I. THE RIEMANN TENSOR (10 POINTS)

**WARNING!** Be careful how you use equations from this problem. The equations in Hartle’s supplement, along with Eqs. (1.1), (1.2), (1.6)-(1.9), and (1.17), are valid in any coordinate system. However, all of the other equations used in this problem (that is, those which are not direct rewritings of the equations just listed) are in general only valid at the origin of a local inertial coordinate system, where \( g_{\alpha\beta,\gamma} = 0 \).

A.

In lecture the Riemann tensor was defined by the equation of geodesic deviation [Eq. (21.19) in Hartle]. By following the steps in the attached supplement by Hartle\(^1\), we find that the general expression for the Riemann tensor is given by

\[
R_{\alpha\beta\gamma\delta} = \Gamma^\sigma_{\beta\delta,\gamma} - \Gamma^\sigma_{\beta\gamma,\delta} + \Gamma^\sigma_{\gamma\epsilon} \Gamma^\epsilon_{\beta\delta} - \Gamma^\sigma_{\delta\epsilon} \Gamma^\epsilon_{\beta\gamma},
\]

(1.1)

B.

Recall that the connection coefficients are related to the metric by

\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (-g_{\beta\gamma,\delta} + g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta}).
\]

(1.2)

So in coordinates where \( g_{\alpha\beta,\gamma} = 0 \) at some point, we have at that point

\[
\Gamma^\alpha_{\beta\gamma} = 0, \quad \Gamma^\alpha_{\beta\gamma,\lambda} = \frac{1}{2} g^{\alpha\xi} (-g_{\beta\gamma,\xi\lambda} + g_{\xi\beta,\gamma\lambda} + g_{\gamma\xi,\beta\lambda}).
\]

(1.3)

Then from the general expression for the Riemann tensor, we have

\[
R_{\alpha\beta\gamma\delta} = g_{\sigma\alpha} (\Gamma^\sigma_{\beta\delta,\gamma} - \Gamma^\sigma_{\beta\gamma,\delta})
\]

\[
= \frac{1}{2} \left[ \delta^\xi_\alpha (-g_{\beta\xi,\gamma\delta} + g_{\xi\beta,\gamma\delta} + g_{\gamma\xi,\beta\delta}) - \delta^\xi_\alpha (-g_{\beta\gamma,\xi\delta} + g_{\xi\beta,\gamma\delta} + g_{\gamma\xi,\beta\delta}) \right]
\]

\[
= \frac{1}{2} (-g_{\beta\delta,\alpha\gamma} + g_{\alpha\beta,\delta\gamma} + g_{\delta\alpha,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\beta,\gamma\delta} - g_{\gamma\alpha,\beta\delta})
\]

\[
= \frac{1}{2} (g_{\beta\delta,\alpha\gamma} + g_{\delta\alpha,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\gamma\alpha,\beta\delta}),
\]

(1.4)

where we have used \( g_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\delta\gamma} \), to cancel two of the terms. Lastly if we use \( g_{\xi\alpha} = g_{\alpha\xi} \), in the second and fourth terms, and then move those terms to the front, we find

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\beta\gamma,\alpha\delta} - g_{\beta\delta,\alpha\gamma} + g_{\beta\gamma,\alpha\delta}).
\]

(1.5)

C.

We want to verify that the following four symmetries of the Riemann tensor\(^2\)

\[
R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta},
\]

(1.6)

\[
R_{\alpha\beta\gamma\delta} = -R_{\alpha\delta\beta\gamma},
\]

(1.7)

\[
R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta},
\]

(1.8)

\[
R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0,
\]

(1.9)

---

\(^1\) This can be found on the companion website for Hartle’s book at [www.aw.com](http://www.aw.com), and is appended at the end of these solutions.

\(^2\) The Riemann tensor (1.1) actually obeys these symmetries in any coordinate system, but this is easier to show in local inertial coordinates.
in a local inertial coordinate system, where Eq. (1.5) is valid.

