Astrometric Properties of a Stochastic Gravitational Wave Background

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Conference on “Cosmology since Einstein”
Hong Kong University of Science and Technology
31 May 2011

Laura Book, EF, PRD 83, 024024 (2011)
Outline of Talk

• Review of stochastic gravitational wave backgrounds: sources and detection methods

• High precision astrometry

• Constraining gravitational waves with astrometry: history, existing upper limits, order of magnitude estimates

• Deflection pattern produced by a stochastic background, correlation function and spectrum

• Optimal data analysis method and implications
Stochastic Gravitational Wave Backgrounds

• Assume:
  - Gaussian
  - Stationary
  - Homogeneous
  - Isotropic

• Spectrum:

\[ \Omega_{gw}(f) = \frac{1}{\rho_c} \frac{d\rho_{gw}}{d \ln f} \sim f^2 H_0^2 h_{\text{rms}}(f)^2 \]
Gravitational Waves Probe the Early Universe

<table>
<thead>
<tr>
<th>Energy</th>
<th>$\log_{10}(E)$</th>
<th>(MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>-5</td>
</tr>
</tbody>
</table>

- Hubble length
- LISA ($10^{-3}$ Hz)
- LIGO mode wavelength (100 Hz)
- $\gamma$
- $\nu$
- $g$

Unprobed Energy Window
Accelerator Energy Frontier
High Precision Astrometry

- Radio VLBI

\[ \delta \theta \sim \frac{\delta t}{D} \sim \frac{\lambda}{D} \delta \phi \]
\[ \sim 100 \ \mu\text{as} \left( \frac{\lambda}{1 \text{ cm}} \right) \left( \frac{D}{10^4 \text{ km}} \right)^{-1} \left( \frac{\delta \phi}{1} \right) \]

SKA
\[ N \sim 10^6 \ \text{quasars} \]
\[ \delta \theta \sim 10 \ \mu\text{as} \]

- Optical Satellites

HIPPARCOS (1989–93)
\[ N \sim 10^6 \ \text{stars} \]
\[ \delta \theta \sim \text{few mas} \]

GAIA (2013)
\[ N \sim 10^6 \ \text{quasars} \]
\[ N \sim 10^9 \ \text{stars} \]
\[ \delta \theta \sim 10 \ \mu\text{as} \]

SKA
\[ N \sim 10^6 \ \text{quasars} \]
\[ \delta \theta \sim 10 \ \mu\text{as} \]

SIM ??
Astrometry and Gravitational Waves: History

- Grav. waves cause angular deflection, \( \delta \theta \sim h \)

- Initial idea (Fakir, 1993)
  \[
  \delta \theta \sim h \sim \frac{1}{b}
  \]

- Does not work (Damour, Kopekin)
  \[
  \delta \theta \sim \int h \sim \frac{1}{b^3}
  \]

- Second idea (Braginsky, 1991): search for stochastic background
  - Find \( \delta \theta(d) \sim h \), not random walk, \( \delta \theta(d) \propto \sqrt{d} \)
  - Gravitational waves cause apparent angular velocities, correlated across the sky
  - Distant sources have small proper motions
  - Search in data for expected statistical deflection pattern
Order of Magnitude Estimates

• Signal:

\[ \delta \theta_{\text{rms}}(f) \sim h_{\text{rms}}(f) \sim \frac{H_0}{f} \sqrt{\Omega_{\text{gw}}(f)} \]
\[ \delta \omega_{\text{rms}}(f) \sim f \delta \theta_{\text{rms}}(f) \sim H_0 \sqrt{\Omega_{\text{gw}}(f)} \]
\[ \sim 10^{-2} \mu \text{as yr}^{-1} \left( \frac{\Omega_{\text{gw}}}{10^{-6}} \right)^{1/2} \]

• Monitor \( N \) sources, time \( T \), accuracy \( \Delta \theta \)

\[ \Rightarrow \omega \lesssim \frac{\Delta \theta}{T \sqrt{N}} \]
\[ \Rightarrow \Omega_{\text{gw}} \lesssim \frac{\Delta \theta^2}{NT^2 H_0^2} \sim 10^{-6} \left( \frac{\Delta \theta}{10 \mu \text{as}} \right)^2 \left( \frac{N}{10^6} \right)^{-1} \left( \frac{T}{1 \text{yr}} \right)^2 \]

• Existing VLBI upper limit

\[ N \sim 10^3, \quad \Omega_{\text{gw}} \lesssim 10^{-3} \]

• Limit applies to

\[ \int_{\ln H_0}^{\ln(1/T)} d\ln f \, \Omega_{\text{gw}}(f) \]

Monday, June 6, 2011
Order of Magnitude Estimates (cont.)

