Physics 6553 : Problem Set 6

Due Thursday, Oct 16, 2008

1. *The Newtonian limit of general relativity:* [10 points] The Newtonian limit is described by a metric of the form

$$ds^2 = -\frac{1}{\varepsilon^2}\left[1 + 2\varepsilon^2\Phi(t,\mathbf{x}) + O(\varepsilon^4)\right]dt^2 + O(\varepsilon^2)dx^i dt + \left[\delta_{ij} + O(\varepsilon^2)\right]dx^i dx^j, \tag{1}$$

where ε is an expansion parameter (essentially a typical velocity of the system divided by the speed of light). Here $\Phi(t,\mathbf{x})$ is the Newtonian potential.

a. Show that the connection coefficients for this metric are of the form $\Gamma^\alpha_{\beta\gamma} = {}^{(0)}\Gamma^\alpha_{\beta\gamma} + O(\varepsilon^2)$, where the Newtonian connection coefficients ${}^{(0)}\Gamma^\alpha_{\beta\gamma}$ are given by ${}^{(0)}\Gamma^i_{tt} = \partial_i\Phi$, with all the other components being zero.

b. By transforming from proper time derivatives to coordinate time derivatives, show that the equation for timelike geodesics can be written as

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{tt} - 2\Gamma^i_{tj}v^j - \Gamma^i_{jk}v^j v^k + \left[\Gamma^t_{tt} + 2\Gamma^t_{tj}v^j + \Gamma^t_{jk}v^j v^k\right]v^i$$

where $v^i = dx^i/dt$. Deduce that the geodesic equation in the Newtonian connection ${}^{(0)}\Gamma^\alpha_{\beta\gamma}$ coincides with the equation of motion of Newtonian gravity.

c. Show that the components of the Ricci tensor are $R_{tt} = \nabla^2\Phi + O(\varepsilon^2)$, $R_{ti} = O(\varepsilon^2)$, and $R_{ij} = O(\varepsilon^2)$.

2. *Geodesic Deviation in Newtonian Gravity:* [5 points] Consider two freely falling observers in Newtonian gravity, whose worldlines are $\mathbf{x} = \mathbf{x}_1(t)$ and $\mathbf{x} = \mathbf{x}_2(t)$. Let $\mathbf{\Delta x}(t) \equiv \mathbf{x}_2(t) - \mathbf{x}_1(t)$ be the separation vector from one observer to the other.

a. Derive from Newtonian gravity the differential equation

$$\frac{d^2\Delta x^i}{dt^2} = \mathcal{E}^i{}_j(t)\Delta x^j(t) \tag{2}$$

which is accurate to first order in $\mathbf{\Delta x}$, where the quantity $\mathcal{E}_{ij}$, called the tidal force tensor, is given by

$$\mathcal{E}_{ij}(t) = -\frac{\partial^2\Phi}{\partial x^i\partial x^j}[\mathbf{x}_1(t), t]. \tag{3}$$

b. We now re-derive this result from general relativity. Starting from the metric (1) in question 1 describing Newtonian gravitational fields, show that $R_{titj} = \Phi_{,ij} + O(\varepsilon^2)$. Insert these Riemann tensor components and the metric (1) into the orthonormal-basis version of the geodesic deviation equation and expand to the leading order in ε to obtain equation (2).

3. *Gauge freedom in the Newtonian Limit:* [15 points] There is a subgroup of the full group of coordinate transformations that preserves the form (1) of the Newtonian metric. Starting from the coordinates (t, x^i) and the metric (1), we transform to another coordinate system $(\bar{t}, \bar{x}^i)$, and we assume that the metric in these coordinates takes the form

$$ds^2 = -\frac{1}{\varepsilon^2}\left[1 + 2\varepsilon^2\bar{\Phi}(\bar{t}, \bar{x}^j) + O(\varepsilon^4)\right] d\bar{t}^2 + O(\varepsilon^2) d\bar{x}^i d\bar{t} + \left[\delta_{ij} + O(\varepsilon^2)\right] d\bar{x}^i d\bar{x}^j, \tag{4}$$

where $\bar{\Phi}$ is the transformed Newtonian potential. The most general coordinate transformation that achieves this (up to constant displacements in time and to time-independent spatial rotations) is

$$x^i(\bar{t}, \bar{x}^j) = \bar{x}^i + z^i(\bar{t}) + O(\varepsilon^2), \qquad t(\bar{t}, \bar{x}^j) = \bar{t} + \varepsilon^2\left[\beta(\bar{t}) + \dot{z}^i(\bar{t})\bar{x}^i\right] + O(\varepsilon^4), \tag{5}$$

where $\beta(\bar{t})$ and $z^i(\bar{t})$ are an arbitrary functions of time. This represents a transformation to an accelerated reference frame. The transformations with $\ddot{z}^i = 0$ are Galilean transformations, parameterized by the constant velocity $\dot{z}^i$.