Start by substituting Eq. (1.5) into the left hand side of the first symmetry (1.6),

\[-R_{\beta\alpha\gamma\delta} = -\frac{1}{2} (g_{\beta\delta,\alpha\gamma} - g_{\beta\gamma,\alpha\delta} - g_{\alpha\delta,\beta\gamma} + g_{\alpha\gamma,\beta\delta})
= \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= R_{\alpha\beta\gamma\delta}. \quad (1.10)\]

The same treatment of the second symmetry (1.7) gives

\[-R_{\alpha\beta\delta\gamma} = -\frac{1}{2} (g_{\alpha\gamma,\beta\delta} - g_{\alpha\delta,\beta\gamma} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= R_{\alpha\beta\gamma\delta}. \quad (1.11)\]

Likewise for the third symmetry (1.8)

\[R_{\gamma\delta\alpha\beta} = \frac{1}{2} (g_{\gamma\beta,\delta\alpha} - g_{\gamma\alpha,\delta\beta} - g_{\delta\beta,\gamma\alpha} + g_{\delta\alpha,\gamma\beta}),
= \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= R_{\alpha\beta\gamma\delta}. \quad (1.12)\]

where we have used both \(g_{\xi\sigma} = g_{\sigma\xi}\), and \(g_{\kappa\lambda,\xi\sigma} = g_{\kappa\lambda,\sigma\xi}\), which we will use freely from here on, in each of the four terms. Finally for the last symmetry (1.9), we have

\[-R_{\alpha\delta\beta\gamma} - R_{\alpha\gamma\delta\beta} = -\frac{1}{2} (g_{\alpha\gamma,\delta\beta} - g_{\alpha\beta,\delta\gamma} - g_{\delta\gamma,\alpha\beta} + g_{\delta\beta,\alpha\gamma})
= \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= R_{\alpha\beta\gamma\delta}. \quad (1.13)\]

D.

We will verify the identity

\[R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\epsilon\gamma,\delta} + R_{\alpha\beta\delta\epsilon,\gamma} = 0, \quad (1.14)\]

which is only valid at the origin of a local inertial frame where \(g_{\alpha\beta,\gamma} = 0\), by direct substitution of the local inertial frame expression for the Riemann tensor (1.5).

\[-R_{\alpha\beta\epsilon\gamma} - R_{\alpha\beta\gamma\epsilon} = -\frac{1}{2} (g_{\alpha\gamma,\beta\epsilon} - g_{\alpha\epsilon,\beta\gamma} - g_{\beta\epsilon,\alpha\gamma} + g_{\beta\gamma,\alpha\epsilon})
= \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta} + g_{\beta\delta,\alpha\gamma})
= R_{\alpha\beta\gamma\delta}. \quad (1.15)\]

In this coordinate system the connection coefficients vanish at the origin, so covariant derivatives are the same as partial derivatives for quantities evaluated at that point:

\[T_{\alpha,\beta} = \nabla_{\beta} T_{\alpha} = \partial_{\beta} T_{\alpha} = T_{\alpha,\beta}, \quad (1.16)\]

where \(T_{\alpha}\) is any tensor. This means that in this coordinate system we are free to rewrite the identity (1.14) as

\[R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\epsilon\gamma,\delta} + R_{\alpha\beta\delta\epsilon,\gamma} = 0. \quad (1.17)\]

But this expression is completely covariant, unlike Eq. (1.14). It contains only tensors (which are always covariant) and covariant derivatives. So it must be true in any coordinate system. The identity (1.17) is known as the Bianchi identity.
II. HARTLE CHAPTER 21, PROBLEM 3 (5 POINTS)

Consider the Schwarzschild spacetime generated by a point mass $M_\odot \sim 10^{33}$ g. Suppose that a one dimensional meter stick is placed on a radial line: the radial coordinate of the far end is $r + \chi/2$, and the near end is at $r - \chi/2$. If the mass of the stick is $\mu \sim 10$ g $\ll M_\odot$, then as long as $r \gg M_\odot \sim 10^5$ cm, the physics is Newtonian and $\chi \sim 10^2$ cm. The radial tidal force on the stick will be a stretching force, and in this limit, using the result from Example 21.1 in Hartle [Eq. (21.9)], the magnitude of the force is

$$F_r = \frac{2M_\odot \mu}{r^3} \chi.$$  

(2.1)

If the meter stick would break under the weight of a $m \sim 10^4$ g on the Earth’s surface, then in the situation described above the stick would break when $F_r \sim M_\odot m/R_\odot^2$, or rather when $r = r_{\text{break}}$, where

$$r_{\text{break}} \sim \left( \frac{2M_\odot \mu R_\odot^2}{M_\odot m} \chi \right)^{1/3} \sim 10^7 \text{ cm}. \quad (2.2)$$

Since $r_{\text{break}} \sim 10^2 M_\odot$, the Newtonian treatment is reasonable. Note however that $r_{\text{break}} \sim 10^{-4} R_\odot$, so the tidal force produced by the sun itself would certainly not be able to stretch an orbiting meter stick to its breaking point.

