Physics 6554 : Problem Set 10
Due Thursday, April 18, 2013

1. [10 points] Decomposition of the 4D covariant derivative in terms of 3D derivatives:
Suppose that spacetime is foliated by a set of spacelike hypersurfaces, on which the unit timelike normal is \( \vec{n} \). We define \( P_{\alpha \beta} = g_{\alpha \beta} + n_\alpha n_\beta \) to be the projection tensor, \( a^\alpha = n^\beta \nabla_\beta n^\alpha \) to the acceleration of the normal vector, \( K_{\alpha \beta} = P_\alpha ^\gamma P_\beta ^\delta \nabla_\gamma n_\delta \) to be the extrinsic curvature tensor, and \( D_\alpha \) to be the intrinsic derivative operator in each hypersurface. Given any dual vector \( \upsilon_\alpha \), we can decompose it uniquely as \( \upsilon_\alpha = v_n_\alpha + w_\alpha \), where \( v = -n_\alpha v^\alpha \) and \( w_\alpha n^\alpha = 0 \). Derive the following formula for the covariant derivative of the dual vector:
\[
\nabla_\gamma \upsilon_\beta = [D_\gamma w_\beta + v K_{\gamma \beta}] + n_\gamma \left[ -(\mathcal{L}_{\vec{n}} w)_\beta + K_{\gamma \sigma} w_\sigma - v a_\beta \right] + \left[ D_\gamma v + K_{\gamma \sigma} w_\sigma \right] n_\beta - n_\gamma n_\beta \left[ a^\sigma w_\sigma + \mathcal{L}_{\vec{n}} v \right].
\]

2. Physics of a Thin Spherical Shell: Consider an initial data set for Einstein equations of the following form. The extrinsic curvature tensor and matter current vanish, the 3-metric is
\[
ds^2 = \psi(r)^4 \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right],
\]
where \( \psi(r) \to 1 \) as \( r \to \infty \), and the mass density describes a thin shell of coordinate radius \( R \): \( \rho(r) = \rho_0 \delta(r - R) \).

a. Solve the initial value equations to show that \( \psi = 1 + M/(2r) \) for \( r \geq R \), where \( M \) is a constant, and that \( \psi = 1 + M/(2R) \) for \( r \leq R \). By comparing with the Newtonian limit in the limit \( r \to \infty \) argue that \( M \) is the ADM mass.

b. Show that the proper (physical) radius of the shell is
\[
R_p = \left( 1 + \frac{M}{2R} \right)^2 R.
\]

c. Define \( M_p \) to be the integral of the mass density with respect to proper volume, and define the gravitational binding energy to be \( E_{\text{bind}} = M - M_p \). Show that this binding energy is given by
\[
\frac{E_{\text{bind}}}{M} = \frac{\lambda - 1 + \sqrt{1 - 2\lambda}}{\lambda},
\]
where \( \lambda = M/R_p \). Show that this reduces to the expected answer in the Newtonian limit: \( E_{\text{bind}} = -M^2/(2R_p) + \cdots \).

d. For a shell of given ADM mass \( M \), what is the smallest possible value of proper size \( R_p \)?

3. Arnowitt-Deser-Misner (ADM) 3+1 decomposition: Consider a spacetime with coordinates \((t,x^i)\) and with metric
\[
ds^2 = -\alpha^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).
\]
Here the lapse function \( \alpha \), the shift vector \( \beta^i \) and the spatial metric \( h_{ij} \) are arbitrary functions of space and time.

a. Show that the components of the inverse metric are \( g^{tt} = -1/\alpha^2 \), \( g^{ti} = \beta^i/\alpha^2 \), and \( g^{ij} = h^{ij} - \beta^i \beta^j/\alpha^2 \), where \( h^{ij} h_{jk} = \delta^i_k \).

b. Show that the unit, future directed normal vector to the \( t = \) constant hypersurfaces is \( \vec{n} = (\partial_t - \beta^i \partial_i)/\alpha \), and that the corresponding 1-form is \( \textbf{n} = -\alpha dt \).

c. Show that the extrinsic curvature tensor \( K_{ij} \) is given by \( 2\alpha K_{ij} = \dot{h}_{ij} - D_i \beta_j - D_j \beta_i \), where \( D_i \) is the intrinsic derivative operator associated with the metric \( h_{ij} \) and dots denote derivatives with respect to \( t \).

d. The acceleration of the unit normals is defined by \( a^\alpha = n^\beta \nabla_\beta n^\alpha \). Show that the contravariant components of the acceleration are \( a^t = 0 \), \( a^i = D^i(\ln \alpha) \).