- **Source Proper motions:**

  **Stars in Milky Way:**
  \[
  \omega \sim \frac{v}{D} \\
  \sim 10^4 \mu \text{ as yr}^{-1} \left( \frac{v}{100 \text{ km s}^{-1}} \right) \left( \frac{D}{1 \text{ kpc}} \right)^{-1}
  \]

  **Quasars:**
  \[
  \omega \sim 0.1 \mu \text{ as yr}^{-1} \left( \frac{v}{10^3 \text{ km s}^{-1}} \right) \left( \frac{D}{3 \text{ Gpc}} \right)^{-1}
  \]

  In fact jets produce apparent proper motion of quasars of order
  \[
  \omega \sim 10 \mu \text{ as yr}^{-1}
  \]
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Angular Deflection Computation

- **Metric:**
  \[ ds^2 = a(\eta)^2 \left[ -d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right] \]

- **Find perturbed geodesic for which**
  - Detection event unchanged
  - Null
  - Emission frequency unchanged
  - Intersects source worldline

\[
\vec{k}_O = \frac{\omega_O}{1 + \delta z} \left\{ \vec{u}_O + \left[ n^i + \delta n^i(n, t) \right] \vec{e}_i \right\}
\]

- **Result:**

\[
x^\alpha(\lambda) = (\eta_0 + \lambda, -n^i \lambda), \quad P^{ij} = \delta^{ij} - n^i n^j
\]

\[
\delta n^i = -\frac{1}{2} P^{ik} n^j h_{jk}(0) + \frac{1}{\lambda_S} P^{ik} n^j \int_0^{\lambda_S} d\lambda \left[ h_{jk}(\lambda) + \frac{1}{2}(\lambda_S - \lambda)n^l h_{jl,k}(\lambda) \right]
\]
Approximations

• Four lengthscales:

• Three approximations:

1) Subhorizon modes only, \( \lambda \ll L_H \)
   ‣ For simplicity
   ‣ No superhorizon contribution expected

2) Distant source limit, \( \lambda \ll D \)
   ‣ Not a restriction for quasars
   ‣ Could be removed
   ‣ Also invoked in pulsar timing, different justification

3) Slowly varying modes only, \( \lambda \gg cT \)
   ‣ For simplicity only. Removed in present work
   ‣ Not invoked in pulsar timing
Simplified Deflection Formula

• Specialize to plane wave mode: \( h_{ij}(\eta, x) = \text{Re} \left[ H_{ij} e^{i\Omega p \cdot x} q(\eta) \right] \)

\[
q'' + 2\frac{a'}{a} q' + \Omega^2 q = 0, \quad q(\eta) \approx \frac{1}{a(\eta)} e^{-i\Omega \eta} \quad (\text{WKB near observer})
\]

• In distant source and subhorizon limits, find

\[
\delta n^i(n) = -\frac{1}{2} n_j h_{ij}(0) + \frac{n^i + p^i}{2(1 + n \cdot p)} [n_j n_k h_{jk}(0)]
\]

• For stochastic background, find deflection correlation function

\[
\langle \delta n_i(n, t) \delta n_j(n', t') \rangle = \int_0^\infty df \, e^{-2\pi f(t-t')} \frac{H_0^2 \Omega_{gw}(f)}{16\pi^2 f^3} H_{ij}(n, n')
\]

where \( H_{ij} = \alpha(\Theta) \left[ A_i A_j - B_i C_j \right] \), \( \cos \Theta = n \cdot n' \),

\[
A = n \times n', \quad B = n \times A, \quad C = -n' \times A,
\]

\[
\alpha(\Theta) = \frac{1}{2 \sin^2 \Theta} (7 \cos \Theta - 5) - \frac{48 \sin^6(\Theta/2)}{\sin^4 \Theta} \ln[\sin(\Theta/2)]
\]
Angular Deflection Power Spectrum

• We compute power spectrum as a function of frequency and angular scale

• Expand angular deflection as

\[ \delta n(n, t) = \sum_{Q=E,B} \sum_{l \geq 2} \sum_{m=-l}^{l} \delta n_{Qlm}(t) Y_{lm}(n) \]

• Find

\[ \langle \delta n_{Qlm}(t) \delta n_{Q'lm'}(t') \rangle = \delta_{QQ'} \delta_{ll'} \delta_{mm'} \int_{0}^{\infty} df \cos[2\pi f(t - t')] S_{Ql}(f) \]

with

\[ S_{Ql}(f) = \frac{H_0^2 \Omega_{gw}(f) \alpha_l}{2\pi f^3} \frac{\alpha_l}{2l + 1}, \]

\[ \alpha_l = \left( \frac{5}{6}, \frac{7}{60}, \frac{3}{100}, \frac{11}{1050}, \ldots \right), \]

\[ \sum_{l=2}^{\infty} \alpha_l = 1 \]
Typical Deflection Pattern
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Optimal Data Analysis Method

• For slowly varying modes \( \delta \dot{n}(n, t) \approx \delta \dot{n}(n, t_0) + \dot{\delta}n(n, t_0)(t - t_0) + \ldots \)