III. HARTLE CHAPTER 21, PROBLEM 5 (8 POINTS)

In Sec. I of the solutions for homework 6, we showed that in the Newtonian limit

$$\Gamma^\alpha_{\beta\gamma} = (0) \Gamma^\alpha_{\beta\gamma} + O(\varepsilon^2) \quad (3.1)$$

where $\varepsilon$ is a formal expansion parameter, and the only non-vanishing $(0) \Gamma^\alpha_{\beta\gamma}$ are

$$(0) \Gamma^i_{tt} = \delta^{ik} \Phi_k, \quad (3.2)$$

where $\Phi$ is the Newtonian gravitational potential. We can derive Eq. (21.27) in Hartle

$$R^i_{tjlt} = \Gamma^i_{tt,j} + (\text{post-Newtonian terms}) = \delta^{ik} \frac{\partial^2 \Phi}{\partial x^k \partial x^j} + (\text{post-Newtonian terms}), \quad (3.3)$$

by direct substitution of Eq. (3.1) into the general expression for the Riemann tensor (1.1). The substitution gives

$$R^i_{tjlt} = \Gamma^i_{tt,j} - \Gamma^i_{tj,t} + \Gamma^t_{jt} \Gamma^i_{tt} - \Gamma^t_{tt} \Gamma^i_{jt} = \Gamma^i_{tt,j} + O(\varepsilon^2) = \delta^{ik} \frac{\partial^2 \Phi}{\partial x^k \partial x^j} + O(\varepsilon^2), \quad (3.4)$$

which is Hartle’s result (3.3) with the post-Newtonian terms represented by terms of order $\varepsilon^2$.

IV. HARTLE CHAPTER 21, PROBLEM 8 (10 POINTS)

We want to compute $R_{tttr}$ for a radially free falling observer in the Schwarzschild spacetime, who started from rest at $r = \infty$. We will then project this onto the observers local orthonormal basis $\{\vec{e}_\alpha\}$ to obtain

$$R_{tttr} = R_{\alpha\beta\gamma\delta} (\vec{e}_i)^\alpha (\vec{e}_j)^\beta (\vec{e}_k)^\gamma (\vec{e}_l)^\delta.$$  

(4.1)

The vectors $\vec{e}_\alpha$ are given in example 21.12. We will only need two of them, $\vec{e}_t$ and $\vec{e}_r$, in which the Schwarzschild coordinate basis $\{\vec{\partial}_\alpha\}$ are [Eqs. (20.81a) and (20.81b) in Hartle]

$$\vec{e}_t = \left(1 - \frac{2M}{r}\right)^{-1} \vec{\partial}_t - \sqrt{\frac{2M}{r}} \vec{\partial}_r, \quad \vec{e}_r = -\sqrt{\frac{2M}{r}} \left(1 - \frac{2M}{r}\right)^{-1} \vec{\partial}_t + \vec{\partial}_r.$$  

(4.2)

This is just Eq. (1.7) from the solutions for homework 6, with the right hand side modified to keep the indices lined up properly.
As written, each of these vectors has two nonvanishing components. So Eq. (4.1), which has four summation indices, appears to represent a sum with $2^4 = 16$ terms, each of which involves a different component of the Riemann tensor. However because of the symmetries of the Riemann tensor (1.6)-(1.9), it will turn out that we can write Eq. (4.1) in terms of only one component of the Riemann tensor. Using the first three symmetries (1.6)-(1.8), we can reduce the number of needed components to six, which are

$$\begin{align*}
R_{ttrr} &= -R_{rtrt} = R_{rrtt} = -R_{rrrt}, \\
R_{rtrr} &= -R_{rrtr} = R_{rrtt} = -R_{rrrt}, \\
R_{trtr} &= -R_{rrtr} = R_{rrtt} = -R_{rrrt}, \\
R_{tttr} &= R_{rtrt}, \quad R_{tttt}, \quad R_{rrrr}.
\end{align*}$$

The last of the symmetries (1.9) can be used to show that all but one of these six vanish. Setting $\alpha = \beta = \gamma = \delta$, to either $t$ or $r$ in Eq. (1.9) gives

$$R_{tttt} = R_{rrrr} = 0.$$  \hspace{1cm} (4.7)

Similarly setting $\alpha = t$ and $\beta = \gamma = \delta = r$, and then vice versa, in Eq. (1.9) gives

$$R_{ttrr} = R_{rrtt} = 0.$$  \hspace{1cm} (4.8)

