NOTE – We use the notion from lecture: four-vectors \( \vec{v} \) are denoted by an arrow, and three-vectors \( \mathbf{v} \) will be in bold. Hartle uses the opposite notation.

NOTE – We use units where \( c = 1 \).

## I. EXISTENCE OF LOCAL LORENTZ FRAMES (10 POINTS)

The transformation law for the metric components has been derived in lecture

\[
g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x'^\alpha} \frac{\partial x^\beta}{\partial x'^\beta} g_{\alpha\beta}. \tag{1.1}
\]

Near \( x^\alpha = 0 \), the point \( P \), our two coordinate systems are related by

\[
x^\alpha = \Lambda^\alpha_{\beta'} x'^{\beta'} + \frac{1}{2} M^\alpha_{\beta'\gamma'} x'^{\beta'} x'^{\gamma'} + O[(x')^3].
\]

Equation (1.2), which defines the tensor \( M \), has the property that any piece of \( M \) which is antisymmetric in the indices \( \beta' \) and \( \gamma' \) does not contribute. So, without loss of generality, we can assume that \( M \) is symmetric on that pair of indices. The derivatives in Eq. (1.1) are then given by

\[
\frac{\partial x^\alpha}{\partial x'^\alpha} = \Lambda^\alpha_{\alpha'} + x'^{\gamma'} M^\alpha_{\alpha'\gamma'} + O[(x')^2]
\]

where we have used \( x'^{\alpha'} = \delta^{\alpha'}_{\beta'} \), and \( M^\alpha_{\beta'\gamma'} = M^\alpha_{\gamma'\beta'} \). We now substitute this and the metric expansion near \( P \)

\[
g_{\alpha\beta}(x^\mu) = g^0_{\alpha\beta} + g^0_{\alpha\beta,\gamma} x^\gamma + O[(x')^2] = g^0_{\alpha\beta} + g^0_{\alpha\beta,\gamma'} \Lambda^\gamma_{\gamma'} x^\gamma' + O[(x')^2],
\]

into Eq. (1.1) to find the expected result (written in a slightly different form then on the problem set)

\[
g_{\alpha'\beta'} = \Lambda^\alpha_{\alpha'} g^0_{\alpha\beta} \Lambda^\beta_{\beta'} + \left( \Lambda^\alpha_{\alpha'} g^0_{\alpha\beta} M^\beta_{\gamma'\beta'} + g^0_{\alpha\beta,\gamma'} \Lambda^\beta_{\gamma'\beta'} + \Lambda^\gamma_{\gamma'} \Lambda^\beta_{\beta'} g^0_{\alpha\beta,\gamma'} \right) x^\gamma' + O[(x')^2]. \tag{1.5}
\]

Now define matrices \( \mathcal{L} \), \( \mathcal{G}^0 \), and \( \mathcal{N} \) such that for each matrix, the element in row \( a \) and column \( b \) is given by

\[
(\mathcal{L})_{ab} = \Lambda^a_{b}, \quad (\mathcal{G}^0)_{ab} = g^0_{ab}, \quad (\mathcal{N})_{ab} = \eta_{ab}, \tag{1.6}
\]

where \( \eta_{ab} \) is the Minkowski metric. The the requirement that the first term in Eq. (1.5) is the Minkowski metric can now be written as a matrix equation

\[
\mathcal{L}^T \mathcal{G}^0 \mathcal{L} = \mathcal{N}, \tag{1.7}
\]

where \( T \) is the transpose operator: \((\mathcal{A}^T)_{ab} = \mathcal{A}_{ba}\), for any matrix \( \mathcal{A} \). This determines \( \Lambda^\gamma_{\gamma'} \) as a function of \( g^0_{\alpha\beta} \) up to a Lorentz transformation. If \( g^0_{\alpha\beta} \) were already the Minkowski metric, then (1.7) is the definition of a Lorentz transformation. The proof that one can solve Eq. (1.7) is a property of quadratic forms; it is detailed for instance in the book `Tensor analysis on manifolds` by Bishop and Goldberg.

