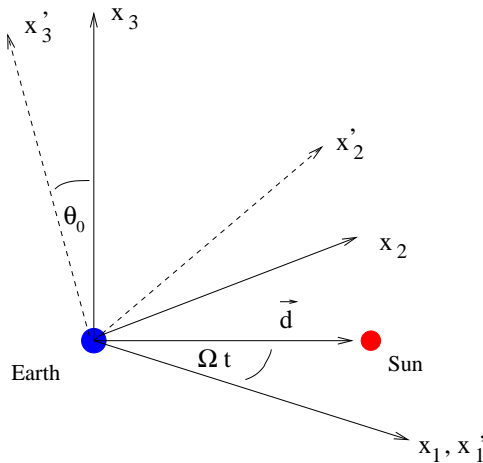


# Physics 318: Problem Set 9

*Due Wednesday, April 9, 2008*

1. The Sun and the Moon exercise a torque on the Earth which leads to a precession of the Earth's axis of rotation with a period of approximately 26,000 years. Today the axis points in the direction of the star Polaris, but in different historical epochs, different stars were considered to be "the polar star". The phenomenon of precession was discovered as early as 343 B.C. by the Babylonian astronomer Cidenas.

To a good approximation, the Earth can be described as a rotationally symmetric ellipsoid, with a half-axis to the pole of  $c = 6357$  km, and a half-axis to the equator of  $a = 6378$  km. Take the center of mass of the Earth as the origin of both the inertial frame (IF)  $(x_1, x_2, x_3)$  and the body coordinate system (BS)  $(x'_1, x'_2, x'_3)$ . Take the displacement vector  $\mathbf{d}$  to the Sun (or the Moon) to be in the  $(x_1, x_2)$  plane of the IF. In the case of the Sun, this plane is called the ecliptic. Describe the rotation of the Earth as a rotation about a principal axis with angular momentum  $\mathbf{L} = I_3 \omega'_3 \mathbf{e}'_3$ . The figure axis ( $x'_3$  axis) is inclined with respect to the  $x_3$  axis by the obliquity angle  $\theta_0 = 23.45^\circ$ .



- a. Describe the Sun (or the Moon) by a point mass  $M$  at a displacement  $\mathbf{d}$  from the center-of-mass of the Earth. In the gravitational field of the Sun (or the Moon), a mass element  $dm = \rho(\mathbf{r})d^3r$  of the Earth at position  $\mathbf{r}$  has the potential energy  $-GMdm/|\mathbf{r} - \mathbf{d}|$ . Show that this implies that the Earth is subject to the torque

$$\mathbf{N} = -GM \int_V d^3r \rho(\mathbf{r}) \mathbf{r} \times \frac{\mathbf{r} - \mathbf{d}}{|\mathbf{r} - \mathbf{d}|^3},$$

where  $V$  is the volume of the Earth. Show that for  $|\mathbf{r}| \ll |\mathbf{d}|$  this relation leads to

$$\mathbf{N} = \frac{3GM}{d^5} \int_V d^3r \rho(\mathbf{r}) (\mathbf{r} \times \mathbf{d})(\mathbf{r} \cdot \mathbf{d}) = 3GM \frac{\mathbf{d} \times (\mathbf{I} \cdot \mathbf{d})}{d^5},$$

where  $d = |\mathbf{d}|$ ,  $\mathbf{I}$  is the moment of inertia tensor, and  $\mathbf{I} \cdot \mathbf{d}$  is the vector whose  $i$ th component is  $I_{ij}d_j$ . Evaluate the result in terms of the principal moments of inertia ( $I_1$  and  $I_3$ ), and the components ( $d'_1, d'_2, d'_3$ ) of  $\mathbf{d}$  in the BS.