Lastly by setting $\alpha = \beta = t$ and $\gamma = \delta = r$, in Eq. (1.9), and then using the second symmetry (1.7), we find

$$R_{tttt} = 0.$$  \hspace{1cm} (4.9)

So the only non-vanishing $t$ and $r$ components of the Riemann tensor are given by $\pm \mathcal{R}$, where

$$\mathcal{R} = R_{ttrt} = -R_{rrtr} = R_{rrrt} = -R_{tttt}.$$  \hspace{1cm} (4.10)

Putting these results into Eq. (4.1) gives

$$R_{t\hat{e}i\hat{e}j} = \mathcal{R} \left\{ \left[ (\hat{e}_i)^\gamma \right]^2 [ (\hat{e}_j)^\gamma ]^2 + [ (\hat{e}_i)^\gamma ]^2 [ (\hat{e}_j)^\gamma ]^2 - 2 (\hat{e}_i)^\gamma (\hat{e}_j)^\gamma (\hat{e}_l)^\gamma (\hat{e}_l)^\gamma \right\}.$$  \hspace{1cm} (4.11)

Then if insert the values for the components of the basis vectors from Eq. (4.2), we find that the quantity in the curly braces simplifies to unity, so that

$$R_{t\hat{e}i\hat{e}j} = \mathcal{R}.$$  \hspace{1cm} (4.12)

We can find $\mathcal{R}$ by computing any of the nonvanishing components (4.10) from the general expression for the Riemann tensor (1.1). For example$^4$,

$$\mathcal{R} = R_{ttrt} = g_{r\hat{e}} R_{rrt} = - \left( 1 - \frac{2M}{r} \right) \left( \Gamma^t_{rr,t} - \Gamma^t_{rt,r} + \Gamma^r_{rrt} - \Gamma^r_{rtt} \right),$$  \hspace{1cm} (4.13)

where the metric components were taken from the Schwarzschild line element

$$ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hspace{1cm} (4.14)

The necessary $\Gamma^\alpha_{\beta\gamma}$ can be found on page 546, in Appendix B, of Hartle: $\Gamma^i_{rr} = 0$, $\Gamma^r_{rr} = Mr^{-2}(1 - 2M/r)^{-1}$, $\Gamma^t_{r\hat{e}} = \delta^t_i \Gamma^i_{r\hat{e}}$, $\Gamma^r_{r\hat{e}} = -\delta^r_i \Gamma^i_{r\hat{e}}$, $\Gamma^i_{t\hat{e}} = \delta^i_t \Gamma^i_{r\hat{e}}$, $\Gamma^t_{t\hat{e}} = \delta^t_r \Gamma^r_{t\hat{e}}$. Substitution into Eq. (4.13) gives

$$\mathcal{R} = \left( 1 - \frac{2M}{r} \right) \left[ \Gamma^t_{r\hat{e},t} + 2 \left( \Gamma^r_{r\hat{e}} \right)^2 \right] = - \frac{2M}{r^2}.$$  \hspace{1cm} (4.15)

$^4$ Notice that for this case (radial free falling observers starting from rest at $r = \infty$, in the Schwarzschild spacetime) $R_{ttrt} = R_{t\hat{e}i\hat{e}j}$. This equality will not in general be valid for other classes of observers or in other spacetimes. The Schwarzschild spacetime has the special feature that its Riemann tensor is invariant under boosts in the radial direction. The same answer is also obtained for the $t\hat{e}i\hat{e}j$ component of the Riemann tensor on the orthonormal basis of a static observer, which is given in Appendix B of Hartle.
This leaves us with the expected result

\[ R_{trtr} = -\frac{2M}{r^3}. \] (4.16)

Many of you incorrectly claimed that instead of resorting to Appendix B, we could have used the local inertial frame expression of the Riemann tensor (1.5) to get

\[ R_{trtr} = \frac{1}{2} (g_{tr,rt} - g_{tt,rr} - g_{rr,tt} + g_{rt,rt}), \] (4.17)

where the metric components are taken from the Schwarzschild line element (4.14). Then since \( g_{tr} = g_{rt} = 0 \), and \( g_{rr,tt} = 0 \), by \( \partial_t r = 0 \), Eq. (4.17) gives

\[ R_{trtr} = -\frac{1}{2} g_{tt,rr} = -\frac{2M}{r^3}, \] (4.18)

leaving us with the correct result (4.16) since \( R^\hat{t}\hat{r} = R = R_{trtr} \). Unfortunately, although this computation yields the correct result, the argument is incorrect! The local inertial frame expressions (1.5) and (4.17) require that the coordinate system used is such that \( g_{\alpha\beta,\gamma} = 0 \) at the point in question, and this is not the case for Schwarzschild coordinates (4.14).

