Physics 6553 : Problem Set 10

Due Tuesday, Nov 18, 2008

1. *Hydrodynamics of relativistic fluids:* [10 points] The stress-energy tensor for a relativistic fluid in terms of the 4-velocity u^a, pressure p and density ρ is

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab}, \tag{1}$$

and the local law of conservation of 4-momentum is

$$\nabla_a T^{ab} = 0. \tag{2}$$

a. By contracting (2) with the four velocity, and by also using the law of conservation of particle number $\nabla_a(nu^a) = 0$, where n is the number density of particles, deduce

$$\frac{d\rho}{d\tau} - \frac{\rho + p}{n}\frac{dn}{d\tau} = 0, \tag{3}$$

where $d/d\tau = u^a \nabla_a$ is derivative with respect to proper time along a fluid elements worldline.

b. By contracting the conservation law (2) with the projection tensor $h_{bc} = g_{bc} + u_b u_c$, deduce that

$$(\rho + p)a_c + \nabla_c p + u_c \frac{dp}{d\tau} = 0, \tag{4}$$

where a_c is the 4-acceleration.

c. Consider steady flows in the absence of gravity, for which there is some Lorentz frame (t, x^i) for which all the hydrodynamic variables are independent of t. Evaluate the time component of (4) in this frame, and combine with (3) to show that

$$\frac{d}{d\tau}\left[u^t \left(\frac{\rho + p}{n}\right)\right] = 0.$$

Show that the non-relativistic limit of this result is Bernouilli's theorem: that the quantity $\mathbf{v}^2/2 + u + p/\rho_M$ is conserved along flow lines, where u is the internal energy per unit mass and ρ_M is the mass density [related to ρ and n by $\rho = \rho_M(1 + u)$ and $\rho_M = mn$].

2. *Falling into a black hole:* [10 points] Consider an observer who starts from rest at large r in Schwarzschild spacetime. Suppose that the observer falls inward along a purely radial trajectory $\theta = \pi/2$, $\phi =$ constant.

a. Show that the observer reaches the coordinate location $r = 2M$ (the "horizon") in a finite amount of her proper time, but that it takes an infinite amount of coordinate time t for her to reach $r = 2M$.

b. As the observer's world line approaches $r = 0$, it asymptotes to the curve (t, θ, ϕ) =const, r variable. Explain why this is required by the light-cone structure near $r = 0$.

c. Show that the curve to which it asymptotes, (t, θ, ϕ) =const, r variable, is a timelike geodesic for $r < 2M$.

d. Show that the basis vectors of the infalling observer's local Lorentz frame near $r = 0$ are related to the Schwarzschild coordinate basis by

$$\vec{e}_0 = -\left(\frac{2M}{r} - 1\right)^{1/2}\frac{\partial}{\partial r}, \qquad \vec{e}_1 = -\left(\frac{2M}{r} - 1\right)^{-1/2}\frac{\partial}{\partial t},$$

$$\vec{e}_2 = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad \vec{e}_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

What are the components of the Riemann tensor in that local Lorentz frame?

e. Show that the tidal forces produced by the Riemann tensor stretch an infalling observer in the radial, $\vec{e}_1$, direction and squeeze the observer in the tangential, $\vec{e}_2$ and $\vec{e}_3$ directions; and show that the stretching and squeezing forces become infinitely strong as the observer approaches $r = 0$.

f. Idealize the body of an infalling observer to consist of a head of mass $\mu = 20$kg and feet of mass $\mu = 20$kg separated by a distance $h = 2$ meters, as measured in the observer's local Lorentz frame, and with the separation direction radial. Compute the stretching force between head and feet, as a function of proper time τ, as the observer falls into the singularity. Assume that the hole has the mass $M = 5 \times 10^9 M_\odot$ which is suggested by astronomical observations for a possible black hole at the center of the nearest giant elliptical galaxy to our own, the galaxy M87. How long before hitting the singularity (at what proper time τ) does the observer die, if he or she is a human being made of flesh, bone, and blood?

3. *Painlevé-Gullstrand coordinates in the Schwarzschild spacetime:* [10 points]

a. Consider an observer radially infalling into a Schwarzschild black hole, such that the velocity of infall starts at zero at $r = \infty$. Show that for such an observer, the four velocity can be written in terms of the gradient of a function

$$u^\alpha = -g^{\alpha\beta}\partial_\beta T_P$$

where

$$T_P(r,t) = t + \int dr \, \frac{\sqrt{1-w(r)}}{w(r)},$$

with $w(r) = 1 - 2M/r$. Here r is the usual Schwarzschild coordinate. Show that (up to an additive constant)

$$T_P = t + 4M\left[g(r) + \frac{1}{2}\ln\left|\frac{g(r)-1}{g(r)+1}\right|\right],$$

where $g(r) = \sqrt{r/(2M)}$.

b. Show that in the Painlevé-Gullstrand coordinates $(T_P, r, \theta, \varphi)$, the metric can be written as

$$ds^2 = -dT_P^2 + \left[dr + \sqrt{\frac{2M}{r}}dT_P\right]^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Show that these coordinates cover both regions I and II of the maximally extended Schwarzschild solution, and that the metric of the constant time surfaces $T_P = $ const is the flat Euclidean metric $dr^2 + r^2 d\Omega^2$. Note that the metric is regular at $r = 2M$.

c. Draw a diagram of the maximally extended Schwarzschild spacetime (in Kruskal coordinates), and sketch on that diagram several surfaces of constant T_P.