a. Show that the new potential is given by

$$\bar{\Phi}(\bar{t}, \bar{x}^j) = \Phi[\bar{t}, \bar{x}^j + z^j(\bar{t})] + \dot{\beta}(\bar{t}) + \ddot{z}^i(\bar{t})\bar{x}_i - \frac{1}{2}\dot{z}^i(\bar{t})\dot{z}_i(\bar{t}).$$

b. To derive the coordinate transformation (5), argue as follows. Let the coordinate transformation to zeroth order in ε be $x^i = x^i(\bar{t}, \bar{x}^j) + O(\varepsilon^2)$, $\quad t = t(\bar{t}, \bar{x}^j) + O(\varepsilon^2)$. Substituting this into the metric expansion (1), show that the leading order expression for the spatial metric is

$$-\frac{1}{\varepsilon^2}\frac{\partial t}{\partial \bar{x}^i}\frac{\partial t}{\partial \bar{x}^j} d\bar{x}^i d\bar{x}^j + O(1).$$

This is in conflict with the expansion (4) unless $\partial t/\partial \bar{x}^i = 0$. Similarly, the leading order expression for the time-time piece of the line element is

$$-\frac{1}{\varepsilon^2}\left(\frac{\partial t}{\partial \bar{t}}\right)^2 d\bar{t}^2 + O(1),$$

which disagrees with the expansion (4) unless $\partial t/\partial \bar{t} = \pm 1$. Assuming that the coordinate transformation preserves the time orientation and neglecting constant displacements in time, deduce that $t = \bar{t} + O(\varepsilon^2)$. Write this relation as $t = \bar{t} + \varepsilon^2\alpha(\bar{t}, \bar{x}^j) + O(\varepsilon^4)$, where the function $\alpha(\bar{t}, \bar{x}^j)$ is as yet undetermined.

c. Show that the leading order expression for the spatial metric is now

$$\delta_{kl}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} d\bar{x}^i d\bar{x}^j + O(\varepsilon^2) = \delta_{ij} d\bar{x}^i d\bar{x}^j + O(\varepsilon^2),$$

Deduce that, for each fixed $\bar{t}$, the function $x^i = x^i(\bar{t}, \bar{x}^j)$ is an isometry of 3-dimensional Euclidean space (i.e. a map that preserves the flat metric δ_{ij}). It follows that this map is thus of the form

$$x^i = R^i{}_j(\bar{t})\bar{x}^j + z^i(\bar{t}) + O(\varepsilon^2)$$

for some time-dependent rotation matrix $R^i{}_j(\bar{t})$ and some time-dependent displacement $z^i(\bar{t})$.

d. Show that the leading order expression for the space-time piece of the line element is now

$$\left\{2\delta_{ik}R^k{}_l(\bar{t})\left[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})\right] - 2\frac{\partial \alpha}{\partial \bar{x}^l}\right\} d\bar{t} d\bar{x}^l + O(\varepsilon^2).$$

and argue that the first term here must vanish in order to be compatible with (4), which gives

$$\delta_{ik}R^k{}_l(\bar{t})\left[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})\right] = \frac{\partial \alpha}{\partial \bar{x}^l}. \tag{6}$$

Show that if $\dot{R}^i{}_j(\bar{t})$ is non-vanishing, it is impossible to find any function $\alpha(\bar{t}, \bar{x}^j)$ which satisfies this equation. Deduce that the rotation matrix is time-independent, and specialize the new coordinate system $\bar{x}^i$ so that $R^i{}_j = \delta^i_j$. Finally solve Eq. (6) for $\alpha(\bar{t}, \bar{x}^j)$ to obtain the coordinate transformation (5).