Saturation of neutron star r-mode instability and gravitational wave signal

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Summary

- Two observational puzzles: (i) neutron stars seem to born with rather large rotation periods compared to the maximum of 0.5 ms–2 ms. Fastest known in supernova remnant: PSR J0537-6910, 16 ms today, ~ 9 ms at birth. (ii) LMXB spin's clustered near 300 Hz, expect $\sim 10^3$ Hz.

- The linear instability of r-modes in newly born neutron stars and LMXBs has caused much excitement over the last few years; potential to explain the above puzzles and a possible strong source of gravitational waves for LIGO.

- We find that nonlinear mode-mode coupling predicts saturation amplitudes $\sim 10^{-3}$, reducing the gravitational wave signal (gr-qc/00101092 & astro-ph/0202345).

- The r-mode instability may be killed by a viscosity enhancement due to Lambda hyperons (Owen and Lindblom, gr-qc/0110558), which I'll ignore for this talk.

Overview of Talk

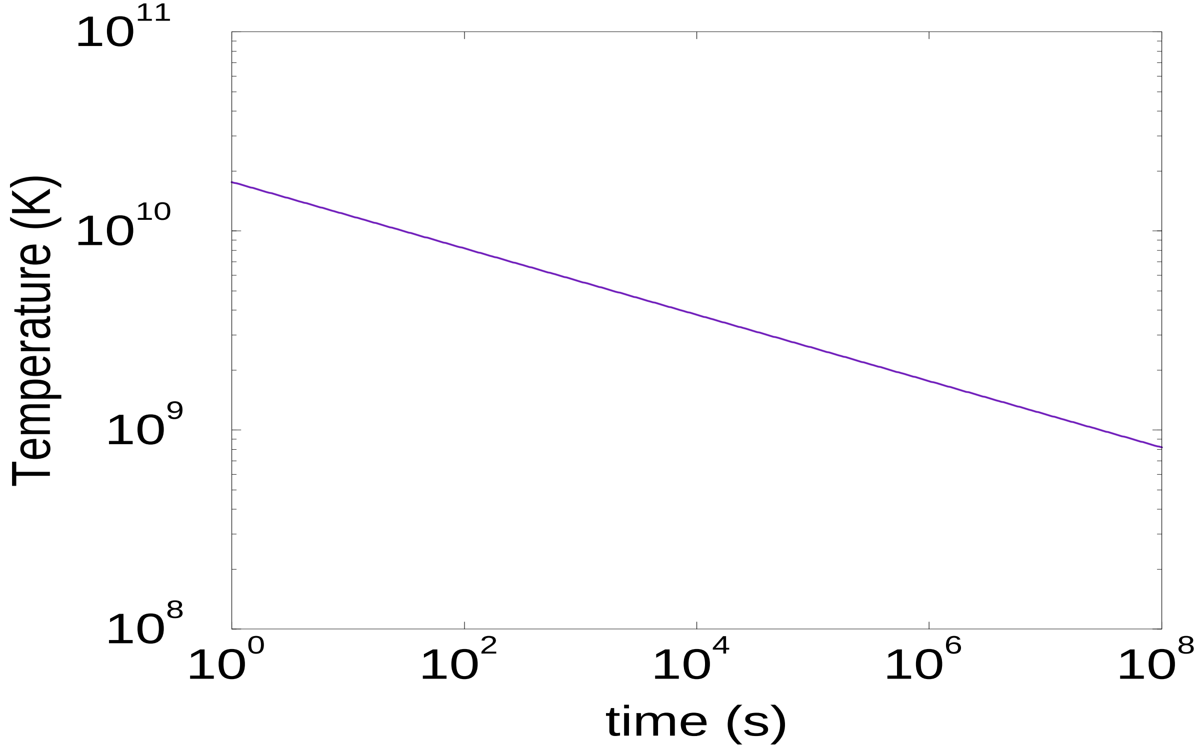
1. Review of r-mode instability.

2. Brief summary of our result and discussion of implications

3. Derivation of our result: (i) Nonlinear perturbation theory in rotating stars, (ii) Inertial stellar modes (iii) parametric instability (iv) turbulent cascade.

4. Conclusions.

The Setting



Physical scales: R $\sim 10 - 15$km, $M \sim 1.4 M_\odot$, $\rho \sim 10^{15}\,\mathrm{g\,cm^{-3}}$, $P \sim 1\,\mathrm{ms}$ (?), $T \sim 10^{10} K$, $T_{\text{crust}} \sim 10^9 K$, $T_{\text{superfluid}} \sim 10^9$, $B \lesssim 10^{12}\,G$.

Stellar Rossby modes (r-modes)

- Mode function

$$\boldsymbol{\xi} = f(r)\mathbf{r} \times \nabla Y_{lm}(\theta, \varphi)\, e^{-i\omega t} + O(\Omega^2),$$

$$\omega = \kappa_l \Omega m + O(\Omega^3),$$

with $\kappa_l = (l+2)(l-1)/(l(l+1)) = 2/3$ for $l = 2$.

- Purely axial, no density perturbation or radial motion at leading order in Ω.

- $\boldsymbol{\xi} \propto \exp im\,(\varphi - \kappa_l \Omega t)$, so "pattern speed" is $\kappa_l \Omega > 0$. All the modes are prograde.

- However, in the frame $\bar{\varphi} = \varphi - \Omega t$ that co-rotates with the star, the modes are retrograde: $\boldsymbol{\xi} \propto \exp im\,[\bar{\varphi} - (\kappa_l - 1)\Omega t]$.

Stellar Rossby modes (cont)

l=2
m=2

l=3
m=3

The Chandrasekhar-Freidman-Schutz (CFS) Instability

Any mode which is retrograde in the co-rotating frame but prograde in the inertial frame grows as a result of its emitting gravitational waves (C, 1970; FS, 1978).

The CFS Instability (cont)

The argument: In the inertial frame $\boldsymbol{\xi} \propto \exp im\,[\varphi - \sigma_I t] + \text{c.c.}$, where σ_I is the inertial-frame pattern speed.

Key physical quantity is mode energy in rotating frame $E_{\text{mode,R}}$, related to mode energy in inertial frame $E_{\text{mode,I}}$ by

$$E_{\text{mode,R}} = E_{\text{mode,I}} - \Omega J_{\text{mode}}.$$

The change due to the emission of a graviton is ($\hbar = 1$)

$$\begin{aligned}
\Delta E_{\text{mode,R}} &= \Delta E_{\text{mode,I}} - \Omega\,\Delta J_{\text{mode}} \\
&= -|m\sigma_I| + \Omega\,\text{sgn}(m\sigma_I)\,m \\
&= |m|\,\text{sgn}(\sigma_I)\,(\Omega - \sigma_I) = -|m|\,\text{sgn}(\sigma_I)\,\sigma_R,
\end{aligned}$$

where $\sigma_R = \sigma_I - \Omega$ is the rotating-frame pattern speed.

Timescales

- Competition between gravitational radiation reaction and various dissipation mechanisms.

- Shear viscosity dominated by n-n scattering

$$\eta = 2 \times 10^{18}\,\mathrm{g\,cm^{-1}\,s^{-1}}\,\rho_{15}^{9/4}\,T_9^{-2},$$

  where $\rho_{15} = \rho/(10^{15}\,\mathrm{g\,cm^{-3}})$ and $T_9 = T/10^9\,\mathrm{K}$.

- Bulk viscosity dominated by modified URCA reactions $e + p + N \to n + N + \nu_e$ and $n + N \to p + N + e + \bar{\nu}_e$, which gives

$$\zeta = 1.5 \times 10^{18}\,\mathrm{g\,cm^{-1}\,s^{-1}}\,\rho_{15}^{2}\,T_9^{6}\nu_{\mathrm{kHz}}^{-2},$$

  where $\nu_{\mathrm{kHz}} = \nu/10^3\,\mathrm{Hz}$.

- Resulting growth/decay rate is (Lindblom, Owen and Morsink, 1998)

$$\dot{E}_{\mathrm{mode}}/2E_{\mathrm{mode}} = \tau_{\mathrm{GW}}^{-1}\,\nu_{\mathrm{kHz}}^{6} - \tau_{\mathrm{S}}^{-1}\,T_9^{-2} - \tau_{\mathrm{B}}^{-1}\,\nu_{\mathrm{kHz}}^{-2}T_9^{6},$$

  with $\tau_{\mathrm{GW}} \sim 20\,\mathrm{s}$, $\tau_{\mathrm{S}} \sim 2 \times 10^8\,\mathrm{s}$, $\tau_{\mathrm{B}} \sim 7 \times 10^8\,\mathrm{s}$.

