1. **Brans-Dicke Gravity:**

General relativity can be summarized by the Einstein-Hilbert action

$$S = S[g_{\mu\nu}, \Psi] = \int d^4x \sqrt{-g} \frac{R}{16\pi G} + S_m[g_{\mu\nu}, \Psi],$$

where $\Psi$ denotes matter fields, and $S_m$ is the action for those matter fields. One of the most popular competitor theories to general relativity is the Brans-Dicke theory of gravity. The idea behind this theory is to imagine that Newton’s constant of gravity $G$ is not an absolute constant but instead can vary through time and space, i.e., is a scalar field. Denoting $1/(16\pi G)$ by $\Phi$ and adding a kinetic term to the action gives the Brans-Dicke action:

$$S_{BD} = S_{BD}[g_{\mu\nu}, \Phi, \Psi] = \int d^4x \sqrt{-g} \left[ \Phi R - \frac{\omega}{\Phi}(\nabla \Phi)^2 \right] + S_m[g_{\mu\nu}, \Psi],$$

where $\omega$ is a dimensionless constant. This theory is in good agreement with all known experiments and observations for $\omega \gtrsim 100,000$, and is a prototype for a class of similar theories obtained as the low energy limit of string theories and from Kaluza-Klein theories.

a. Show that the field equations obtained from the action (2) are

$$G_{\alpha\beta} = \frac{1}{2\Phi} T_{\alpha\beta}[g_{\mu\nu}, \Psi] + \frac{\omega}{\Phi^2} \left[ \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} (\nabla \Phi)^2 \right] + \frac{1}{\Phi} \left[ \nabla_\alpha \nabla_\beta \Phi - g_{\alpha\beta} \nabla_\gamma \nabla^\gamma \Phi \right],$$

and

$$\nabla_\alpha T^{\alpha\beta}[g_{\mu\nu}, \Psi] = 0,$$

where

$$T^{\alpha\beta}[g_{\mu\nu}, \Psi](x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta S_m}{\delta g_{\alpha\beta}(x)},$$

is the usual matter stress-energy tensor and $T = g_{\alpha\beta} T^{\alpha\beta}$.

b. Compute the exterior gravitational field of a static, spherically symmetric source, in a post-Newtonian expansion. Thereby deduce the values of the Eddington-Schiff parameters $\beta$ and $\gamma$ for this theory.

c. Gravitational waves in this theory are described by solutions of the linearized versions of Eqs. (3) and (4) with no matter present (i.e., with $T^{\alpha\beta} = 0$). To obtain these solutions, it is useful to perform the following transformation rather than working directly with Eqs. (3) and (4). Define a new metric

$$\hat{g}_{\alpha\beta} = e^{-2\Omega} g_{\alpha\beta},$$

where $\Omega$ is some function. Show that the Ricci scalars of the old and new metrics are related by

$$R = e^{-2\Omega} \left[ \hat{R} - 6 \hat{\nabla}_\alpha \hat{\nabla}^\alpha \Omega - 6 \hat{g}^{\alpha\beta} \hat{\nabla}_\alpha \Omega \hat{\nabla}_\beta \Omega \right],$$

where $\hat{\nabla}_\alpha$ is the derivative operator associated with $\hat{g}_{\alpha\beta}$. Now choose $\Omega = -\ln(\Phi/\Phi_0)/2$, where $\Phi_0$ is the background cosmological value of $\Phi$ today, and insert this formula into the action (2). Show that, up to a boundary term, the resulting action can be written as

$$S_{BD} = \frac{1}{2} m_p^2 \int d^4x \sqrt{-\hat{g}} \left[ \hat{R} - \left( \omega + \frac{3}{2} \right)(\hat{\nabla} \chi)^2 \right] + S_m[e^{-\chi} \hat{g}_{\alpha\beta}, \Psi].$$
where $\chi = \ln(\Phi/\Phi_0)$ and we have defined $m_p^2 = 2\Phi_0$. The metric $\tilde{g}_{\alpha\beta}$ is called the Einstein frame metric; it is not the physical metric which is $g_{\alpha\beta} = e^{-\chi} \tilde{g}_{\alpha\beta}$. Nevertheless using the Einstein frame metric can be useful in computations. In particular, from the action (6) we see that when there is no matter present, the theory reduces to the usual theory of GR plus a minimally coupled scalar field (in terms of the variables $\tilde{g}_{\alpha\beta}$, $\chi$).

d. Argue that the linearized equations of motion obtained from the action (6) in vacuum are

\[ \partial_\alpha \partial^\alpha \tilde{h}_{\mu\nu} = 0 \]

and

\[ \partial_\alpha \partial^\alpha \chi = 0, \]

where $\tilde{g}_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}$. As in general relativity, we can choose the coordinate system to make $\tilde{h}_{\alpha\beta}$ be purely spatial, transverse and traceless. Deduce that the physical metric is, to linear order,

\[ g_{\alpha\beta}(x) = \eta_{\alpha\beta} - \chi(x)\eta_{\alpha\beta} + \tilde{h}_{\alpha\beta}(x). \]

Define, in a particular Lorentz frame, the gravitational wave field $H_{ij}$ by

\[ \ddot{H}_{ij} = -2R_0\eta_{i0}\eta_{j0}. \]

Show that $H_{ij}$ has a pure trace component in addition to the usual transverse-traceless components. Describe qualitatively how the resulting forces would distort a spherical swarm of test particles.

2. **Extrinsic Curvature:**

a. Show that the extrinsic curvature tensor of a spacelike hypersurface with unit normal $\vec{n}$ is $K_{\alpha\beta} = \mathcal{L}_{\vec{n}} P_{\alpha\beta}/2$, where $P_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$ is the projection tensor into the surface.

b. Compute the extrinsic curvature of a $\tau =$ constant slice of the spacetime given by

\[ ds^2 = -d\tau^2 + a(\tau)^2 \gamma_{ij}(x^k)dx^i dx^j, \]

where $a(\tau)$ is some function of time and $\gamma_{ij}$ is a fixed, three dimensional metric.