Physics 6553 : Problem Set 2
Due Thursday , Sept 13, 2012

Reading: Chapters 2 and 3 of Wald. You may also find it useful to consult chapters 9-11, 13-15 of MTW, sections 4.2 to 4.9 and chapter 10 of Weinberg, sections 2.2-2.5, 3.1-3.4, 3.6, 3.7 and 3.10 of Carroll, or chapters 2,7,8,20, 21.1 to 21.3 of Hartle.

1. [5 points] Conserved Quantities in Special Relativity: In lecture we defined the four momentum $\vec{P} = P^\alpha(t)\vec{e}_\alpha$ by the formula, in a given Lorentz frame $(t, \mathbf{x})$,

$$P^\alpha(t) = \int d^3x T^{\alpha0}(t, \mathbf{x}),$$

where $T^{\alpha\beta}$ is the stress energy tensor that satisfies the local law of conservation of stress energy $\partial_\alpha T^{\alpha\beta} = 0$.

a. We showed in class that $\vec{P}$ is independent of $t$. Show also that it is independent of the Lorentz frame in which it is computed. You can assume that $T^{\alpha\beta}$ goes to zero at spatial infinity faster than $1/|\mathbf{x}|$.

b. Show that the antisymmetric tensor

$$J^{\alpha\beta}(t) = \int d^3x \left[ x^{\alpha} T^{\beta0}(t, \mathbf{x}) - x^{\beta} T^{\alpha0}(t, \mathbf{x}) \right]$$

is similarly conserved, and that the tensor $J^{\alpha\beta} \vec{e}_\alpha \otimes \vec{e}_\beta$ is frame independent. This is the total angular momentum of the system.

2. [10 points] A Lorentz covariant, scalar theory of gravity: Consider the theory of gravity discussed in lecture, where gravity is mediated by a scalar field $\Phi$. The field equation for $\Phi$ is

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta \Phi = -4\pi G T^{\alpha}_{\alpha},$$

The equation of motion for a particle with rest mass $m$ and four momentum $p^\alpha = dx^\alpha/d\lambda$ with $p^2 = -m^2$ is

$$\frac{dp^\alpha}{d\lambda} = -(m^2 \eta^{\alpha\beta} + p^\alpha p^\beta) \partial_\beta \Phi + mf^\alpha,$$

where $f^\alpha$ is the non-gravitational four force.

a. Show that the equation of motion conserves rest mass as long as $\vec{f} \cdot \vec{p} = 0$.

b. Show that a photon which passes near the Sun is not deflected by the Sun’s gravitational field, by taking the $m \to 0$ limit of the equation of motion with $\vec{f} = 0$ and showing that in this limit the quantity $e^\Phi \vec{p}$ is conserved. This prediction is in disagreement with observations.
3. **Jacobi identity:** [5 points] On a manifold $M$, let $\vec{u}, \vec{v}$ and $\vec{w}$ be three vector fields. Prove that

$$ [\vec{u}, [\vec{v}, \vec{w}]] + [\vec{v}, [\vec{w}, \vec{u}]] + [\vec{w}, [\vec{u}, \vec{v}]] = 0. $$

4. **Geometric interpretation of commutator of vector fields:** [10 points] Let $\vec{v}, \vec{w}$ be any two vector fields on a manifold $M$. For any real number $\varepsilon$, define $\varphi_\varepsilon : M \to M$ to be the mapping which moves any point $P$ along the integral curve of $\vec{v}$ which passes through $P$ by an increment $\Delta s = \varepsilon$ of the curve’s parameter $s$. Similarly define $\psi_\varepsilon$ using $\vec{w}$ instead of $\vec{v}$.

   a. Show that for sufficiently small $\varepsilon$ and $\varepsilon'$, $\varphi_\varepsilon$ has the property that

   $$ \varphi_\varepsilon \circ \varphi_{\varepsilon'} = \varphi_{\varepsilon + \varepsilon'}.$$

   [Hint: consider the properties of the differential equation which defines $\varphi_\varepsilon$.] Deduce that for sufficiently small $\varepsilon$, $\varphi_\varepsilon$ is a bijective mapping from $M$ to $M$ (a “diffeomorphism”).

   b. Consider the mapping $T_\varepsilon$ from $M$ to $M$ given by

   $$ T_\varepsilon = \varphi_{-\varepsilon} \circ \psi_{-\varepsilon} \circ \varphi_\varepsilon \circ \psi_\varepsilon. $$

   (1)

   Show that for any point $P$ and any coordinate system $x^\alpha$,

   $$ x^\alpha[T_\varepsilon(P)] = x^\alpha(P) + k\varepsilon^2 s^\alpha(P) + O(\varepsilon^3), $$

   (2)

   where $k$ is a constant to be determined, and $\vec{s} = [\vec{v}, \vec{w}]$. Draw a diagram illustrating the geometric meaning of Eqs. (1) and (2).