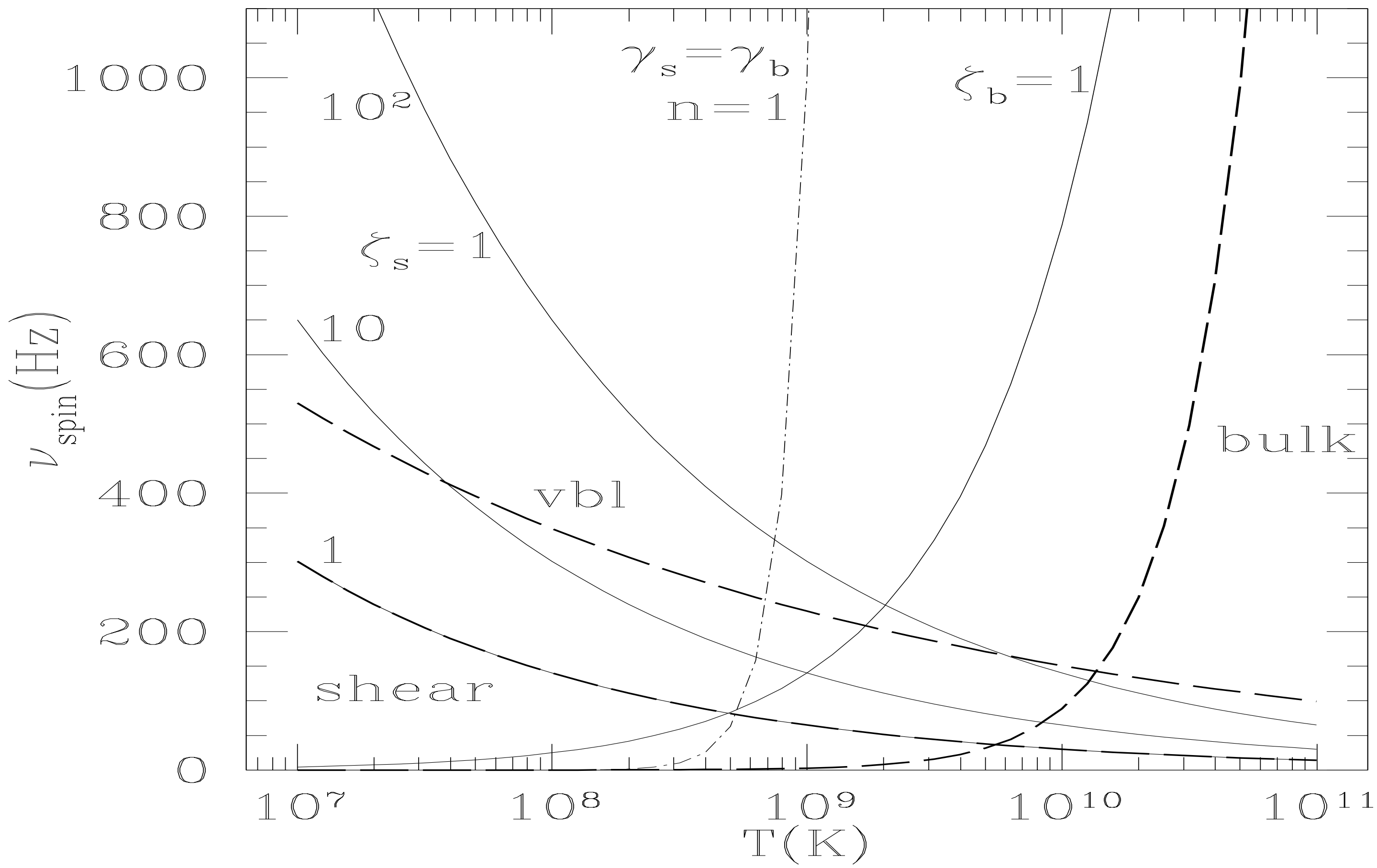
Timescales (cont)

- If a crust crystallizes, a viscous boundary layer of width $\sim 1\,\mathrm{cm}$ should form at the fluid-crust interface, enhancing the dissipation (Bildsten and Ushomirsky, 2000). The contribution to $\dot{E}_{\mathrm{mode}}/2E_{\mathrm{mode}}$ from shear viscosity is changed to

$$-\tau_{\mathrm{VBL}}^{-1}\, T_9^{-1} \nu_{\mathrm{kHz}}^{1/2}$$

with $\tau_{\mathrm{VBL}} \sim 6 \times 10^4\,\mathrm{s}$.

Instability Window



What happens to a newly born neutron star?

- Nonlinear hydrodynamic effect cause the dimensionless mode amplitude $\alpha \sim \boldsymbol{\xi}/R$ to saturate and hold steady at a value α_s of order unity (Lindblom et al 1998, Owen et al 1998).

- The mode grows until at $\alpha \sim 1$, strong shocks form and the mode energy rapidly dissipates (Lindblom, Tohline and Vallisneri, 2000).

- We believe the energy drains out of the r-mode into stellar inertial modes, forming a turbulent cascade. The resulting, steady saturation amplitude is

$$\alpha_s \sim \frac{1}{\sqrt{\tau_{\mathrm{GW}}\Omega}} \sim 6 \times 10^{-3}\, \nu_{\mathrm{kHz}}^{5/2}.$$

Numerical hydrodynamic simulations

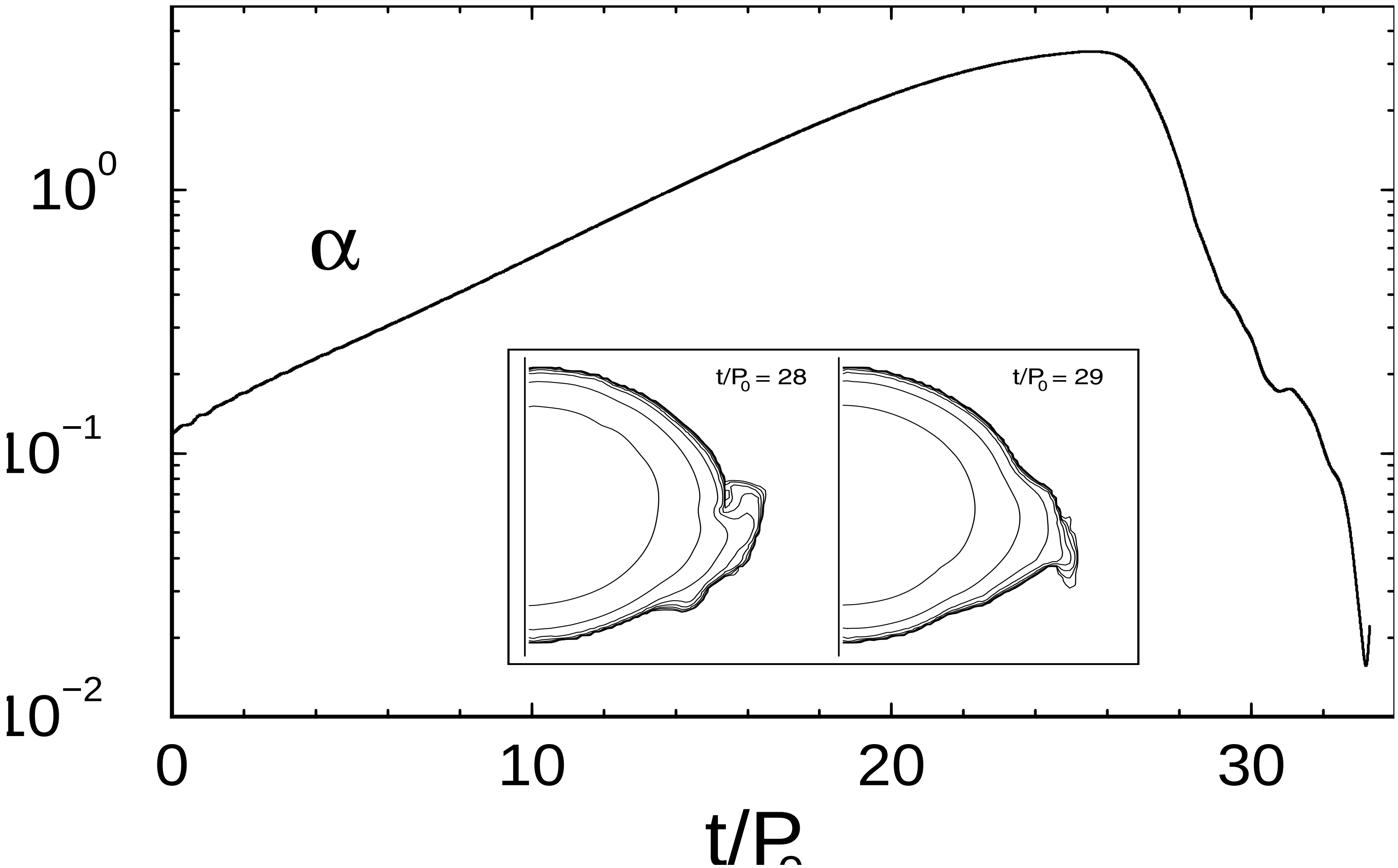
- Difficulties for numerical simulations: (i) 3D problem (ii) efold time is $\sim 10^4$ dynamical times (iii) relevant modes have $\gtrsim 100$ nodes.

- Stergioulas and Font (2000) evolved a star for ~ 25 rotation periods with $\alpha \sim 0.1$ initially; no discernible energy leakage from r-mode to other modes.

- Lindblom, Tohline and Vallisneri (2000) artificially boosted the radiation reaction force to give a growth time of ~ 3 rotation periods. In their simulations, shocks due to breaking waves halted the growth at $\alpha \sim 1$.

- Saturation at $\alpha \ll 1$ not excluded by either simulation.

Lindblom, Tohline and Vallisneri (2000)



Implications of saturation at low amplitude

- Following Owen et al (1998), we use energy and angular momentum conservation to evolve the three degrees of freedom $\alpha(t)$, $\Omega(t)$ and $T(t)$. We assume that

$$\alpha(t) = \alpha_s(T, \Omega) = \text{fixed}$$

  once $\alpha > \alpha_s$, while in the instability window. We evolve the temperature $T(t)$ using neutrino cooling and rmode heating.