V. HARTLE CHAPTER 21, PROBLEM 11 (8 POINTS)

The line element for the hyperbolic plane was given in problem 12, of chapter 8 in Hartle

\[ ds^2 = y^{-2} dx^2 + y^{\frac{1}{2}} dy^2, \quad (y \geq 0). \] (5.1)

We want to calculate the Ricci curvature scalar \( R \), which is the fully contracted Riemann tensor

\[ R = g^{\beta\delta} R^\gamma_{\beta\gamma\delta}. \] (5.2)

From the general form of the Riemann tensor (1.1), and from \( g^{\alpha\beta} = y^{2} \delta^{\alpha\beta} \), we have

\[ R = y^{2}\delta^{\beta\delta} \left( \Gamma^\gamma_{\beta\delta,\gamma} - \Gamma^\gamma_{\beta\gamma,\delta} + \Gamma^\gamma_{\gamma\epsilon} \Gamma^\epsilon_{\beta\delta} - \Gamma^\gamma_{\delta\epsilon} \Gamma^\epsilon_{\beta\gamma} \right). \] (5.3)

Instead of computing the \( \Gamma^\alpha_{\beta\gamma} \) directly from their definition, we will first compute [Eq. (1.2) on the solutions to homework 6]

\[ g_{\gamma\lambda} \ddot{x}^\lambda = \left( \frac{1}{2} g_{\alpha\beta,\gamma} - g_{\gamma\alpha,\beta} \right) \dot{x}^\alpha \dot{x}^\beta, \] (5.4)

where the overdot represents a derivative with respect to the curve parametrization. Setting \( \gamma \) to \( x \) and \( y \) gives

\[ \ddot{x} = \frac{2}{y} \dot{x} \dot{y}, \quad \ddot{y} = -\frac{1}{y} \dot{x}^2 + \frac{1}{y^2}. \] (5.5)

Comparing these to the geodesic equation \( \ddot{\alpha} = -\Gamma^\alpha_{\beta\gamma} \dot{\beta} \dot{\gamma} \dot{\alpha} \) gives

\[ \Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -\Gamma^y_{xx} = -\frac{1}{y}, \quad \Gamma^x_{xx} = \Gamma^y_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0. \] (5.6)

Substituting these into the expression for the Ricci curvature scalar (5.3) gives

\[ R = -2. \] (5.7)

This confirms that the hyperbolic plane (5.1) has constant negative curvature.
Supplement to Chapter 21:
Deriving the Equation of Geodesic Deviation
and a Formula for the Riemann Tensor

The equation of geodesic deviation (21.19) relates the acceleration of the separation vector \( \chi \) between two nearly geodesics to the Riemann curvature exhibited explicitly in (21.20). This supplement works through more of the details of the derivation that was sketched in Sectin 21.2. It parallels the conceptually similar but algebraically simpler derivation of the Newtonian geodesic equation (21.5).

The separation four-vector \( \chi(\tau) \) connects a point \( x^\alpha(\tau) \) on one geodesic (the fiducial geodesic) to a point \( x^\alpha(\tau) + \chi^\alpha(\tau) \) on a nearby geodesic at the same proper time. Since there is no unique way of relating proper time on one geodesic to proper time on another there are many different separation vectors. The exact choice will not be important for us except to assume that the difference is small for nearby geodesics so that \( \chi \) is small.

The separation acceleration that is the left hand side of the equation of geodesic deviation is

\[
\mathbf{w} \equiv \nabla_u \nabla_v \chi = \nabla_u \mathbf{v}
\]

(1)

where \( u \) is the four-velocity of the fiducial geodesic and \( v \equiv \nabla_u \chi \). The coordinate basis components of \( \mathbf{w} \) and \( \mathbf{v} \) can be calculated using (20.54) for the covariant derivatives. Thus, as in (21.17) and (21.18),

\[
\nu^\alpha \equiv (\nabla_u \chi)^\alpha = u^\beta \nabla_\beta \chi^\alpha = \frac{d\chi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\delta} u^\beta \chi^\delta,
\]