• Data: average positions, velocities, accelerations \( n_A, \omega_A, \alpha_A, 1 \leq A \leq N \)

• Model: \[ \omega_A = \sum_{Qlm} Y^Q_{lm}(n_A) \delta \dot{n}_{Qlm}(t_0) + \delta \omega_A, \]

or \( \omega = M \cdot s + \delta \omega, \) with \( \langle ss^T \rangle = \Sigma^s, \langle \delta \omega \delta \omega^T \rangle = \Sigma^w \)

• Noise \( \delta \omega \) due to measurement error and source proper motion

• Detection statistic: \( Q = \text{tr} \left[ \omega^T \cdot A \cdot \omega \right] \) for some \( A \) with \( \text{tr} \left[ A \cdot \Sigma^w \right] = 1 \)

• Choose \( A \) to maximize the signal to noise ratio. Result is

\[
\frac{S^2}{N^2} \equiv \frac{\langle Q \rangle^2}{\langle Q^2 \rangle_{\text{no signal}}} = \frac{1}{2} \text{tr} \left[ D_{TF}^2 \right], \quad D = (\Sigma^w)^{-1} \cdot M \cdot \Sigma^s \cdot M^T
\]
Prediction for Sensitivity of Astrometry

\[
\frac{S^2}{N^2} = \sum_{l \geq 2} \frac{N^2 H_0^4 \alpha_l^2}{2l + 1} \left\{ \left[ \int d\ln f \frac{\Omega_{gw}(f)}{\sigma_\omega} \right]^2 + \left[ \int d\ln f (2\pi f)^2 \Omega_{gw}(f) \right]^2 \right\} + \ldots
\]

• Parameters:
  
  ‣ Measurement error: \( \sigma_\omega \sim 10 \mu \text{ as yr}^{-1} \), \( \sigma_\alpha \sim 10 \mu \text{ as yr}^{-2} \)
  
  ‣ Stellar motion: \( \sigma_\omega \sim 10^4 \mu \text{ as yr}^{-1} \), \( \sigma_\alpha \sim 10^{-3} \mu \text{ as yr}^{-2} \)
  
  ‣ Quasar motion: \( \sigma_\omega \sim 10 \mu \text{ as yr}^{-1} \), \( \sigma_\alpha \sim 0 \)

• Pristine sources:
  
  ‣ Dominated by measurement error
  
  ‣ Quasars today
  
  ‣ Velocity data best

• Non-pristine sources:
  
  ‣ Dominated by proper motion errors
  
  ‣ Stars today, quasars in future?
  
  ‣ Acceleration data also useful
Prediction for Sensitivity of Astrometry (cont.)

- Generalizing analysis to include time dependence of deflections gives

$$\frac{S^2}{N^2} = \sum_{i \geq 2} \frac{N^2 H_0^4 \alpha_i^2}{(2l + 1)\sigma_\omega^4} \left\{ \left[ \int_{\ln(1/T)}^{\ln(1/T)} d\ln f \Omega_{gw}(f) \right]^2 + \frac{9}{2\pi^4} \int_{\ln(1/T)}^{\ln(1/T)} d\ln f \frac{\Omega_{gw}(f)^2}{f^5 T^5} \right\}$$
Conclusions

• Future optical (GAIA) and radio (SKA) astrometry data can provide interesting constraints on stochastic gravitational waves, at the level of $\Omega_{gw} \sim 10^{-6}$ over a broad band.

• The limit is scale invariant at frequencies low compared to the observation time, but rises as $\Omega_{gw} \propto f^{5/2}$ at high frequencies.

• For GAIA, using the acceleration data from the $\sim 10^9$ stars may be competitive with the velocity data from the $\sim 10^6$ quasars at certain frequencies.
Conclusions (cont.)

- **The future:** As technology improves, how far can this technique be pushed? Achieving the inflation-signal regime of $\Omega_{gw} \sim 10^{-14}$ would require $\Delta \theta \sim 10^{-4} \times 10 \mu$ as $\sim$ n as. Even if this were achieved there are several other challenges:

  **Background from Scalar Modes:** Scalar perturbations give a $l=2$, electric signal equivalent to $\Omega_{gw,\text{eff}} \sim \int d\ln k \delta_{\text{rms}}(k)^2 (H_0/k)^4 \sim 10^{-9}$, but do not contribute to the $l=2$, magnetic signal at linear order.

  **Narrow Band:** Acceleration rather than velocity data of quasars would have to be used, giving a narrowband limit.

  **Solar System Modeling:** Extremely accurate models of the Solar System would be required to subtract from the data kinematic effects (acceleration and rotation of our reference frame) and light bending by the Sun and planets.

  **Black Hole Background:** A stochastic background of gravitational waves from coalescing supermassive black holes might be present at frequencies of $f \sim \text{yr}^{-1}$ and swamp any early Universe signal.