- Physical and canonical angular momenta differ (Levin and Ushomirsky) so use physical quantities. Allow dissipation in other modes than the $l = m = 2$ r mode.

Implications of saturation at low amplitude (cont)

- In units with $M = R = G = 1$, equations are

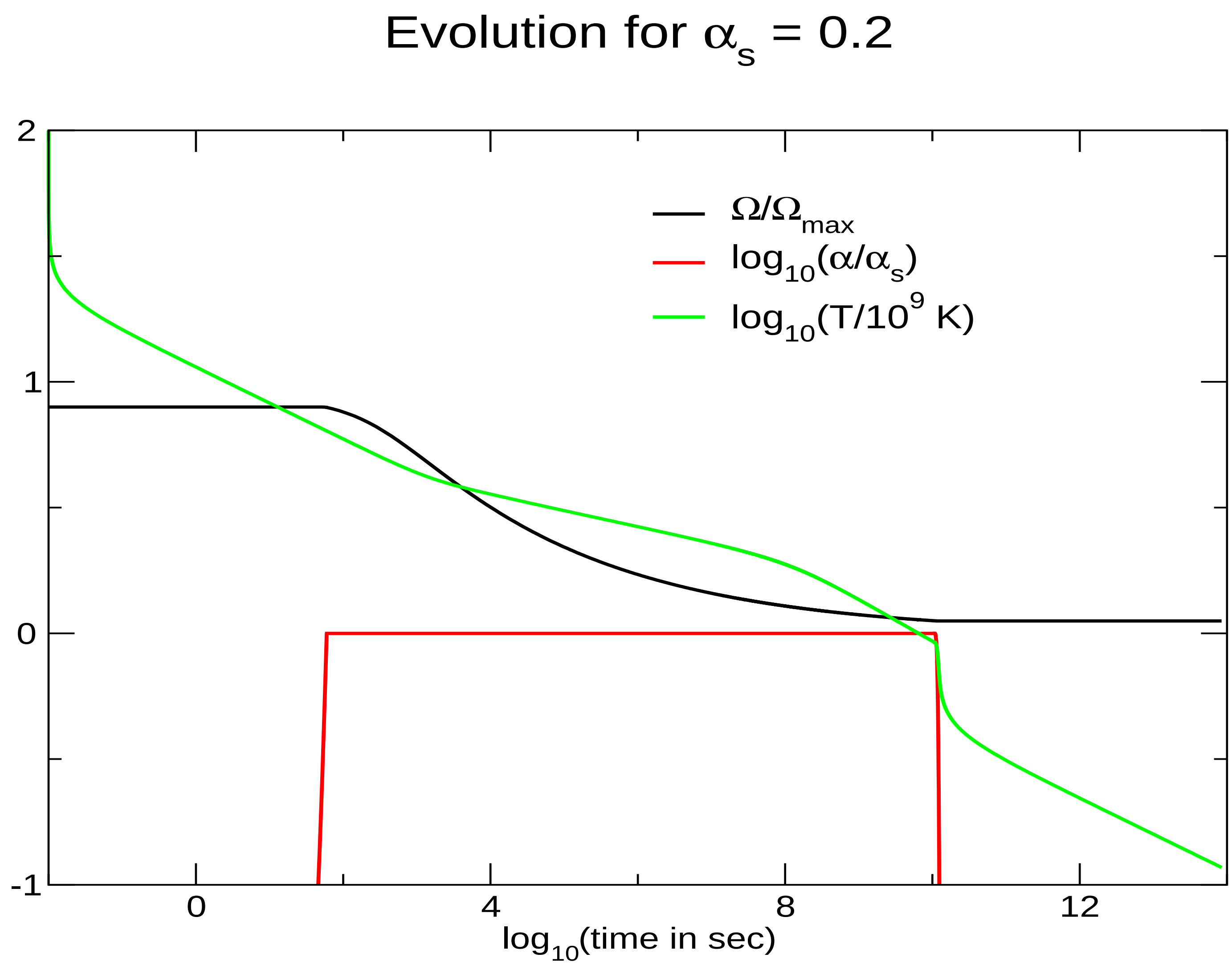
$$\frac{d}{dt}\left[I\Omega + \frac{1}{4}\alpha^2\Omega\tilde{J}\right] = -\frac{3}{\tau_{\mathrm{GR}}}\alpha^2\Omega\tilde{J}.$$

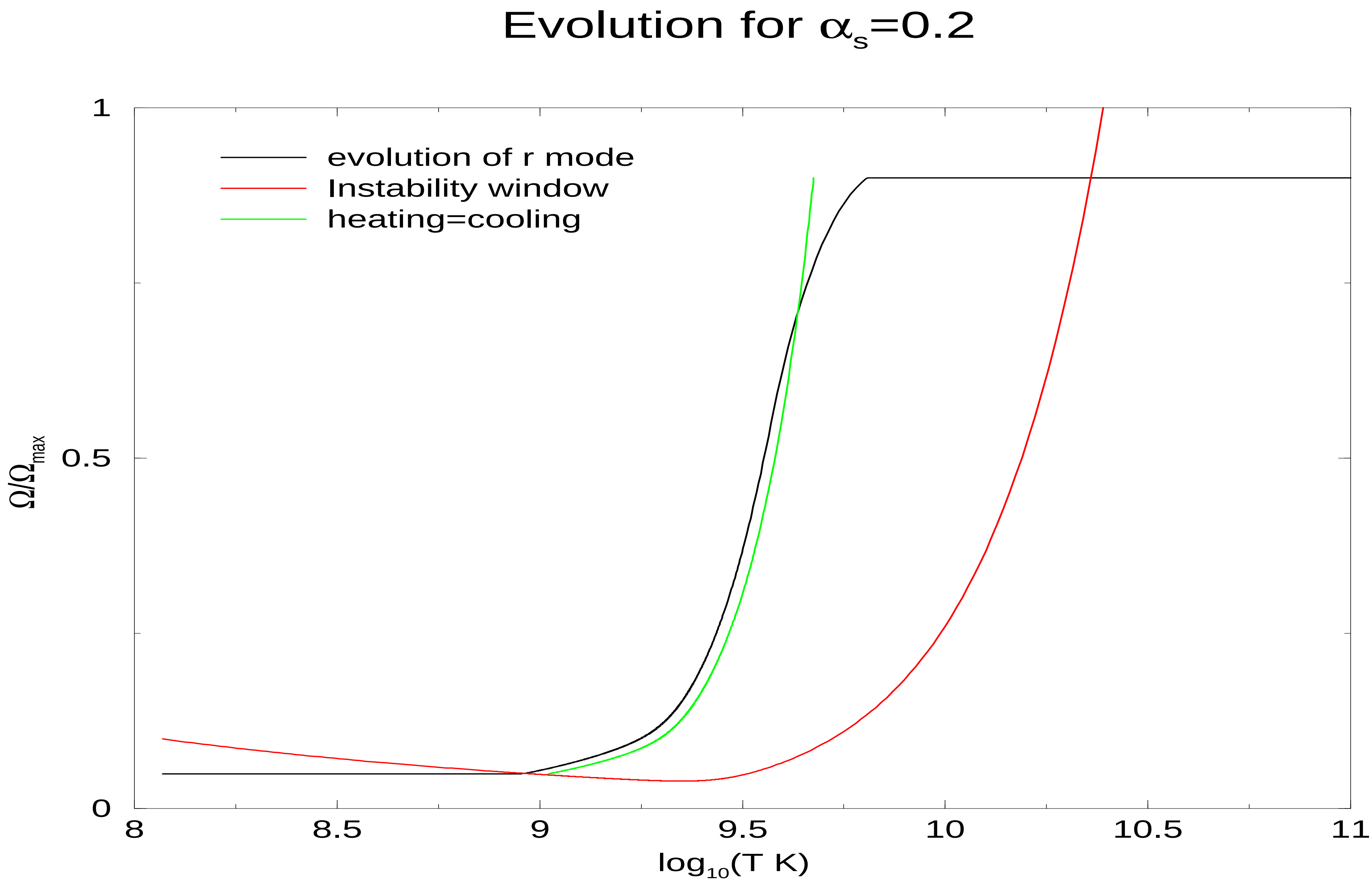
$$\frac{d}{dt}\left[\frac{1}{2}I\Omega^2 + \frac{3}{4}\alpha^2\Omega^2\tilde{J}\right] = -2\left(\frac{4}{3\tau_{\mathrm{GR}}} + \frac{2}{3\tau_V}\right)\left(\frac{3}{4}\alpha^2\Omega^2\tilde{J}\right) - P_{\mathrm{diss,other}}.$$