- b. At a given time  $t$ , choose the axes of the BS such that  $\mathbf{e}_1 = \mathbf{e}'_1$ . Then convince yourself that in the BS the vector  $\mathbf{d}$  has components

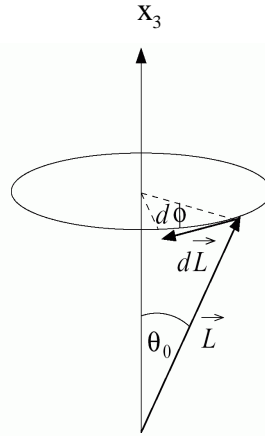
$$(d'_1, d'_2, d'_3) = d(\cos \Omega t, \cos \theta_0 \sin \Omega t, -\sin \theta_0 \sin \Omega t),$$

where  $\Omega$  is the frequency of the Earth's orbit around the Sun (or the Moon), and we assume that this orbit is circular ( $d$  is constant). Show that the torque  $\mathbf{N}$  is given by

$$\mathbf{N} = \frac{3GM}{d^3} (I_3 - I_1) [\cos \theta_0 \sin^2 \Omega t (\mathbf{e}'_3 \times \mathbf{e}_3) + \sin \Omega t \cos \Omega t (\mathbf{e}_3 - \cos \theta_0 \mathbf{e}'_3)].$$

Note that this expression is independent of our choice of the  $x'_1$  and  $x'_2$  axes. Since the precession frequency of the figure axis is much smaller than  $\Omega$ , it is justified to average the torque over time. Show that this gives

$$\langle \mathbf{N} \rangle = \frac{3GM}{2d^3} (I_3 - I_1) \cos \theta_0 (\mathbf{e}'_3 \times \mathbf{e}_3).$$



- c. Consider the equation of motion in the IF  $d\mathbf{L}/dt = \langle \mathbf{N} \rangle$ , and show that  $|\mathbf{L}|$  is constant. Use this fact to argue, based on the above figure, that the precession frequency  $\dot{\phi}$  of the angular momentum vector around the  $x_3$  axis is given by

$$\dot{\phi} = -\frac{3GM}{2\omega_3 d^3} \frac{I_3 - I_1}{I_3} \cos \theta_0.$$

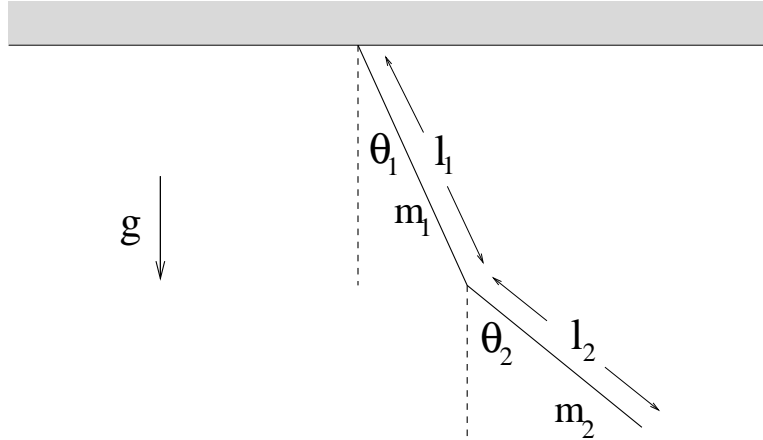
Use Kepler's third law to eliminate the quantity  $GM/d^3$  in favor of the frequency  $\Omega$  and the mass ratio  $M_{\text{Earth}}/M$ .

- d. In order to obtain a numerical estimate of the precession frequency, show that

$$\frac{I_3 - I_1}{I_3} = \frac{a^2 - c^2}{2a^2} \approx 3.3 \times 10^{-3}.$$

Assume that the Sun, Earth and Moon all move in a plane (which is true to a good approximation), and that the effects of their interactions simply add up. Derive an expression for the precession period as a function of the mass ratio  $M_{\text{Earth}}/M_{\text{Moon}} \approx 81$  (you can set  $M_{\text{Earth}}/M_{\text{Sun}} = 0$ ), the period  $T_{\text{Sun}} = 1$  yr for the Earth's orbit around the Sun, and the period  $T_{\text{Moon}} \approx 27.3$  days for the orbit of the Moon. Evaluate the result numerically. Which effect (Sun or Moon) is more important?

