Physics 6553 : Problem Set 2

Due Thursday , Sept 17, 2008

1. [5 points] *Conserved Quantities in Special Relativity:* In lecture we defined the four momentum $\vec{P} = P^\alpha(t)\vec{e}_\alpha$ by the formula, in a given Lorentz frame $(t, \mathbf{x})$,

$$P^\alpha(t) = \int d^3x T^{\alpha 0}(t, \mathbf{x}),$$

where $T^{\alpha\beta}$ is the stress energy tensor that satisfies the local law of conservation of stress energy $\partial_\alpha T^{\alpha\beta} = 0$.

a. We showed in class that $\vec{P}$ is independent of t. Show also that it is independent of the Lorentz frame in which it is computed. You can assume that $T^{\alpha\beta}$ goes to zero at spatial infinity faster than $1/|\mathbf{x}|$.

b. Show that the antisymmetric tensor

$$J^{\alpha\beta}(t) = \int d^3x \left[x^\alpha T^{\beta 0}(t, \mathbf{x}) - x^\beta T^{\alpha 0}(t, \mathbf{x})\right]$$

is similarly conserved, and that the tensor $J^{\alpha\beta}\vec{e}_\alpha \otimes \vec{e}_\beta$ is frame independent. This is the total angular momentum of the system.

2. [10 points] *A Lorentz covariant, scalar theory of gravity:* Consider the theory of gravity discussed in lecture, where gravity is mediated by a scalar field Φ. The field equation for Φ is

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = -4\pi G T^\alpha{}_\alpha.$$

The equation of motion for a particle with rest mass m and four momentum $p^\alpha = dx^\alpha/d\lambda$ with $\vec{p}^2 = -m^2$ is

$$\frac{dp^\alpha}{d\lambda} = -(m^2\eta^{\alpha\beta} + p^\alpha p^\beta)\partial_\beta\Phi + mf^\alpha,$$

where f^α is the non-gravitational four force.

a. Show that the equation of motion conserves rest mass as long as $\vec{f} \cdot \vec{p} = 0$.

b. Show that a photon which passes near the Sun is not deflected by the Sun's gravitational field, by taking the $m \to 0$ limit of the equation of motion with $\vec{f} = 0$ and showing that in this limit the quantity $e^\Phi \vec{p}$ is conserved. This prediction is in disagreement with observations.

3. *Jacobi identity:* [5 points] On a manifold M, let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vector fields. Prove that

$$[\vec{u},[\vec{v},\vec{w}]] + [\vec{v},[\vec{w},\vec{u}]] + [\vec{w},[\vec{u},\vec{v}]] = 0.$$

4. *Geometric interpretation of commutator of vector fields:* [10 points] Let $\vec{v}$, $\vec{w}$ be any two vector fields on a manifold M. For any real number ε, define $\varphi_\varepsilon : M \to M$ to be the mapping which moves any point $\mathcal{P}$ along the integral curve of $\vec{v}$ which passes through $\mathcal{P}$ by an increment $\Delta s = \varepsilon$ of the curve's parameter s. Similarly define ψ_ε using $\vec{w}$ instead of $\vec{v}$.

a. Show that for sufficiently small ε and ε', φ_ε has the property that

$$\varphi_\varepsilon \circ \varphi_{\varepsilon'} = \varphi_{\varepsilon+\varepsilon'}.$$

[Hint: consider the properties of the differential equation which defines φ_ε]. Deduce that for sufficiently small ε, φ_ε is a bijective mapping from M to M (a "diffeomorphism").

b. Consider the mapping T_ε from M to M given by

$$T_\varepsilon = \varphi_{-\varepsilon} \circ \psi_{-\varepsilon} \circ \varphi_\varepsilon \circ \psi_\varepsilon. \tag{1}$$

Show that for any point $\mathcal{P}$ and any coordinate system x^α,

$$x^\alpha[T_\varepsilon(\mathcal{P})] = x^\alpha(\mathcal{P}) + k\varepsilon^2 s^\alpha(\mathcal{P}) + O(\varepsilon^3), \tag{2}$$

where k is a constant to be determined, and $\vec{s} = [\vec{v},\vec{w}]$. Draw a diagram illustrating the geometric meaning of Eqs. (1) and (2).