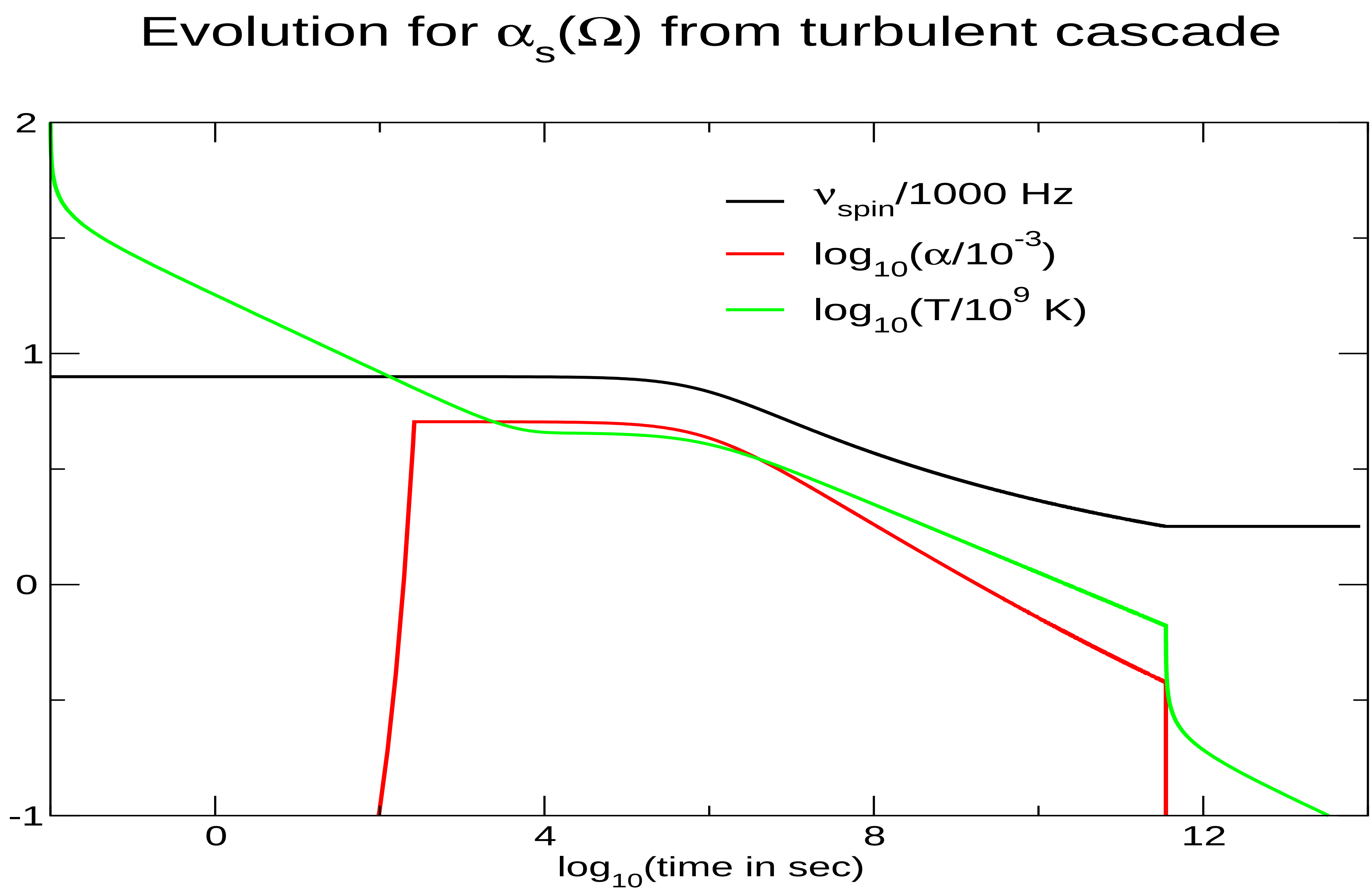
and

$$\frac{d}{dt}E_{\mathrm{int}}(T) = -L_{\mathrm{cool},\nu}(T) + f_{\mathrm{rmode}}\frac{2}{\tau_V}\left(\frac{2}{3}\right)\left(\frac{3}{4}\alpha^2\Omega^2\tilde{J}\right) + f_{\mathrm{other}}P_{\mathrm{diss,other}},$$

where $f_{\mathrm{rmode}} = f_{\mathrm{other}} = \tau_V/\tau_S$.


Evolution for α_s = 0.2
Ω/Ω_max
log10(α/α_s)
log10(T/10^9 K)
2
1
0
-1
0
4
8
12
log10(time in sec)


Evolution for α_s=0.2
evolution of r mode
Instability window
heating=cooling
Ω/Ω_{max}
1
0.5
0
8
8.5
9
9.5
10
10.5
11
$\log_{10}$(T K)


Evolution for $\alpha_s(\Omega)$ from turbulent cascade
ν_{spin}/1000 Hz
$\log_{10}(\alpha/10^{-3})$
$\log_{10}(T/10^9\ K)$
2
1
0
-1
0
4
8
12
$\log_{10}$(time in sec)

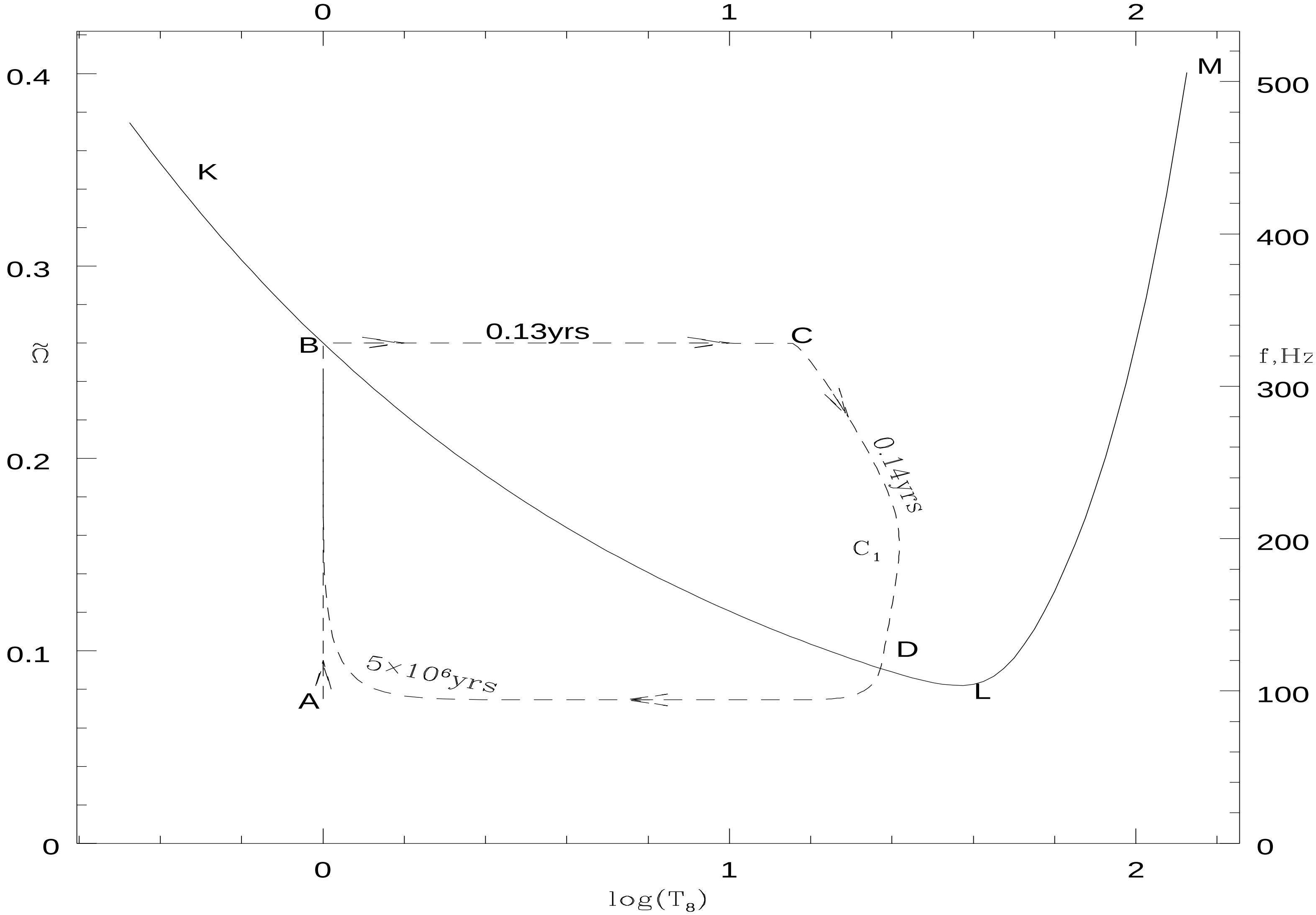
Implications of saturation at low amplitude (cont)

- Original scenario: star spins from ~ 1 ms to ~ 10 ms in ~ 1 yr, providing possible explanation for low spins of young neutron stars (Lindblom et al, 1998).

- We find time taken to exit the region of instability is $\sim 10^3 - 10^4$ yrs. Final spin is $\sim 250\,\mathrm{Hz}$, although magnetic dipole spindown dominates for $\nu \leq 400\,\mathrm{Hz} B_{12}^{2/9}$. Explanation for young NS spins is now less compelling.

- The r-mode instability can still explain the observed clustering of spin periods of LMXBs at ~ 300 Hz (Bildsten 1998, Andersson et al, 1999). However the gravithermal instability of Levin (1999) probably applies, with spin ups over $\sim 10^7$ years and spin downs over $\sim 10^4$ years, making the gravitational wave signal hard to observe (Heyl, 2001).

Gravithermal instability (Levin, 1998)



Detectability of gravitational wave signal

- Lindblom and Owen showed

$$\frac{S^2}{N^2} = \frac{4G}{5\pi m c^3 D^2} \int df \, \frac{1}{f S_h(f)} \frac{dJ}{df} = (5.4)^2 \left(\frac{10\mathrm{Mpc}}{D}\right)^2 \left(\frac{1}{\nu_{\mathrm{khz,f}}^2} - \frac{1}{\nu_{\mathrm{khz,i}}^2}\right).$$

- We estimate $\nu_{\mathrm{f}} \sim 660$Hz after 1 year, so nominally only loose a factor of 3 in SNR. However cannot integrate for 1 year.

