Physics 318: Problem Set 6

Due Wednesday, March 5, 2008

1. Consider a particle of mass μ moving in the attractive Coulomb potential $V(\rho) = -\alpha/\rho$ where α is a positive constant.

a. Show that when the energy E of the particle is negative (corresponding to bound elliptical orbits), the relation between time t and radius ρ is given by

$$t = \sqrt{\frac{\mu}{2|E|}} \int d\rho \frac{\rho}{\sqrt{a^2\varepsilon^2 - (\rho - a)^2}}.$$

Here

$$\varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu\alpha^2}}, \qquad a \equiv \frac{\alpha}{2|E|}$$

are the eccentricity and semi-major axis of the ellipse, and l is the angular momentum of the orbit.

b. By making the change of variables $\rho = a - a\varepsilon\cos\xi$ in the integral, deduce that the relation between t and ρ can be written in the parametric form

$$t(\xi) = \sqrt{\frac{\mu a^3}{\alpha}}(\xi - \varepsilon\sin\xi), \qquad \rho(\xi) = a(1 - \varepsilon\cos\xi).$$

These are called Kepler's equations.

2.

a. For motion in a central potential $V(\rho)$, show that the change $\Delta\varphi$ in the angle φ while ρ changes from its maximum value $\rho_{\max}$ to its minimum value $\rho_{\min}$ and back again is

$$\Delta\varphi = -2\frac{\partial}{\partial l}\int_{\rho_{\min}}^{\rho_{\max}} d\rho \left\{2\mu\left[E - V(\rho)\right] - \frac{l^2}{\rho^2}\right\}^{1/2}. \qquad (1)$$

b. Suppose that the potential is a Coulomb potential plus a small inverse cubic term:

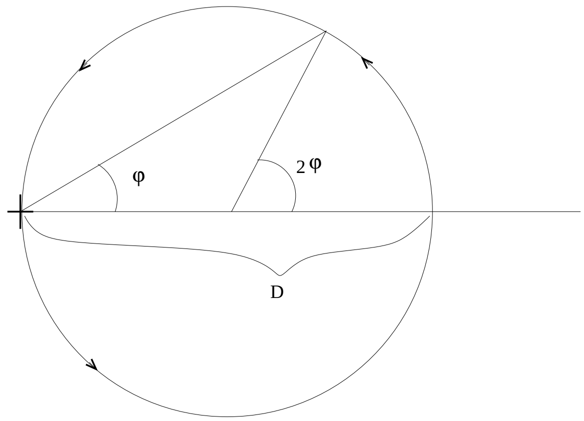
$$V(\rho) = -\frac{\alpha}{\rho} + \frac{\beta}{\rho^3}.$$

By substituting into Eq. (1), expanding to linear order in β, and changing the variable of integration from ρ to φ (using the relation $\rho = \rho(\varphi)$ for $\beta = 0$), show that $\Delta\varphi = 2\pi - 6\pi\beta/(\alpha p^2) + O(\beta^2)$, where $p = l^2/(\mu\alpha)$ is the semi-latus rectum of the orbit.

c. The leading order correction to Newtonian gravity from general relativity is of the above form, where $\alpha = Gm_2m_1$ and $\beta = -GMl^2/(\mu c^2)$, where G is Newton's constant of gravitation, c is the speed of light, l is the orbital angular momentum, and $M = m_1 + m_2$ and $\mu = m_1m_2/M$ are the total and reduced masses. Evaluate the perihelion shift for Mercury's orbit around the Sun. The Sun's mass is 2.00×10^{30} kg, Mercury's mass is 3.30×10^{23} kg, the semi-major axis of the orbit is 5.79×10^{10} m, and the eccentricity is 0.206. Express your answer in arc seconds per century. [The agreement between the prediction of general relativity and the observed perihelion precession is one of the three classical tests of general relativity.]

3.

a. Show that if a particle moves in a circular orbit under the influence of an attractive central force directed toward a point such a circular orbit, then the force varies as the inverse-fifth power of from that point. The diagram explains the geometry of such an orbit.

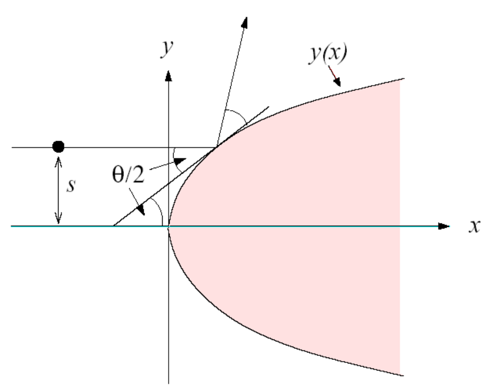


b. Show that for the orbit described, the total energy of the particle is zero.

c. Find the period of the motion.

d. If the circle is in the xy plane, find $\dot{x}$, $\dot{y}$ and v as a function of the angle around the circle, and show that all three quantities become infinite as the particle passes through the center of force.

4. [Problem due to Prof. Neubert] Consider the elastic scattering of a beam of particles with initial velocities $v_\infty \mathbf{e}_x$ on a rigid surface, whose shape is obtained by rotating the curve $y(x) = \sqrt{ax}$, for $x \geq 0$, about the x-axis. Here a is a positive constant. Assume that the particles bounce elastically off the surface.



Use a geometrical argument to find a relation between the impact parameter s (the distance from the x-axis) and the scattering angle θ. Derive the differential scattering cross section $d\sigma/d\Omega$, and show that it has the same angular dependence as Rutherford scattering.