

# Physics 6553 : Problem Set 5

Due Thursday , Oct 9, 2008

1. *Curvature on a Non-coordinate Bases:* [10 points] Consider a manifold together with a set of vectors  $\vec{e}_{\hat{\alpha}}$  that form a basis at every point. The components of the metric on this basis are  $g_{\hat{\alpha}\hat{\beta}}$ . The connection coefficients  $\Gamma_{\hat{\beta}\hat{\alpha}}^{\hat{\gamma}}$  are defined by the equation  $\nabla_{\vec{e}_{\hat{\alpha}}} \vec{e}_{\hat{\beta}} = \Gamma_{\hat{\beta}\hat{\alpha}}^{\hat{\gamma}} \vec{e}_{\hat{\gamma}}$ , and the commutation coefficients  $c_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}$  are defined by  $[\vec{e}_{\hat{\alpha}}, \vec{e}_{\hat{\beta}}] = c_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} \vec{e}_{\hat{\gamma}}$ .

- a. Using the identity  $\nabla_{\vec{u}} \vec{v} - \nabla_{\vec{v}} \vec{u} = [\vec{u}, \vec{v}]$ , show that  $c_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} = \Gamma_{\hat{\beta}\hat{\alpha}}^{\hat{\gamma}} - \Gamma_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}}$ .
- b. By acting on  $g_{\hat{\alpha}\hat{\beta}} = \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}}$  with the differential operator  $\nabla_{\vec{e}_{\hat{\gamma}}}$ , show that

$$\Gamma_{\hat{\alpha}\hat{\gamma}}^{\hat{\sigma}} g_{\hat{\sigma}\hat{\beta}} + \Gamma_{\hat{\beta}\hat{\gamma}}^{\hat{\sigma}} g_{\hat{\sigma}\hat{\alpha}} = g_{\hat{\alpha}\hat{\beta}, \hat{\gamma}},$$

where the quantity on the right hand side is  $\vec{e}_{\hat{\gamma}}(g_{\hat{\alpha}\hat{\beta}})$ .

- c. By combining the results of parts a. and b., show that the connection coefficients are given by

$$\Gamma_{\hat{\beta}\hat{\alpha}}^{\hat{\gamma}} = \frac{1}{2} g^{\hat{\alpha}\hat{\mu}} \left[ g_{\hat{\mu}\hat{\beta}, \hat{\gamma}} + g_{\hat{\mu}\hat{\gamma}, \hat{\beta}} - g_{\hat{\beta}\hat{\gamma}, \hat{\mu}} + c_{\hat{\mu}\hat{\beta}\hat{\gamma}} + c_{\hat{\mu}\hat{\gamma}\hat{\beta}} - c_{\hat{\beta}\hat{\gamma}\hat{\mu}} \right]$$

where  $c_{\hat{\alpha}\hat{\beta}\hat{\gamma}} \equiv g_{\hat{\mu}\hat{\gamma}} c_{\hat{\alpha}\hat{\beta}}^{\hat{\mu}}$ . The first three terms in square brackets vanish for the case of an orthonormal basis, where  $g_{\hat{\alpha}\hat{\beta}}$  are constants, and the last three terms vanish for the case of a coordinate basis, for which  $c_{\hat{\alpha}\hat{\beta}}^{\hat{\gamma}} = 0$ .

- d. Show that the components of the Riemann tensor are

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}}^{\hat{\delta}} = \Gamma_{\hat{\gamma}\hat{\alpha}, \hat{\beta}}^{\hat{\delta}} - \Gamma_{\hat{\gamma}\hat{\beta}, \hat{\alpha}}^{\hat{\delta}} + \Gamma_{\hat{\gamma}\hat{\alpha}}^{\hat{\sigma}} \Gamma_{\hat{\sigma}\hat{\beta}}^{\hat{\delta}} - \Gamma_{\hat{\gamma}\hat{\beta}}^{\hat{\sigma}} \Gamma_{\hat{\sigma}\hat{\alpha}}^{\hat{\delta}} + c_{\hat{\alpha}\hat{\beta}}^{\hat{\sigma}} \Gamma_{\hat{\gamma}\hat{\sigma}}^{\hat{\delta}}.$$

Note that, unlike the case for a coordinate basis, the ordering of the indices on the connection coefficients in this formula is important, since the connection coefficients are no longer symmetric.

[Hint: You should be able to do this problem without ever referring to a coordinate system or to coordinate components of tensors.]