- Brady & Creighton (BC) show limited to $\sim 10^6$ s integration, get LIGO II range of 8 Mpc using 200 Hz $\rightarrow$ 186 Hz. Repeating BC's analysis for our case gives optimum for 1 Teraflop at 10 stacks and 3×10^5 integration time (corresponding to $\Delta\nu_{\mathrm{spin}} \sim 0.5$ Hz), giving range of ~ 200 kpc.

- Also possibly limited by phase wandering of r-mode.

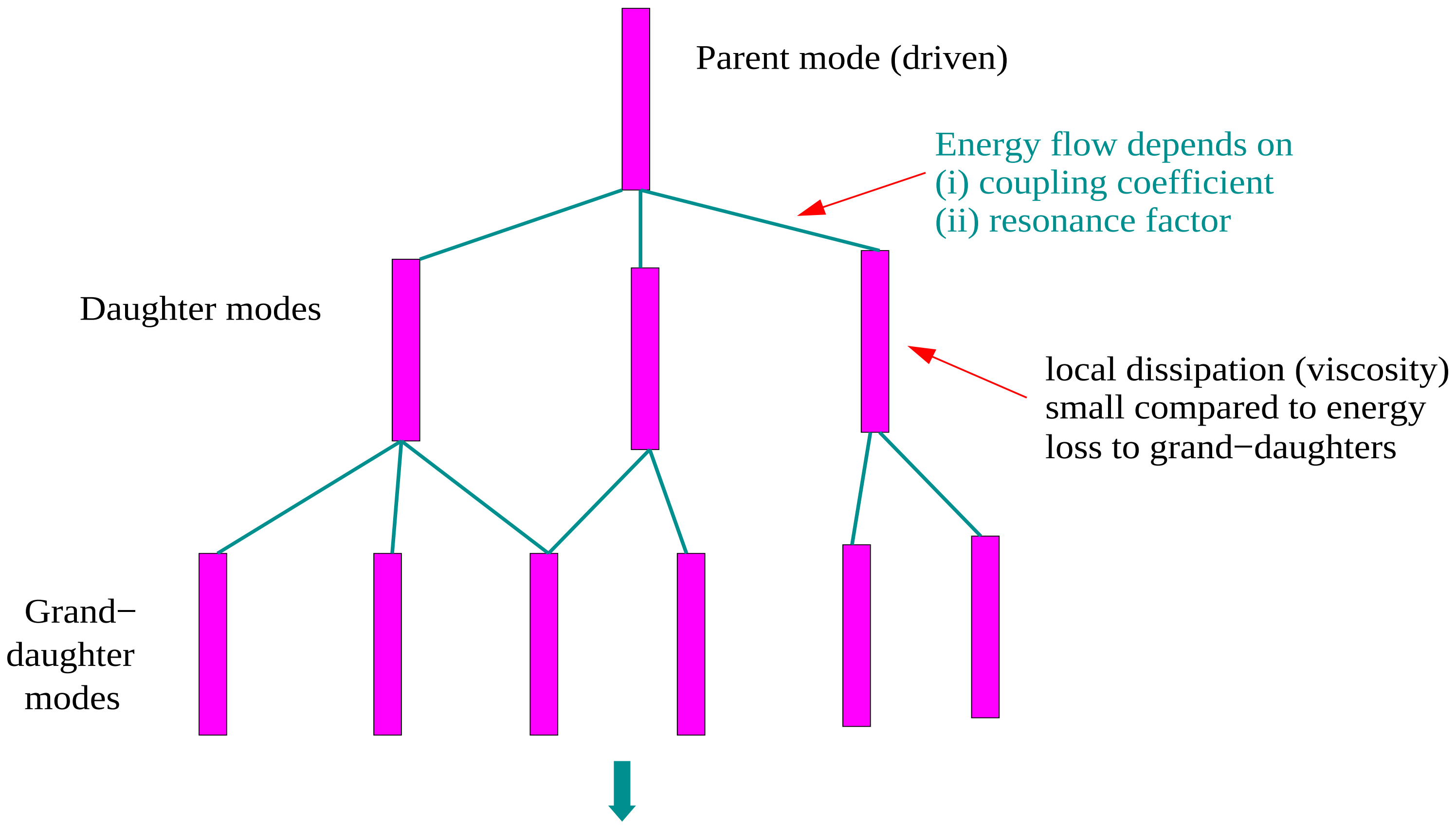
Our approach: 2nd order perturbation theory

- Assume all mode amplitudes remain small, so keep only the leading order nonlinear terms. The star is modeled as a set of harmonic oscillators with cubic coupling terms (3-mode couplings) in the Hamiltonian.

- This approach has been used to predict saturation amplitudes for g-modes driven by turbulence in white dwarfs, giving good agreement with observations (Wu and Goldreich, 1999).

- Two regimes (i) weak driving, saturated via parameteric instability; (ii) strong driving, saturated by development of turbulent cascade (weak turbulence).

Overall picture



At small scales, scaling as in turbulent cascade

Formalism

- The linear fluid equation of motion for rotating frame Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ is $\ddot{\boldsymbol{\xi}} + \mathbf{B} \cdot \dot{\boldsymbol{\xi}} + \mathbf{C} \cdot \boldsymbol{\xi} = 0$, where $\mathbf{B} \cdot \boldsymbol{\xi} = 2\boldsymbol{\Omega} \times \boldsymbol{\xi}$ and $\mathbf{C}$ is a Hermitian differential operator. Solutions $\exp[-i\omega_\alpha t]\boldsymbol{\xi}_\alpha(\mathbf{x})$ where

$$\left[-\omega_\alpha^2 - i\omega_\alpha \mathbf{B} \cdot \boldsymbol{\xi}_\alpha + \mathbf{C}\right] \cdot \boldsymbol{\xi}_\alpha = 0.$$

- Phase space mode expansion:

$$\left[\begin{array}{c} \boldsymbol{\xi}(t) \\ \dot{\boldsymbol{\xi}}(t) \end{array} \right] = \sum_\alpha c_\alpha(t) \left[\begin{array}{c} \boldsymbol{\xi}_\alpha \\ -i\omega_\alpha \boldsymbol{\xi}_\alpha \end{array} \right] + c_\alpha(t)^* \left[\begin{array}{c} \boldsymbol{\xi}_\alpha^* \\ i\omega_\alpha \boldsymbol{\xi}_\alpha^* \end{array} \right],$$

and

$$c_\alpha(t) = \frac{1}{b_\alpha} \left\langle \boldsymbol{\xi}_\alpha \, , \, \omega_\alpha \boldsymbol{\xi}(t) + i\dot{\boldsymbol{\xi}}(t) + i\mathbf{B} \cdot \boldsymbol{\xi}(t) \right\rangle .$$

- Equations of motion to second order:

$$\dot{c}_\alpha + i\omega_\alpha c_\alpha + \gamma_\alpha c_\alpha = i\omega_\alpha \sum_{\beta,\gamma} \left[\kappa_{\bar{\alpha}\beta\gamma} c_\beta c_\gamma + \kappa_{\bar{\alpha}\bar{\beta}\gamma} c_\beta^* c_\gamma + \kappa_{\bar{\alpha}\beta\bar{\gamma}} c_\beta c_\gamma^* + \kappa_{\bar{\alpha}\bar{\beta}\bar{\gamma}} c_\beta^* c_\gamma^* \right] .$$

Formalism (cont)

- The coupling coefficients are

$$\begin{aligned}\kappa_{\alpha\beta\gamma} &= \frac{1}{2}\int d^3x p\Big\{3(\Gamma_1 - 1)\nabla_j\xi^i_\alpha\nabla_i\xi^j_\beta\nabla_k\xi^k_\gamma + 2\nabla_j\xi^i_\alpha\nabla_k\xi^j_\beta\nabla_i\xi^k_\gamma + \\ &+ \left[(\Gamma_1 - 1)^2 + \frac{\partial\Gamma_1}{\partial \ln \rho}\right] \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\alpha\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\beta\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\gamma\Big\} \\ &- \frac{1}{2}\int d^3x\rho\left[3\xi^i_\alpha\xi^j_\beta\nabla_i\nabla_j\delta\phi_\gamma + \xi^i_\alpha\xi^j_\beta\xi^k_\gamma\nabla_i\nabla_j\nabla_k\phi\right]\end{aligned}$$

  symmetrized over α, β, γ, where $\Gamma_1 = d\ln p/d\ln\rho$.

- Task: (i) compute modes $\boldsymbol{\xi}_\alpha(\mathbf{x})$, frequencies ω_α, viscous damping rates γ_α, and coupling coefficients $\kappa_{\alpha\beta\gamma}$. (ii) Solve nonlinear equation of motion to determine evolution of r-mode amplitude.

- Its crucial to *not* to attempt a brute force simulation including all possible modes, as relevant modes are high l, n. Much effort goes into analyzing which modes are important.

Approximations

- Compute modes and coupling coefficients to leading order in Ω.

