Physics 318: Problem Set 7

Due Wednesday, March 12, 2008

1. A comet of mass m approaches the Sun of mass M, attracted by its gravitational field. The comet is idealized as a point mass while the Sun has radius R. The comet has an initial energy $E > 0$ when far from the Sun.

a. Write down a formula for the the trajectory of the comet, in terms of the semi-latus rectum a and eccentricity ε of the comet's motion. For what values of a and ε will the comet crash into the Sun?

b. Use the condition from part a. to determine the minimal impact paramter s which allows the comet to pass the sun without crashing into it. Express your answer in terms of the energy E, Newton's constant of gravitation G, the mass m of the comet, the mass M of the Sun, and the radius R of the sun.

c. Compute the total cross section σ for the comet to crash into the Sun.

d. How does the cross-section behave in the limit $G \to 0$? Explain in physical terms this limiting behavior. How does the cross section behave as the Sun becomes more massive?

2. Two particles of masses m_1 and m_2 collide head on. The initial velocity in the laboratory frame of particle 1 is u_1, while particle 2 is at rest. The particles interact through a potential energy

$$V(r) = V_0 \frac{a^2}{r^2},$$

where $V_0 > 0$ and a are constants.

a. Calculate the total energy and the total angular momentum of the two particles in the center-of-mass frame, in terms of u_1, m_1 and m_2.

b. Make a sketch of the initial and final states of the two particles in the lab frame and in the center-of-mass frame.

c. Find the distance of closest approach of the two particles.

d. Calculate the velocity of particle 2 in the lab frame after the collision.

3. Consider the motion of a particle of mass μ in the potential $V(r) = kr^2/2$ with $k > 0$. This is called the spherical harmonic oscillator potential. Angular momentum conservation allows us to choose the coordinate system such that the motion takes place in the xy plane, so that $z(t) = 0$.

a. Find the motion $\mathbf{r}(t)$ by solving the Lagrange equations in Cartesian coordinates, using the fact that the Lagrangian can be written as the sum of three uncoupled terms:

$$\mathcal{L} = \sum_{i=1}^{3} \left[\frac{1}{2}\mu\dot{x}_i^2 - \frac{1}{2}kx_i^2 \right],$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$. Describe the resulting orbit in the xy plane by an equation of the form $f(x, y) = 0$ for some function f.

b. Solve the same problem in polar coordinates (ρ, φ), using the conservation of energy E and angular momentum l. Evaluate the integral giving $\varphi = \varphi(\rho)$ derived in lecture by making the substitution $u = 1/\rho^2$, and find the equation $\rho = \rho(\varphi)$ of the orbit. Relate the constants E and l to two parameters characterizing the shape of the orbit. Make a sketch of $\rho(\varphi)$ together with the circles $\rho = \rho_{\text{min}}$ and $\rho = \rho_{\text{max}}$. What is the angular difference $\Delta\phi$ between two successive points of closest approach to the center?

4. In this problem you will derive a formula for the matrix that describes a rotation about an angle Φ in the direction of the unit vector $\mathbf{n}$. We decompose the position vector $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp},$$

where $\mathbf{r}_{\parallel} = (\mathbf{r} \cdot \mathbf{n})\mathbf{n}$ is the component of $\mathbf{r}$ parallel to $\mathbf{n}$, and $\mathbf{r}_{\perp}$ is the component perpendicular to $\mathbf{n}$. Argue using a diagram that the effect of the rotation is to make the replacements

$$\mathbf{r}_{\parallel} \to \mathbf{r}_{\parallel}$$

and

$$\mathbf{r}_{\perp} \to \cos\Phi\, \mathbf{r}_{\perp} + \sin\Phi\, \mathbf{n} \times \mathbf{r}_{\perp}.$$

Deduce that the rotation is given by $\mathbf{r} \to \mathbf{A} \cdot \mathbf{r}$, where the matrix $\mathbf{A}$ has components

$$a_{ij}(\mathbf{n}, \Phi) = n_i n_j + \cos\Phi\, [\delta_{ij} - n_i n_j] - \sin\Phi\, \epsilon_{ijk} n_k.$$