Physics 6553 : Problem Set 9

Due Thursday, Nov 6, 2008

1. *Conservation of Energy for Newtonian Fluids:* [5 points] The energy density of a Newtonian fluid, neglecting gravity, is

$$\mathcal{E} = \rho_M \left(\frac{1}{2}\mathbf{v}^2 + u \right)$$

where u is the internal energy per unit mass, ρ_M is mass density and $\mathbf{v}$ is velocity. One might expect the energy flux to be be $\mathcal{F} = \mathcal{E}\mathbf{v}$. However this expression omits an important contribution. Consider an element of surface area dA orthogonal to the fluid velocity $\mathbf{v}$. A fluid element that crosses dA during a time dt moves through a distance $dl = vdt$, and as it moves, the fluid behind this element exerts a force pdA on it. That force, acting through the distance dl, feeds an energy $dE = (pdA)dl = pvdAdt$ across dA; the corresponding energy flux across dA has magnitude $dE/(dAdt) = pv$ and points in the $\mathbf{v}$ direction, so it contributes $p\mathbf{v}$ to the energy flux $\mathcal{F}$. Thus the total energy flux is

$$\mathcal{F} = \rho_M \mathbf{v} \left(\frac{1}{2}\mathbf{v}^2 + u + p/\rho_M \right).$$

In this problem you will derive the local law of energy conservation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F} = \rho_M T \frac{ds}{dt}, \tag{2}$$

where $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the comoving time derivative, T is temperature and s is entropy per unit mass.

a. By combining the Euler and continuity equations, show that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_M v^2 \right) + \nabla \cdot \left(\frac{1}{2}\rho_M v^2 \mathbf{v} \right) = -(\mathbf{v} \cdot \nabla)p.$$

b. By combining the first law of thermodynamics in the form $du = Tds - pd(1/\rho_M)$ with the definition $h = u + p/\rho_M$ of the enthalpy h per unit mass, show that $dp = \rho_M dh - \rho_M T ds$.

c. Use the result of part b to eliminate the gradient of pressure term on the right hand side of part a. Now use the continuity equation and the first law of thermodynamics again to derive the energy conservation equation (2).

2. *Newtonian Limit of Fluid Equations:* [10 points] Consider a Newtonian spacetime described by the metric

$$ds^2 = -c^2\left[1 + 2\Phi(t,\mathbf{x})/c^2 + O(c^{-4})\right]dt^2 + O(c^{-2})dx^i dt$$
$$+ \left[\delta_{ij} - 2\Phi(t,\mathbf{x})\delta_{ij}/c^2 + O(c^{-4})\right]dx^i dx^j, \tag{1}$$

where Φ is the Newtonian potential. For a Newtonian fluid with mass density ρ_M, pressure p and 3-velocity v^i, argue that the components of the stress energy tensor in the (t, x^i) coordinate system are, to Newtonian order,

$$T^{tt} = \rho_M, \qquad T^{ti} = \rho_M v^i, \qquad T^{ij} = \rho_M v^i v^j + p\delta^{ij}.$$

[Hint: write down the stress tensor in the rest frame of the fluid. Then boost into the lab frame and take the $c \to \infty$ limit of the Lorentz transformation.] Using stress-energy conservation $\nabla_\alpha T^{\alpha\beta} = 0$, the Newtonian metric (1), and working to leading order in $1/c^2$ deduce the equations of Newtonian hydrodynamics: the continuity equation

$$\dot{\rho} + \partial_i(\rho v^i) = 0,$$

and Euler's equation

$$\dot{v}^i + v^j \partial_j v^i = -(\partial_i p)/\rho - \partial_i \Phi.$$

Here dots indicate $\partial/\partial t$.

3. *Numerical Models of Neutron Stars:* [10 points] Neutron stars are configurations of cold matter at the endpoint of thermonuclear evolution having central densities roughly in the range $10^{14}\,\mathrm{g\,cm^{-3}} < \rho < 10^{16}\,\mathrm{g\,cm^{-3}}$. In this density range (which is about the density of an atomic nucleus), the bulk of the material in the star can be roughly approximated as a degenerate Fermi gas of neutrons. The equation of state for such a gas takes the form

$$p = \frac{3^{2/3}\pi^{4/3}\hbar^2}{5m_{\mathrm{n}}^{8/3}}\rho^{5/3},$$

where $\hbar$ is the reduced Planck constant and m_{n} is the neutron mass. Integrate the Tolman-Oppenheimer-Volkoff equation numerically using Mathematica, Maple, or whatever else you prefer. Compute the total mass M (in Solar Masses) and the total radius R (in km) for a sequence of central densities ρ_{c}; e.g., $\rho_{\mathrm{c}} = 10^{14}, 3\times10^{14}, 10^{15}, 3\times10^{15}$, and $10^{16}\,\mathrm{g\,cm^{-3}}$. Plot the resulting mass-radius relation (analogous to Fig. 6.1 of Wald) for stellar models based on this equation of state.