Physics 6553 : Problem Set 3

Due Thursday , Sept 24, 2008

1. [10 points] *Local representations of metrics:*

a. Let $\mathcal{P}$ be a point in a manifold M, and let $\vec{e}_\alpha(\mathcal{P})$ be an arbitrary basis of the tangent space $T_\mathcal{P}(M)$. Show that one can always find a coordinate system $\{x^\alpha\}$ such that

$$\vec{e}_\alpha(\mathcal{P}) = \left(\frac{\partial}{\partial x^\alpha}\right)_\mathcal{P}.$$

b. By combining part a. with Q5 from homework 1, show that if g is a metric on a manifold M, given a point $\mathcal{P}$, one can always find coordinates x^α such that $g_{\alpha\beta}(\mathcal{P})$ is diagonal with diagonal elements ± 1.

c. Show that one can further specialize the choice of coordinate system $\{x^\alpha\}$ such that

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}(\mathcal{P}) = 0.$$

Such coordinates are called, in the context of general relativity, local Lorentz coordinates adapted to the point $\mathcal{P}$. [Hint: start with a general coordinate system with $x^\alpha(\mathcal{P}) = 0$, use a coordinate transformation of the form

$$y^\alpha(x^\beta) = a^\alpha + b^\alpha{}_\beta x^\beta + c^\alpha{}_{\beta\gamma} x^\beta x^\gamma + O(x^3),$$

compute the metric in the new coordinate system, and choose the constants a^α, $b^\alpha{}_\beta$ and $c^\alpha{}_{\beta\gamma}$ suitably].

2. [10 points] *Differentiating tensors:*

a. Let f be a function and $\vec{v} = v^\alpha \partial_\alpha$ be a vector field on a manifold M. Define, in each coordinate system $\{x^\alpha\}$, the quantities

$$w_\alpha = f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}$$

and

$$s^\alpha{}_\beta = v^\alpha{}_{,\beta} \equiv \frac{\partial v^\alpha}{\partial x^\beta}.$$

Show that w_α is a tensor, i.e., that it transforms between coordinate systems according to the tensor transformation law, but that $s^\alpha{}_\beta$ does not.

b. Let M be a manifold, and let x^α and $x^{\bar\alpha}$ be two overlapping coordinate systems defined on M. Let ∇ be a connection on M. Show that the coefficients of the connection $\Gamma^\alpha_{\beta\gamma}$ on the coordinate system x^α are related to the corresponding coefficients $\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma}$ by

$$\Gamma^{\bar\alpha}_{\bar\beta\bar\gamma} = \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\bar\beta}}\frac{\partial x^\gamma}{\partial x^{\bar\gamma}}\Gamma^\alpha_{\beta\gamma} + \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^{\bar\beta}\partial x^{\bar\gamma}}.$$

3. [6 points] *Non-coordinate Bases:*

a. Give an example of two linearly independent, nowhere-vanishing vector fields in $\mathbf{R}^2$ whose commutator does not vanish. Such fields form a basis of the tangent space at each point, but it is not a coordinate basis for any choice of coordinates. Can you find a similar basis on the 2-sphere?

b. Let $\vec{e}_{\hat\alpha} = e^\mu_{\hat\alpha}\partial_\mu$ be a non-coordinate basis with dual basis of 1 forms $\theta^{\hat\alpha}$, and define the connection coefficients $\Gamma^{\hat\alpha}_{\hat\beta\hat\gamma}$ by

$$\nabla \vec{e}_{\hat\alpha} = \Gamma^{\hat\beta}_{\hat\alpha\hat\gamma}\, \vec{e}_{\hat\beta} \otimes \theta^{\hat\gamma}.$$

Show that for any vector field $\vec{v} = v^{\hat\alpha}\vec{e}_{\hat\alpha}$,

$$(\nabla \vec{v})^{\hat\beta}_{\hat\gamma} = \left(e^\sigma_{\hat\gamma}\partial_\sigma\right) v^{\hat\beta} + \Gamma^{\hat\beta}_{\hat\alpha\hat\gamma} v^{\hat\alpha}.$$