2. [10 points] *Spherically symmetric spacetime:* Consider the spacetime with coordinates  $(t, r, \theta, \varphi)$  and metric

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

We define an orthonormal basis by  $\vec{e}_{\hat{t}} = e^{-\Phi} \partial_t$ ,  $\vec{e}_{\hat{r}} = e^{-\Lambda} \partial_r$ ,  $\vec{e}_{\hat{\theta}} = (1/r) \partial_\theta$  and  $\vec{e}_{\hat{\varphi}} = (1/r \sin \theta) \partial_\varphi$ . Show that the nonvanishing, orthonormal-basis components of the Einstein tensor are

$$G_{\hat{t}\hat{t}} = \frac{1}{r^2} \frac{d}{dr} [r(1 - e^{-2\Lambda})], \quad G_{\hat{r}\hat{r}} = -\frac{1}{r^2} (1 - e^{-2\Lambda}) + \frac{2}{r} e^{-2\Lambda} \Phi'(r),$$

and

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \frac{1}{r} e^{-2\Lambda} [\Phi' + r(\Phi')^2 - \Lambda'(1 + r\Phi') + r\Phi''].$$

**3.** [10 points] *Geometric interpretation of the Riemann Tensor:* Let  $S$  be a two dimensional surface in a manifold parameterized by two coordinates  $s$  and  $t$ , so that  $x^\alpha = x^\alpha(t, s)$ . Let  $\mathcal{P}$  be the point  $s = 0, t = 0$ , and for any  $\varepsilon > 0$  consider the closed curve  $\Gamma_\varepsilon$  in  $M$  consisting of the four segments  $x^\alpha = x^\alpha(u)$  for  $0 \leq u \leq \varepsilon$  with

$$(i) \quad x^\alpha(u) = x^\alpha(u, 0), \quad (ii) \quad x^\alpha(u) = x^\alpha(\varepsilon, u), \\ (iii) \quad x^\alpha(u) = x^\alpha(\varepsilon - u, \varepsilon), \quad (iv) \quad x^\alpha(u) = x^\alpha(0, \varepsilon - u).$$

That is,  $\Gamma_\varepsilon$  starts at  $\mathcal{P}$ , moves along the  $s = 0$  curve, then along the  $t = \varepsilon$  curve, then back along the  $s = \varepsilon$  curve, then completes the quadrilateral back to  $\mathcal{P}$ . Let  $\Lambda_\varepsilon$  be the holonomy of the curve  $\Gamma_\varepsilon$ , that is, the linear map from  $T_{\mathcal{P}}(M)$  to  $T_{\mathcal{P}}(M)$  obtained by parallel transporting vectors around  $\Gamma_\varepsilon$ . Show that for any vector  $v^\alpha$  at  $\mathcal{P}$ ,

$$(\Lambda_\varepsilon \vec{v})^\alpha = v^\alpha + \varepsilon^2 R_{\gamma\beta\delta}{}^\alpha t^\gamma s^\beta v^\delta + O(\varepsilon^3),$$

where  $s^\alpha$  is the vector  $\partial/\partial s$  at  $\mathcal{P}$  and  $t^\alpha$  is the vector  $\partial/\partial t$  at  $\mathcal{P}$ .

**4.** [5 points] *Properties of the Riemann tensor:* Consider tensors over an  $n$ -dimensional vector space  $V$  with  $n \geq 3$ . Let  $\mathcal{S}$  be the space of  $(0, 4)$  tensors  $R_{abcd}$  satisfying the symmetries  $R_{(ab)cd} = 0$  and  $R_{abcd} = R_{cdab}$ . In lecture we argued that the dimension of  $\mathcal{S}$  is  $N(N+1)/2$  where  $N = n(n-1)/2$ . Let  $\mathcal{R}$  be the subspace of  $\mathcal{S}$  consisting of tensors that satisfy the additional symmetry  $R_{[abc]d} = 0$ ; the dimension of  $\mathcal{R}$  is then the number of independent components of the Riemann tensor.

- Show that  $\mathcal{S} = \mathcal{R} \oplus \mathcal{T}$ , where  $\mathcal{T}$  is the space of completely antisymmetric  $(0, 4)$  tensors on  $V$ . [Hint: show that every element of  $\mathcal{S}$  can be expressed uniquely as the sum of an element of  $\mathcal{R}$  and an element of  $\mathcal{T}$ .]
- Argue that the dimension of  $\mathcal{T}$  is  $n(n-1)(n-2)(n-3)/4!$ , and deduce that the number of independent components of the Riemann tensor is  $n^2(n^2-1)/12$ .
- Show that the Weyl tensor  $C_{abcd}$  defined by

$$R_{abcd} = C_{abcd} + \frac{2}{n-2} [g_{a[c}R_{d]b} - g_{b[c}R_{d]a}] - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}$$

has the same symmetry properties as the Riemann tensor  $R_{abcd}$  and is traceless on all pairs of indices.