(t, \mathbf{x})$ must be independent of $\mathbf{x}$, and (ii) the function is given by $t'(t, \mathbf{x}) = t - t_0$ for some constant t_0.

b. Now fix a value of t, and define the function $f_i(\mathbf{x}) = x'_i(t, \mathbf{x})$. We assume that the transformation preserves distances, which implies that for every pair of points $\mathbf{x}_1$ and $\mathbf{x}_2$, $|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)| = |\mathbf{x}_1 - \mathbf{x}_2|$. By squaring this equation and differentiating with respect to $x_{1\,j}$ and then with respect to $x_{2\,k}$ show that

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial x_j}(\mathbf{x}_1) \frac{\partial f_i}{\partial x_k}(\mathbf{x}_2) = \delta_{jk},$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. Multiply across by the inverse matrix of the matrix $\partial f_i / \partial x_j(\mathbf{x}_1)$, and then argue that since the left hand side is independent of $\mathbf{x}_1$ and the right hand side is independent of $\mathbf{x}_2$, both sides must be independent of both $\mathbf{x}_1$ and $\mathbf{x}_2$. Deduce that

$$x'_i(t, \mathbf{x}) = \sum_{j=1}^{3} \alpha_{ij}(t) x_j - d_i(t), \tag{2}$$

where $d_i(t)$ is an arbitrary function of time and the matrix α_{ij} is orthogonal.

c. Next we impose the fact that the transformation should preserve the form of Newton's first law. Suppose that $x_i = x_i(t)$ is the path of a particle in the original frame, with velocity $v_i(t) = dx_i/dt$ and acceleration $a_i(t) = dv_i/dt$. Show using Eq. (2) that the acceleration as measured in the new frame is

$$\frac{d^2 x'_i}{dt'^2} = \sum_{j=1}^{3} [\ddot{\alpha}_{ij} x_j + 2\dot{\alpha}_{ij} v_j + \alpha_{ij} a_j] - \ddot{d}_i.$$

Deduce that in order to preserve Newton's first law we must have

$$\sum_{j=1}^{3} [\ddot{\alpha}_{ij} x_j + 2\dot{\alpha}_{ij} v_j] - \ddot{d}_i = 0.$$

Argue that since this equation must hold for all values of $\mathbf{x}$ and $\mathbf{v}$, that $\dot{\alpha}_{ij} = 0$ and $\ddot{d}_i = 0$. Deduce the usual formula for Galilean transformations.