2. Consider a double pendulum consisting of a uniform thin rigid rod of mass  $m_1$  and length  $l_1$  that hangs from a pivot point, making an angle of  $\theta_1$  with the vertical, together with a second uniform thin rigid rod of length  $l_2$  and mass  $m_2$  connected to the first rod at its end, making an angle  $\theta_2$  to the vertical. The joint between the two rods can pivot freely, and the double pendulum is in a uniform gravitational field  $g$ .



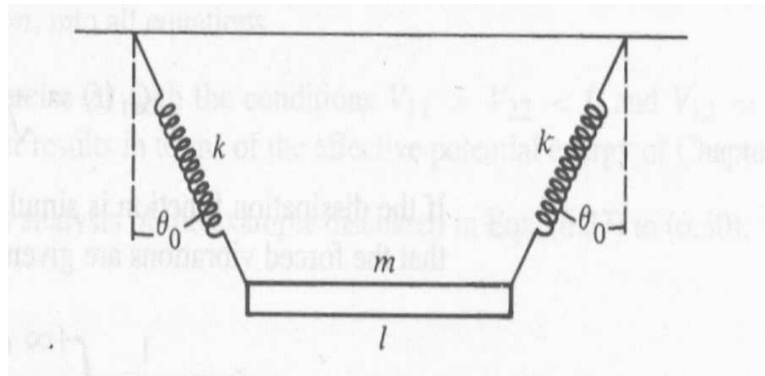
- a. Argue that the kinetic energy of the first rod is  $m_1(\dot{x}_1^2 + \dot{y}_1^2)/2 + I_1\dot{\theta}_1^2/2$ , where  $(x_1, y_1)$  are the coordinates of its center of mass, and  $I_1$  is its moment of inertia about its center of mass. Similarly argue that the kinetic energy of the second rod is  $m_2(\dot{x}_2^2 + \dot{y}_2^2)/2 + I_2\dot{\theta}_2^2/2$ . Deduce that the Lagrangian for the system is

$$\mathcal{L} = \frac{1}{6}(m_1 + 3m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2l_2^2\dot{\theta}_2^2 + \frac{1}{2}m_2l_1l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

$$+ \frac{1}{2}(m_1 + 2m_2)gl_1 \cos \theta_1 + \frac{1}{2}m_2gl_2 \cos \theta_2.$$

- b. For the special case where  $m_1 = m_2$  and  $l_2 = 4l_1/3$ , expand the Lagrangian to second order in  $\theta_1$  and  $\theta_2$  (small oscillations) and their derivatives. Derive the explicit form of the symmetric matrices  $\mathbf{T}$  and  $\mathbf{V}$  introduced in lectures.
- c. Find the two eigenfrequencies  $\omega^{(1)}$  and  $\omega^{(2)}$  and the corresponding eigenvectors  $\mathbf{a}^{(1)}$  and  $\mathbf{a}^{(2)}$ . Deduce expressions for the normal coordinates  $Q_1$  and  $Q_2$  in terms of  $\theta_1$  and  $\theta_2$ . Make a sketch of the oscillations corresponding to the normal modes of the system.
- d. Find the general solution of the equation of motion for small oscillations, in terms of normal coordinates. For the initial conditions  $\theta_1(0) = \theta_2(0) = \theta_0$ ,  $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$ , deduce the solutions  $\theta_1(t)$  and  $\theta_2(t)$ . Make a sketch of the resulting motion.

3. A uniform rigid thin bar of length  $l$  and mass  $m$  is suspended by two identical springs of equilibrium length  $b$  and spring constant  $k$ , as shown in the diagram. The angle between the springs and the vertical in equilibrium is  $\theta_0$ . Consider only motions of this system in the plane of the figure.



- Argue, based on the total number of degrees of freedom and constraints, that this 2D system should have three normal modes of oscillations. Find these three normal modes of small oscillations (neglect gravity).
- Is there a conserved angular momentum for this system? Why?