Physics 318: Problem Set 8

Due Wednesday April 2, 2008

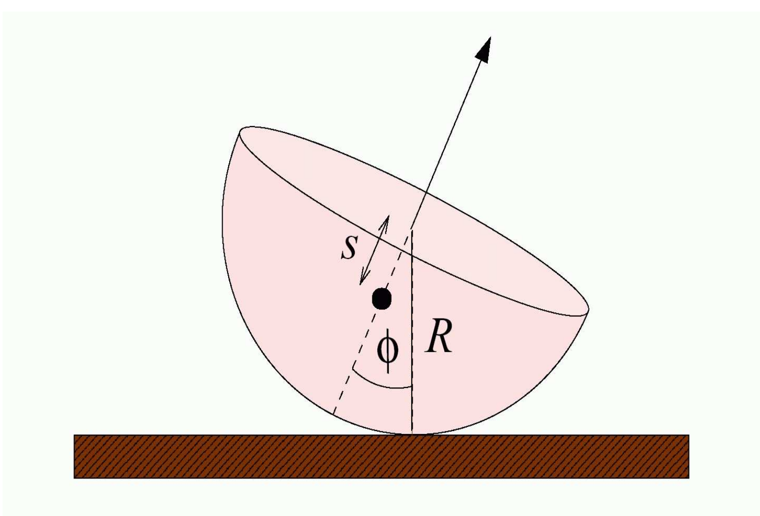
1. *Moments of inertia:*

a. The components of the moment of inertial tensor I'_{ij} of a rigid body depend on the choice of the origin of the body coordinate system (x'_1, x'_2, x'_3). Consider two body coordinate systems with parallel axes. The origin of the first coordinate system is the center of mass of the rigid body, and the origin of the second is displaced from the center of mass by a displacement $\mathbf{a}$. Find the relation between the components I'_{ij} in the two systems.

b. Determine the principal moments of inertia about the center of mass for the following homogeneous bodies: (i) a thin rod of length l and mass M; (ii) a circular cylinder of radius R, height h and mass M; (iii) a rectangular parallelepiped of sides a, b and c and mass M; (iv) a circular cone of height h and base radius R and mass M; and (v) an ellipsoid of radii a, b and c of mass M.

c. Suppose a rigid body is constrained so that it can only rotate about a fixed axis that passes through the origin of the body coordinate system in the direction of the fixed unit vector $\mathbf{n}$. Show that the kinetic energy of the body is $T = I(\mathbf{n})\omega^2/2$, where ω is the angular velocity about the axis, and $I(\mathbf{n}) = I'_{ij} n'_i n'_j$. The quantity $I(\mathbf{n})$ is called the moment of inertia about the axis.

2. A solid hemisphere of constant density, mass M and radius R rolls without slipping on the surface of a horizontal plane: (The hemisphere is not rotating about its axis of symmetry).



a. Find the distance s of the center of mass from the planar surface of the hemisphere. Calculate the moment of inertial with respect to an axis through the center of mass and orthogonal to the axis of symmetry of the hemisphere.

b. Derive the Lagrangian of the system, using ϕ as the generalized coordinate, and write down the equation of motion. Compute the frequency of small oscillations about the equilibrium position.

3. Consider a symmetric, spinning top in a uniform gravitational field g. As in lectures, let the distance between the center of mass and the support point be s, let the moments of inertial be I_1 and I_3, and let the mass be M.

a. Use the substitution $u = \cos\theta$ to write the energy conservation law

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + U_{\text{eff}}(\theta)$$

derived in lectures in the form $\dot{u}^2 = P(u)$, where $P(u)$ is a third order polynomial in u. Find the explicit form of this polynomial.

b. The initial conditions for the top at $t = 0$ are such that the angle between the figure axis and the vertical is $\theta(0) = \theta_0$, and the spinning frequency about the figure axis is $\dot{\psi}(0) = \omega_0$. In addition assume that $\varphi(0) = \psi(0) = 0$ and that $\dot{\theta}(0) = \dot{\varphi}(0) = 0$. Use these conditions to calculate the conserved quantities p_φ and p_ψ and E' and insert their values into the polynomial $P(u)$. Show that up to an overall constant, $P(u)$ can be expressed as a function of $u_0 = \cos\theta_0$ and of the dimensionless parameter

$$\varepsilon = \frac{2MgsI_1}{I_3^2 \omega_0^2}.$$

What is the physical meaning of this ratio?

c. Perform a graphical discussion of the polynomial $P(u)$ as a function of u and calculate its zeros analytically. Assume that $\varepsilon \ll 1$ (fast-spinning top). What is the allowed region in u? What is the width of this region? What happens in the limit of a free top (i.e. for $g \to 0$)?

d. Insert the ansatz

$$u(t) = u_0 - \varepsilon(1 - u_0^2)x(t)$$

into the equation $\dot{u}^2 = P(u)$ and derive the corresponding equation for $\dot{x}^2$ in the limit $\varepsilon \to 0$. Solve this equation by taking a derivative with respect to t. Use this result to compute $\theta(t)$ to the first non-trivial order in ε. Interpret the results.

e. Find and solve the equations of motion for $\varphi(t)$ and $\psi(t)$ to the first non-trivial order in ε. Interpret the results. What form of motion do you obtain in the (θ, φ) plane?