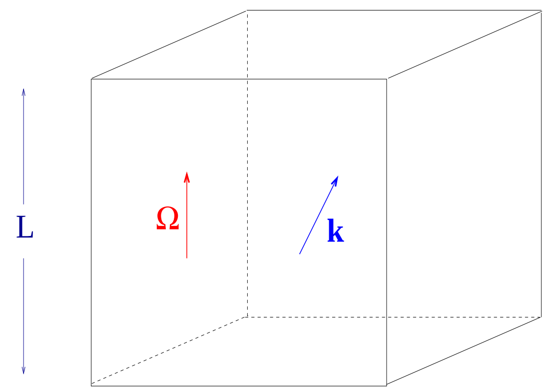
- Neglect buoyancy forces.

- Use Newtonian gravity, except for the assumed driving of the $l = m = 2$ r-mode.

- Consider only coupling to inertial modes, modes for which $\omega \propto \Omega$.

- Use WKB approximation for sea of inertial modes, and also Cowling approximation.

Inertial modes in a torus



Incompressible fluid, periodic BCs, no gravity. Modes are $\delta p \propto \exp\left[i\mathbf{k}\cdot\mathbf{x} - i\omega t\right]$, where $\mathbf{k} = 2\pi(n_x, n_y, n_z)/L$. Dispersion relation is

$$\omega = 2\Omega\frac{|k_z|}{|\mathbf{k}|}\left[1 + O\left(\frac{\Omega^2}{c_s^2 k^2}\right)\right].$$

Stellar inertial modes

- Previous numerical studies (eg Friedman & Lockitch 1999) did not consider the WKB regime. We extend the treatment of Bryan (1899) and Lindblom and Ipser (1999) to compute analytic formulae for the modes in WKB regime.

- Use a frequency dependent coordinate system $\varphi, \theta_1, \theta_2$ related to cylindrical polar coordinates R, φ, z by

$$R = \frac{r_*}{1-\mu^2} \sin\theta_1 \sin\theta_2 \qquad z = \frac{r_*}{|\mu|} \cos\theta_1 \cos\theta_2,$$

  were r_* is the stellar radius and $\mu = \omega/(2\Omega)$.

- Quantum numbers n, k, m; modes are

$$\delta p \propto \frac{\cos(p\theta_1 + \alpha)\cos(p\theta_2 + \alpha)}{\sqrt{\rho \sin\theta_1 \sin\theta_2}} e^{im\varphi},$$

  where $p = \sqrt{n(n+1) - m(m+1)}$, and dispersion relation is

$$\omega_{knm} \approx 2\Omega \left[\frac{k\pi}{n} + \frac{m}{n^2}\right].$$

Coupling coefficients and damping rates

- The coupling will be very small unless approximate momentum conservation laws are satisfied:

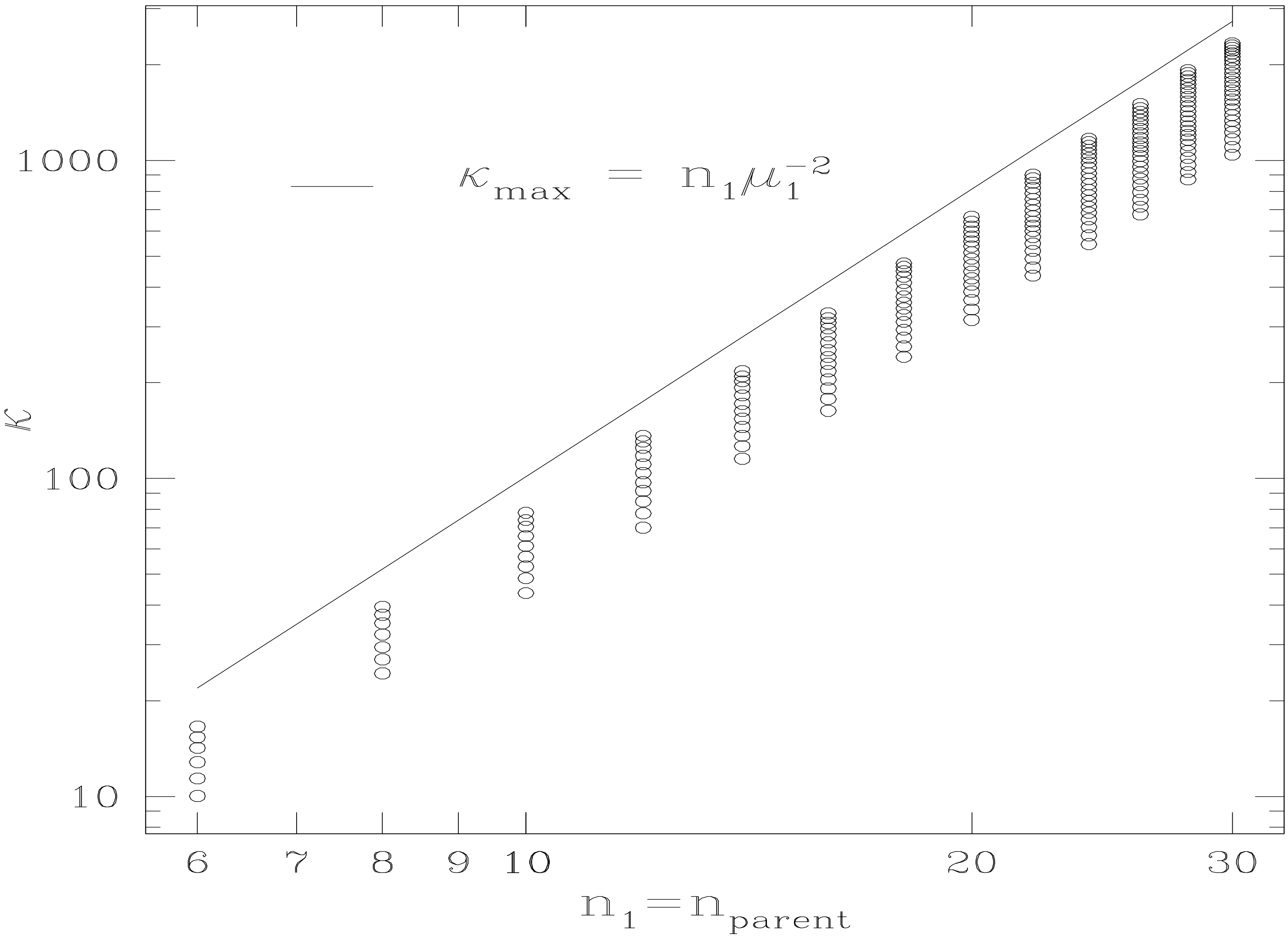
$$m_1 + m_2 + m_3 = 0 \qquad |n_1 \pm n_2 \pm n_3| \le (\text{few}) \qquad |k_1 \pm k_2 \pm k_3| \le (\text{few}).$$

- We can derive analytical approximation for the case $\omega_2 \sim \omega_3 \sim -\omega_1/2$

$$\max_{2,3}\ \kappa_{123} \sim n_1 \mu_1^{-2}.$$

- Damping rates: $\gamma_{\text{bulk}} \approx 1.7 \times 10^{-10}\,\text{sec}^{-1}\ T_9^6 \nu_{\text{kHz}}^{-2} \frac{\ln k}{\mu^2}.$

$$\gamma_{\text{shear}} \approx 3.8 \times 10^{-9}\,\text{sec}^{-1}\, T_9^{-2} \frac{n^2}{1-\mu^2}.$$



$\kappa_{max} = n_1 \mu_1^{-2}$
1000
100
10
κ
6
7
8
9
10
20
30
$n_1 = n_{parent}$

Mode triplets

- In other similar systems with unstable or driven modes, the saturation amplitude is often set by a parametric instability threshold.

