Physics 6554 : Problem Set 6
Due Thursday, March 14, 2013

1. Asymptotic Energy in Static Spacetimes: Consider a static spacetime of the form
\[ ds^2 = -\alpha(x^k)^2 dt^2 + h_{ij}(x^k)dx^idx^j. \]
Assume that the stress energy tensor vanishes near spatial infinity. We define \( \vec{\xi} = \partial/\partial t \) to be the timelike Killing vector field, and \( \vec{n} = \vec{\xi}/\alpha \) to be the unit normal to the spatial slices \( t = \text{constant} \).

a. Show that the asymptotic energy defined by
\[ E = -\frac{1}{8\pi} \int_S \epsilon_{abcd} \nabla^c \xi^d, \]
where \( S \) is any 2-surface near spacelike infinity, is given by
\[ E = 2 \int d^3x \sqrt{-h} (T_{cd} - g_{cd}T_{ab}g^{ab}/2) n^c \xi^d. \]

b. What does this formula reduce to in the Newtonian limit?

2. Noether’s Theorem: For a Lagrangian 4-form \( L \) which depends on some background fields \( \psi \) and dynamical fields \( \phi \), we defined in lecture the 4-form \( E \) and the symplectic potential 3-form \( \Theta \) by considering the variation \( \phi \to \phi + \delta \phi \), for which
\[ \delta L(\psi, \phi, \delta \phi) = E(\psi, \phi) \cdot \delta \phi + d\Theta(\psi, \phi, \delta \phi). \]
For any vector field \( \vec{\xi} \) with respect to which the Lie derivatives \( L_{\vec{\xi}} \psi \) of the background fields vanish, we argued that the Noether current 3-form \( j = \Theta(\psi, \phi, L_{\vec{\xi}} \phi) - \vec{\xi} \cdot L \) is a closed form, corresponding to a conserved current.

a. Take the background field \( \psi \) to be a flat, Minkowski metric \( \eta_{ab} \), the dynamical field \( \phi \) to be a scalar field, and the Lagrangian 4-form to be \( L = \epsilon_{\alpha\beta\gamma\delta} L(\phi, \partial_\alpha \phi) \) for some Lagrangian density \( L \). Let \( \vec{\xi} \) be any Killing vector of Minkowski spacetime. Show that the Hodge dual of the Noether current obtained from the above version of Noether’s theorem is just (up to a constant) \( T^\alpha \xi_\alpha \), where \( T^\alpha_\beta \) is the canonical stress energy tensor obtained from the Lagrangian \( L \),
\[ T^\alpha_\beta = -\partial^\alpha \phi \frac{\partial L}{\partial (\partial_\beta \phi)} + \eta^\alpha_\beta L. \]

b. Now take the Lagrangian to be the Einstein-Hilbert action for general relativity with no matter, and compute the quantities \( E = F_{abcd} \) and \( \Theta = \Theta_{abc} \) explicitly.

c. Next take \( \vec{\xi} \) to be an arbitrary vector field. Show that the Noether current for general relativity is proportional to (*\( j^a = -\nabla^a \nabla_b \xi^b + \nabla^b \nabla_b \xi^a \)). Show explicitly that this current is conserved whenever the equations of motion \( G_{ab} = 0 \) are satisfied.
3. Covariant form of Newtonian gravity:

In this problem we will derive a coordinate-independent form of Newtonian gravity, called Newton-Cartan theory. We define the dimensionless parameter $\varepsilon = 1/c^2 = L^2/c^2T^2$, where $L$ and $T$ are the lengthscale and timescale characterizing some isolated physical system. We consider a one-parameter family of metrics $g_{ab}(\varepsilon)$ and stress-energy tensors $T^{ab}(\varepsilon)$ that satisfy Einstein's equation (in units with $L = T = 1$) $G^{ab}[g_{cd}(\varepsilon)] = 8\pi\varepsilon^2T^{ab}(\varepsilon)$. We assume that the stress-energy tensor and connection $\nabla_a$ can be expanded as power series in $\varepsilon$ near $\varepsilon = 0$:

$$T^{ab}(\varepsilon) = T^{ab} + O(\varepsilon), \quad \nabla_a = D_a + O(\varepsilon)$$

where $T^{ab}$ is the Newtonian stress-energy tensor and $D_a$ is the Newtonian connection. We also assume that the contravariant metric can be similarly expanded:

$$g^{ab}(\varepsilon) = h^{ab} + \varepsilon k^{ab} + O(\varepsilon^2),$$

where $h^{ab}$ has signature $(0,+,+,+)$. We denote by $t_a$ the one-form for which $h^{ab}t_b = 0$ (this is defined up to a choice normalization at each point), and we assume that $k^{ab}t_a$ does not vanish anywhere.

