

Physics 6553 : Problem Set 1

Due Thursday , Sept 10, 2008

1. [5 points] *Double Dual of a Vector Space:* Given a finite dimensional vector space V , we defined a map from V to V^{**} taking \vec{v} to $F_{\vec{v}}$ where

$$F_{\vec{v}}(\mathbf{w}) = \mathbf{w}(\vec{v})$$

for any dual vector \mathbf{w} in V^* . Show that this mapping is linear and bijective, so that V and V^{**} are naturally isomorphic. Did you need to use the fact that V is finite dimensional?

2. [10 points] *Properties of Tensor Product:* Given finite dimensional vector spaces U , V , and W , establish the followign natural isomorphisms:

$$(U \otimes V)^* \simeq U^* \otimes V^*$$

$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W).$$

In the second case, show that both of these spaces are naturally isomorphic to the set of linear maps from $U^* \times V^* \times W^*$ to the real numbers.

3. [5 points] *Reinterpretation of tensors:* Show that the set of multilinear maps

$$T : V \times \dots \times V \times V^* \times \dots \times V^* \rightarrow V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$$

is naturally isomorphic to the tensor space $T_{s+s'}^{r+r'}(V)$. Here on the left hand side there are s copies of V and r copies of V^* , and on the right hand side there are r' copies of V and s' copies of V^* .

4. [5 points] *Outer products:* Suppose that A is a tensor in $T_s^r(V)$ and B is a tensor in $T_{s'}^{r'}(V)$. Using the natural isomorphism $T_s^r(V) \otimes T_{s'}^{r'}(V) \simeq T_{s+s'}^{r+r'}(V)$, show that the tensor product of A and B is given by the formula

$$(A \otimes B)(\mathbf{w}^1, \dots, \mathbf{w}^{r+r'}, \vec{v}_1, \dots, \vec{v}_{s+s'}) = A(\mathbf{w}^1, \dots, \mathbf{w}^r, \vec{v}_1, \dots, \vec{v}_s) \\ \times B(\mathbf{w}^{r+1}, \dots, \mathbf{w}^{r+r'}, \vec{v}_{s+1}, \dots, \vec{v}_{s+s'}).$$

5. [5 points] *Quadratic Forms:* Suppose that q is a nondegenerate, symmetric tensor in $T_2^0(V)$. Show that there exists a choice of basis \vec{e}_α of V for which the matrix of components $q_{\alpha\beta}$ is diagonal with all the diagonal components either +1 or -1. [Hint: one method is to use induction in the dimension of V .]