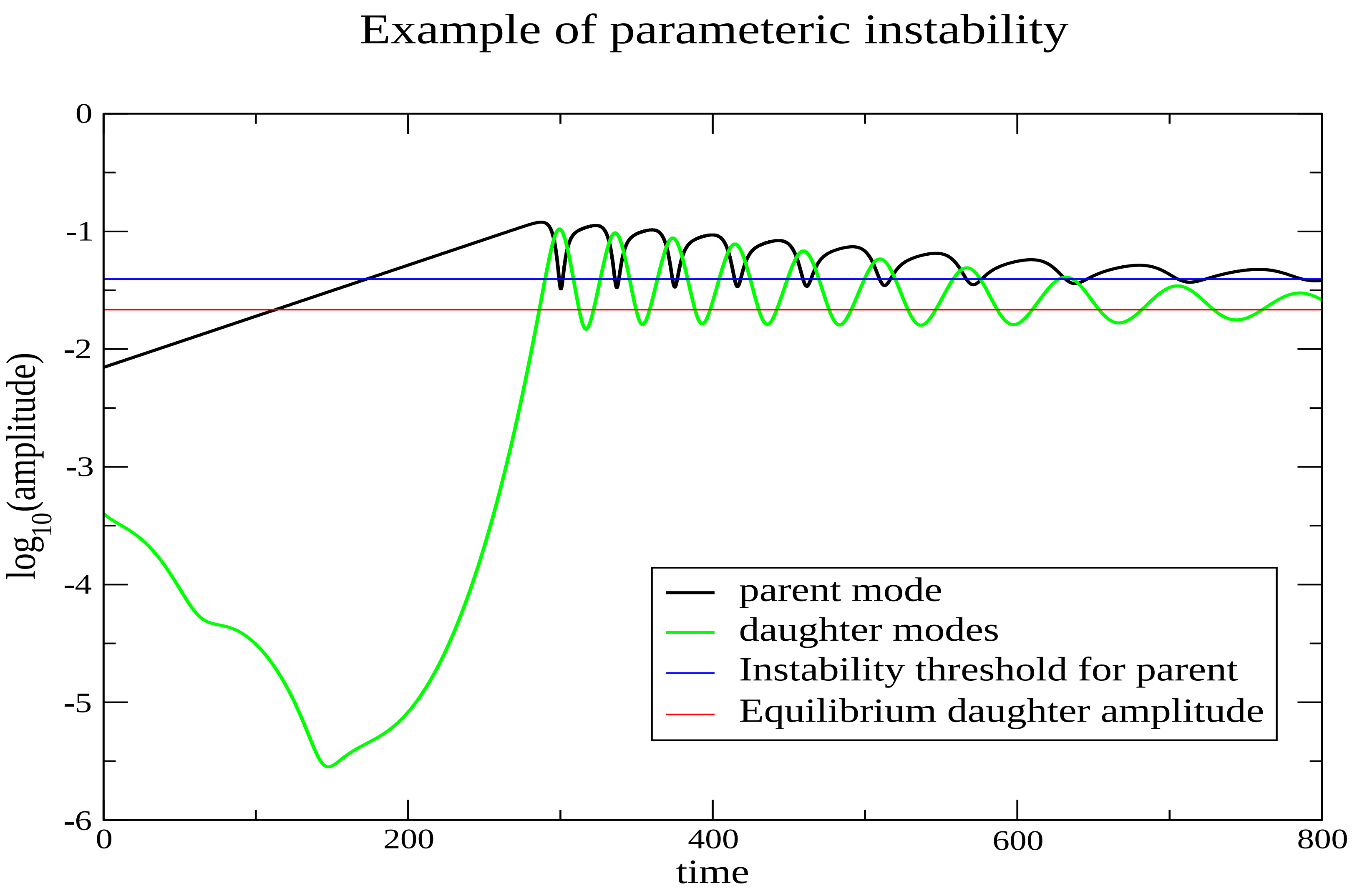
- Consider two daughter modes with frequencies ω_1 and ω_2 , and damping rates γ_1 and γ_2 , such that $\delta\omega \equiv w_r - w_1 - w_2 \approx 0$. When the amplitude of the $l = m = 2$ r-modes exceeds the threshold value given by

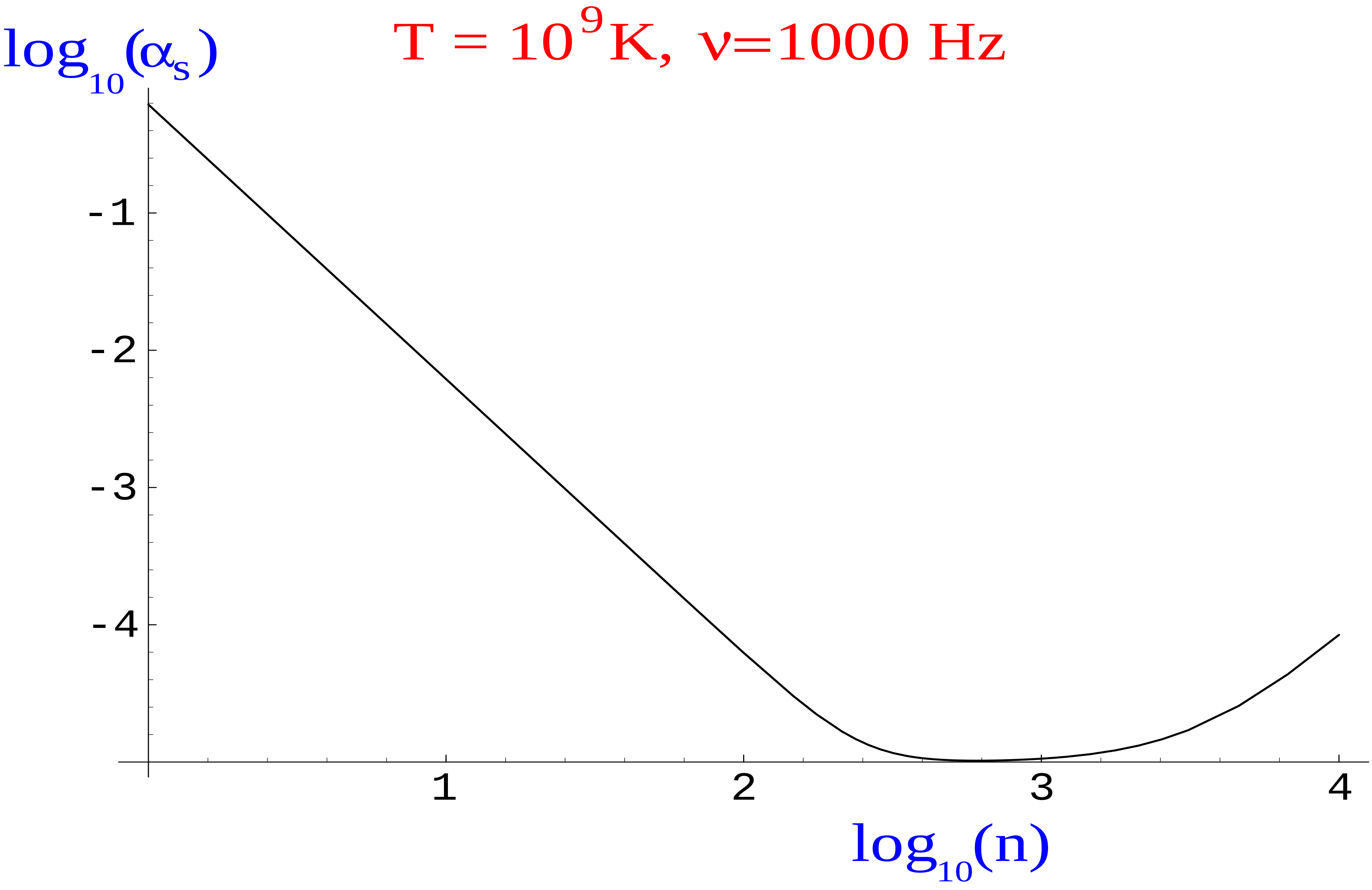
$$\alpha_{\rm th}^2(\tau) = \frac{\bar{\gamma}_1\bar{\gamma}_2}{4|\kappa_{r12}|^2\omega_1\omega_2}\left[1 + \left(\frac{\delta\omega}{\bar{\gamma}_1 + \bar{\gamma}_2}\right)^2\right],$$

  where $\bar{\gamma}_{1,2} = \gamma_{1,2} + 1/\tau$, the daughter modes grow exponentially with growth time τ.

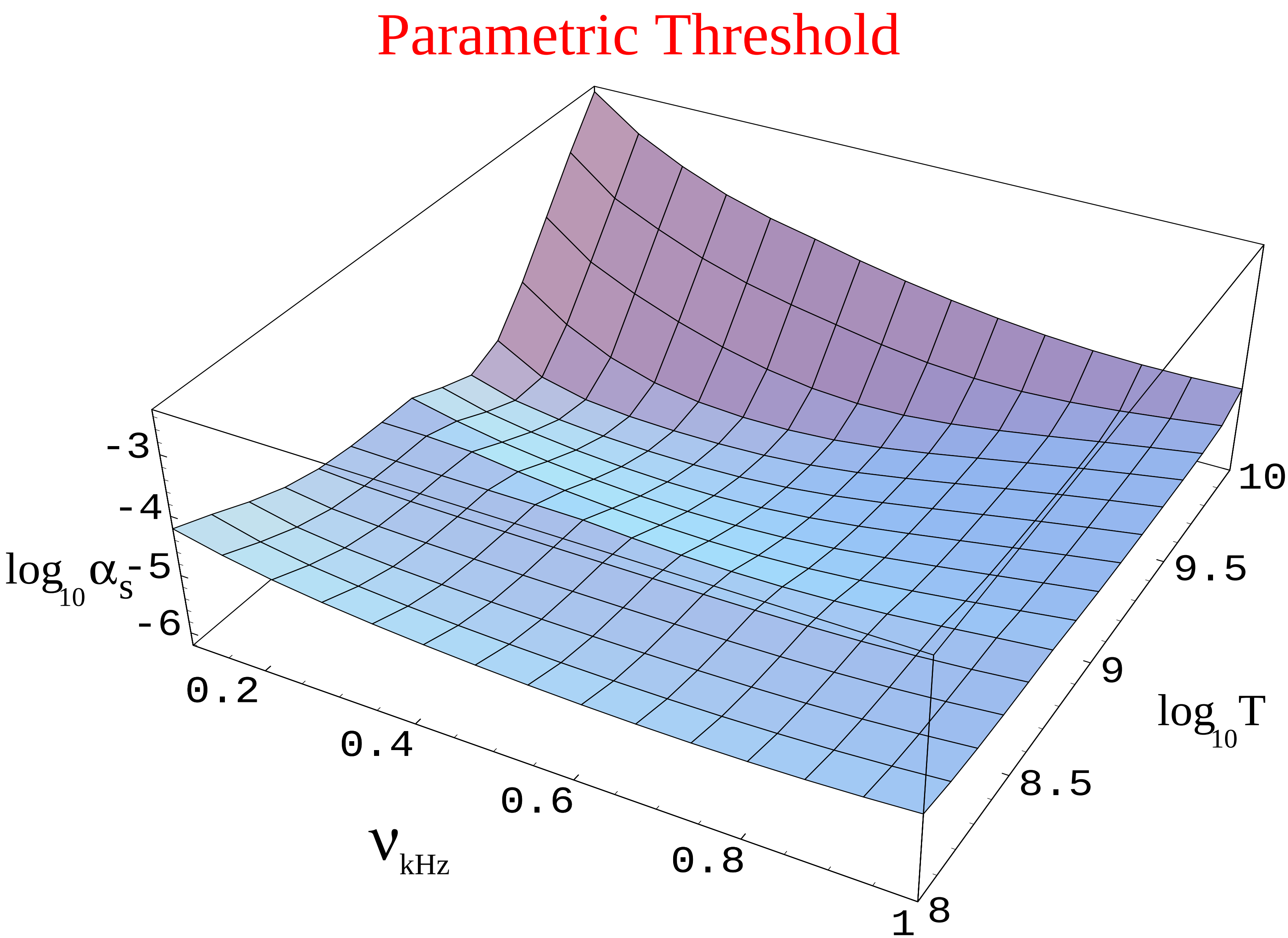
- If the daughter modes' damping is large enough, $\gamma_1 + \gamma_2 \geq \gamma_p$, there is an attractor equilibrium solution with