a. By adjusting the normalization of the 1-form $t_a$, show that the covariant metric has the expansion $g_{ab}(\varepsilon) = -t_at_b/\varepsilon + p_{ab} + O(\varepsilon)$ for some tensor field $p_{ab}$. [Hint: choose a basis of vectors $e_a^\alpha$ for which $h^{ab} = \delta^\alpha_\beta e^a_\beta e^b_\alpha$ and invert $g^{ab}(\varepsilon)$ on this basis.]

b. Starting from the equations of general relativity, derive the following equations involving the tensor fields $t_a$, $h^{ab}$, and $T^{ab}$ and the connection $D_a$ which define the Newton-Cartan theory: (i) an orthogonality condition $h^{ab}t_b = 0$; (ii) the relations $D_a h^{bc} = 0$, $D_at_b = 0$ which enforce the compatibility of the fields $h^{ab}$ and $t_a$ with the connection; (iii) a relation called the Trautman condition $h^{[a}R_{f(bc)]d} = 0$ (where $R_{abc}^d$ is the Riemann tensor associated with the connection $D_a$); (iv) a field equation $R_{ab} = 4\pi t_at_bT^{cd}$, where $R_{ab} = R_{abc}^c$; and (v) a stress-energy conservation equation $D_a T^{ab} = 0$. [Hint: These relations can all be derived covariantly, without choosing coordinate systems, by writing down appropriate relations that are valid for all $\varepsilon$ and taking the limit as $\varepsilon \to 0$. Also, for the Trautman condition, it is useful to first derive the identity $g^{[a}(\varepsilon)R_{f(bc)]d}[\nabla_c(\varepsilon)] = 0$ in general relativity.]

c. In the remainder of this problem we will show that the six equations in part b. can be used to obtain the usual formulation of Newtonian gravity. Show first from the compatibility relations (ii) that there exists a function $t$ for which $t_a = D_at$. Then choose a set of coordinates $x^a$ for which $x^0 = t$, and show using the orthogonality condition (i) that in these coordinates $t_a dx^a = dt$ and $h^{00} = h^{0i} = 0$. In this coordinate system we define a 3-metric $h_{ij}$ by $h_{ij} h^{jk} = \delta^k_i$. Using the compatibility conditions (ii) show that the only nonzero components of the connection $D_a = \Gamma^i_{lm} = h^{ik}(h_{kl,m} + h_{km,l} - h_{lm,k})/2$, $\Gamma^l_{it} = h^{ik}(h_{kl,0} - \epsilon_{klm}B_m)/2$, and $\Gamma^0_{ij} = h^{ik}\Phi_k$, where the quantities $\Phi_k$ and $B_m$ are still undetermined.

d. Deduce that the spatial components of the Ricci tensor of the connection $D_a$ coincide with the components of the 3-dimensional Ricci tensor computed from the 3-metric $h_{ij}$. Argue using the field equation (iv) that this Ricci tensor must vanish, so that the 3-metric $h_{ij}$ must be flat. Therefore, at each time $t$ we can choose spatial coordinates $x^i$ so that $h_{ij} = \delta_{ij}$.

e. Show from the field equation (iv) that $R_{0i} = 0$, and deduce that $\epsilon_{ijk}\partial_j B_k = 0$. Show also from the Trautman condition (iii) that $\partial_i B_i = 0$, from which it follows that the field $B_i$ satisfies Laplace's equation, $B_{i,j} = 0$. Assuming that $R_{a}^{\alpha\beta\gamma} \to 0$ as $r \to \infty$, show that the only allowed nontrivial solutions to Laplace's equation are those with $B_i = \text{constant}$. Show that these solutions can be eliminated by transforming to a uniformly rotating coordinate system. Therefore, in a suitably adjusted coordinate systems, $B_i = 0$.

f. Show using the Trautman condition (iii) that $\Phi_{i,j} - \Phi_{j,i} = \epsilon_{ijk}B_k = 0$, so that $\Phi_i = \partial_i \Phi$ for some function $\Phi$, which will be the Newtonian potential. Finally deduce the standard equations of Newtonian gravity, $\Phi_{kk} = 4\pi T^{00}$, $\partial_a T^{a0} = 0$, and $\partial_a T^{ai} + \Phi, T^{00} = 0$. 