(2)

\[
w^\alpha \equiv (\nabla_u v)^\alpha = u^\beta \nabla_\beta v^\alpha = \frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\delta\epsilon} u^\delta v^\epsilon.
\]

(3)

Here, expressions like \( u^\beta (\partial \chi^\alpha / \partial x^\beta) \) have been written \( d\chi^\alpha / d\tau \) following the general relation for any function \( f \) [cf. (21.14)].

\[
\frac{df}{d\tau} = \frac{df(x^\alpha(\tau))}{d\tau} = \frac{\partial f}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} = u^\alpha \frac{\partial f}{\partial x^\alpha}.
\]

(4)

The derivation consists of evaluating \( w^\alpha \) by substituting (2) into (3) and using the geodesic equation for the fiducial and nearby geodesic. Substituting (2) into (3)
gives

\[ w^\alpha = \frac{d^2 \chi^\alpha}{d\tau^2} + \frac{d}{d\tau} \left( \Gamma^\alpha_{\beta\gamma} u^\beta \chi^\gamma \right) + \Gamma^\alpha_{\delta\epsilon} \frac{d}{d\tau} \left( \frac{d\chi^\epsilon}{d\tau} + \Gamma^e_{\beta\gamma} u^\beta \chi^\gamma \right) \]

\[ = \frac{d^2 \chi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma} u^\beta \frac{d\chi^\gamma}{d\tau} + \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} u^\delta u^\beta \chi^\gamma + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{d\tau} \chi^\gamma + \Gamma^\epsilon_{\delta\epsilon} \Gamma^e_{\beta\gamma} u^\beta u^\delta \chi^\gamma. \] \hspace{1cm} (5)

Here, (4) has been used and the freedom to relabel dummy indices has been employed to group equal terms together, for instance

\[ \Gamma^\alpha_{\beta\gamma} u^\beta \frac{d\chi^\gamma}{d\tau} = \Gamma^\alpha_{\delta\epsilon} \frac{d\chi^e}{d\tau}. \] \hspace{1cm} (6)

To evaluate (5) note that

\[ \chi^\alpha(\tau) + \chi^\alpha(\tau) \text{ obeys the geodesic equation } (8.14) \]

\[ \frac{d^2 (x^\alpha + \chi^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} (\chi^\delta + \chi^\delta) \frac{d(x^\beta + \chi^\beta)}{d\tau} \frac{d(x^\gamma + \chi^\gamma)}{d\tau} = 0. \] \hspace{1cm} (7)

When \( \chi \) vanishes this is the geodesic equation for the fiducial geodesic. Since \( \chi \) is small, we need to keep only the first order term in (7) in an expansion in \( \chi^\alpha \).

[Compare the transition from the Newtonian (21.3) to (21.5).] Using \( u^\alpha = dx^\alpha / d\tau \), (4), and \( \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \) this is

\[ \frac{d^2 \chi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma} u^\beta \frac{d\chi^\gamma}{d\tau} + \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} u^\delta u^\beta \chi^\gamma = 0. \] \hspace{1cm} (8)

Again, the freedom to relabel dummy indices has been used to group equal terms together.

Eq. (8) can be used to eliminate the second derivative of \( \chi^\alpha \) from the expression (5) for \( w^\alpha \). The geodesic equation (8.15) can be used to eliminate the derivative of \( u^\beta \). The result is

\[ w^\alpha = -\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} u^\beta u^\gamma \chi^\delta + \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} u^\beta u^\delta \chi^\gamma - \Gamma^\alpha_{\beta\gamma} \Gamma^\beta_{\delta\epsilon} u^\delta u^\gamma \chi^\epsilon + \Gamma^\epsilon_{\delta\epsilon} \Gamma^e_{\beta\gamma} u^\beta u^\delta \chi^\gamma. \] \hspace{1cm} (9)

All the terms involving \( d\chi^\alpha / d\tau \) have luckily canceled since there is no equation to eliminate that. Using the freedom to relabel dummy indices a common factor \( u^\beta \chi^\gamma u^\delta \) can be identified in each term. The result is

\[ w^\alpha = -\left( \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\gamma_{\epsilon\beta} \Gamma^\epsilon_{\beta\gamma} - \Gamma^\epsilon_{\delta\epsilon} \Gamma^e_{\beta\gamma} \right) u^\beta \chi^\gamma u^\delta \equiv -R^\alpha_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta. \] \hspace{1cm} (10)

Thus, we derive both the geodesic equation (21.19) and the explicit expression for the Riemann tensor (21.20).