$$\alpha^2 = \alpha_{\rm eq}^2 = \frac{\gamma_1\gamma_2}{4|\kappa_{r12}|^2\omega_1\omega_2}\left[1 + \left(\frac{\delta\omega}{\gamma_1 + \gamma_2 - |\gamma_p|}\right)^2\right].$$


Example of parameteric instability
0
-1
-2
-3
-4
-5
-6
0
200
400
600
800
time
log10(amplitude)
parent mode
daughter modes
Instability threshold for parent
Equilibrium daughter amplitude

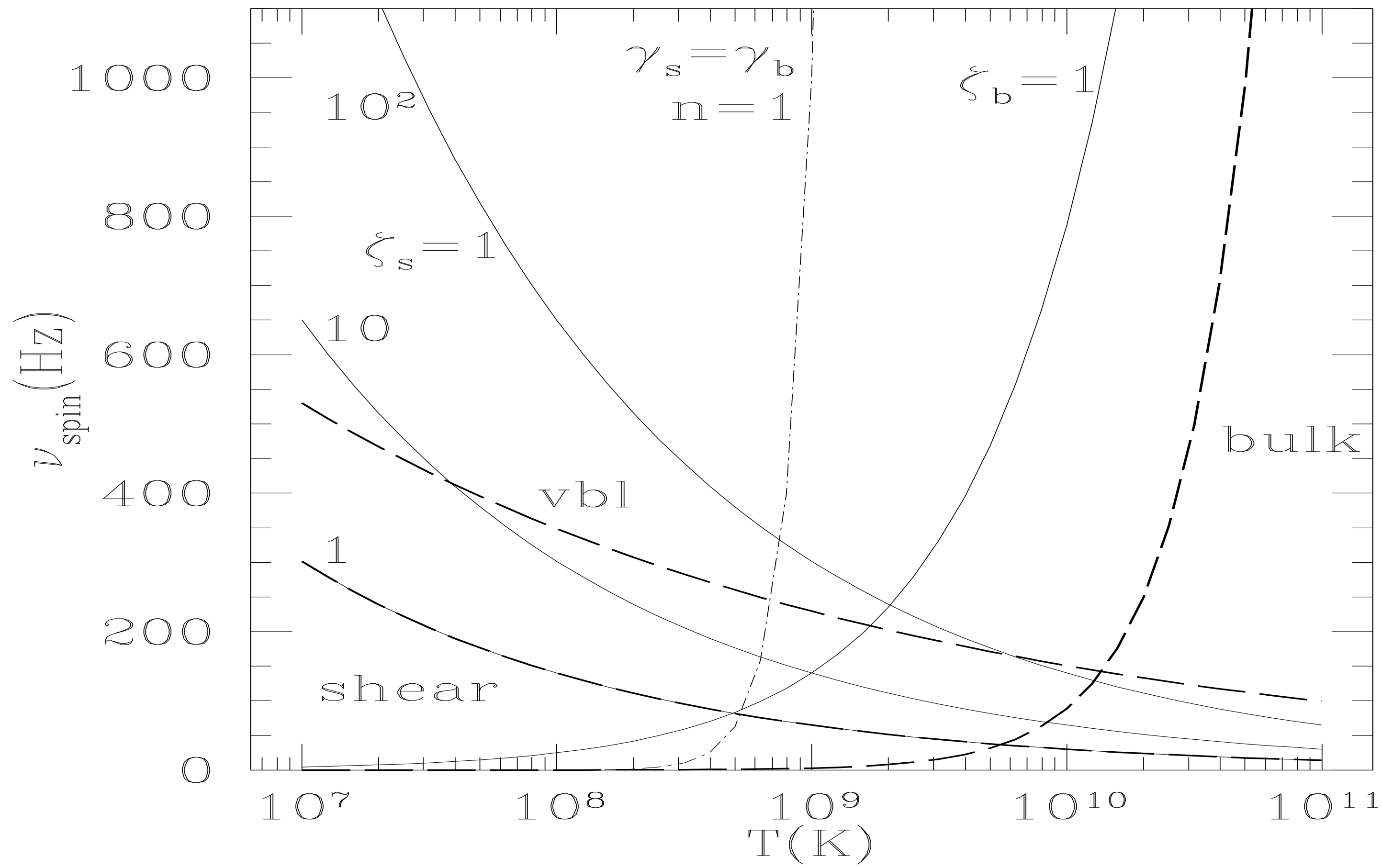


$\log_{10}(\alpha_s)$
$T = 10^9$K, ν=1000 Hz
-1
-2
-3
-4
1
2
3
4
$\log_{10}(n)$



Parametric Threshold
-3
-4
log10 αs -5
-6
0.2
0.4
0.6
0.8
1
νkHz
8
8.5
9
9.5
10
log10 T

Instability Window



Turbulent cascade

- Throughout most of the Ω, T plane, once the r-mode amplitude exceeds the parametric threshold, a large number of inertial modes are excited. The parametric threshold is not a robust upper bound. We estimate the resulting r-mode amplitude using weak turbulence theory (Zhakarov, 1992).

- By (i) assuming random phase approximation, and (ii) averaging over times long compared to mode frequencies but short compared to damping times, we can derive equations for the "occupation numbers" $N_\alpha = |c_\alpha|^2/\omega_\alpha$:

$$
\begin{aligned}
\dot{N}_\alpha \;=\; & 4\pi \sum_{\beta\gamma} \omega_\alpha\omega_\beta\omega_\gamma \,\mathrm{sign}(\Gamma) \left\{ |\kappa_{\alpha\beta\gamma}|^2 \delta_\Gamma(\omega_\alpha+\omega_\beta+\omega_\gamma) \left[N_\beta N_\gamma + N_\alpha N_\gamma + N_\alpha N_\beta\right] \right. \\
& + |\kappa_{\alpha\bar{\beta}\gamma}|^2 \delta_\Gamma(\omega_\alpha-\omega_\beta+\omega_\gamma) \left[N_\beta N_\gamma - N_\alpha N_\gamma + N_\alpha N_\beta\right] \\
& + |\kappa_{\alpha\beta\bar{\gamma}}|^2 \delta_\Gamma(\omega_\alpha+\omega_\beta-\omega_\gamma) \left[N_\beta N_\gamma + N_\alpha N_\gamma - N_\alpha N_\beta\right] \\
& \left. + |\kappa_{\alpha\bar{\beta}\bar{\gamma}}|^2 \delta_\Gamma(\omega_\alpha-\omega_\beta-\omega_\gamma) \left[N_\beta N_\gamma - N_\alpha N_\gamma - N_\alpha N_\beta\right] \right\} - 2\gamma_\alpha N_\alpha,
\end{aligned}
$$

where $\Gamma = \gamma_\alpha + \gamma_\beta + \gamma_\gamma$ and $\delta_\Gamma(\omega) \equiv |\Gamma|/\pi(\omega^2+\Gamma^2)$.

Turbulent cascade (cont)

- We make the approximations (i) neglect damping; (ii) use a continuum approximation

$$\sum_{\alpha} \to \int d^3\mathbf{k} = \int dk_z d^2\mathbf{k}_\perp;$$

(iii) use the asymptotic scaling

$$\kappa(ak_{z1}, b\mathbf{k}_{\perp 1}, ak_{z2}, b\mathbf{k}_{\perp 2}, ak_{z3}, b\mathbf{k}_{\perp 3}) = \frac{b^3}{a^2}\kappa(k_{z1}, \mathbf{k}_{\perp 1}, k_{z2}, \mathbf{k}_{\perp 2}, k_{z3}, \mathbf{k}_{\perp 3})$$

to compute the steady state solution

$$N(k_z, \mathbf{k}_\perp) = N_0 k_z^{-1/2} |\mathbf{k}_\perp|^{-7/2}.$$

- By matching onto the r-mode driving region in phase space, we can obtain the normalization constant N_0. The resulting value of r-mode energy is $E_{\mathrm{rmode}} = MR^2\Omega^2/2 \times \alpha_e \gamma_{\mathrm{GR}}/\Omega$, where we estimate $\alpha_e \sim 0.1$.

Conclusions

- We believe the low saturation amplitude result is robust.

- The r-modes are no longer as promising gravitational wave sources, but may still be relevant for the spin evolution of neutron stars.

- Illustrates the limitations of numerical simulations for complex problems.