Now consider the order \( x^\gamma' \) term in Eq. (1.5). We want to choose the components of \( M \) so that this term vanishes. First define matrices \( \mathcal{M}_{\gamma'}, \mathcal{Z}_{\gamma'}, \) and \( \mathcal{S}_{\gamma'} \)

\[
(\mathcal{M}_{\gamma'})_{ab} = M^b_{\gamma'}, \quad (\mathcal{Z}_{\gamma'})_{ab} = (\mathcal{G}^0 M_{\gamma'})_{ab} = g^0_{ab} M^\gamma_{\gamma'}, \quad (S_{\gamma'})_{ab} = -\Lambda^a_{\alpha} \Lambda^\beta_{\beta} \Lambda^\gamma_{\gamma'} g^0_{\alpha\beta,\gamma'}.
\]

so that the “vanishing requirement” can be written as the following matrix equation

\[
\mathcal{L}^T \mathcal{Z}_{\gamma'} + \mathcal{Z}_{\gamma'}^T \mathcal{L} = \mathcal{S}_{\gamma'}. \tag{1.9}
\]
Inserting the suggested solution $\mathcal{Z}_\gamma = (\mathcal{L}^T)^{-1} \mathcal{S}_\gamma / 2$, where the superscript $-1$ denotes an inverse matrix, gives

$$S + S \left[ (\mathcal{L}^T)^{-1} \right]^T = 2S,$$

(1.10)

which is an equality since $[(\mathcal{L}^T)^{-1}]^T = \mathcal{L}^{-1}$. So we have shown that the suggested solution is indeed a solution. Then since $\mathcal{Z}_\gamma = \mathcal{G}^0 \mathcal{M}_\gamma$, we have

$$\mathcal{M}_\gamma = \frac{1}{2} \left( \mathcal{G}^0 \right)^{-1} (\mathcal{L}^T)^{-1} \mathcal{S}_\gamma,$$

(1.11)

Which determines $M^\alpha_{\beta \gamma}$ as a function of $g^0_{\alpha \beta}$ and $g^0_{\alpha \beta \gamma}$.

Now Eq. (1.5) takes the form

$$g_{\alpha \beta} = \eta_{\alpha \beta} + O[(x^\mu)^2].$$

(1.12)

This is simply an expansion of the metric $g_{\alpha \beta}$ near $P$, similar to Eq. (1.4). Since there is no order $x^\gamma$ term, the first derivatives $g_{\alpha \beta \gamma}$ vanish at $P$, and the primed frame is a local Lorentz frame. Note that equations (1.7) and (1.11) could now be used in Eq. (1.2) to determine the local Lorentz coordinates $x^{\alpha'}$, strictly in terms of the original coordinates $x^\alpha$, the original components of the local metric $g^0_{\alpha \beta}$, and the derivatives $g^0_{\alpha \beta \gamma}$.

II. ALTERNATIVE ACTION PRINCIPLE FOR TIMELIKE GEODESICS (8 POINTS)

Consider the functional

$$S[x^\mu(\tau)] = \int d\tau \ g_{\alpha \beta}(x^\mu) \dot{x}^\alpha \dot{x}^\beta = \int d\tau L(x^\mu, \dot{x}^\mu),$$

(2.1)

where an overdot represents $d/d\tau$. We want to determine the function $x^\mu(\tau)$ which extremizes $S$. This function will satisfy the Euler-Lagrange equation

$$L_{,\alpha} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\alpha}.$$  

(2.2)

From Eq. (2.1) we have

$$L_{,\alpha} = g_{\alpha \beta}(\dot{x}^\alpha \dot{x}^\beta), \quad \frac{\partial L}{\partial \dot{x}^\alpha} = 2g_{\lambda \mu} \dot{x}^\mu, \quad \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\alpha} = 2g_{\lambda \mu} \ddot{x}^\mu + 2g_{\lambda \mu, \gamma} \dot{x}^\gamma \dot{x}^\mu,$$

(2.3)

where we have used $g_{\alpha \beta} = g_{\beta \alpha}$, and $x^{\alpha'}_{, \beta} = \delta^\alpha_{\beta}$. Substituting into Eq. (2.2) then gives

$$g_{\lambda \mu} \ddot{x}^\mu = \frac{1}{2} \left( g_{\alpha \beta, \lambda} \dot{x}^\alpha \dot{x}^\beta - 2g_{\lambda \mu, \gamma} \dot{x}^\gamma \dot{x}^\mu \right).$$

(2.4)

Contracting with $g^{\lambda \sigma}$ gives

$$\ddot{x}^\sigma = \frac{1}{2} g^{\lambda \sigma} \left( g_{\alpha \beta, \lambda} \dot{x}^\alpha \dot{x}^\beta - 2g_{\lambda \mu, \gamma} \dot{x}^\gamma \dot{x}^\mu \right).$$

(2.5)

To clean up a bit, rename dummy indices $\mu \rightarrow \alpha$ and $\gamma \rightarrow \beta$ in the second term on the right hand side

$$\ddot{x}^\sigma = \frac{1}{2} g^{\lambda \sigma} \left( g_{\alpha \beta, \lambda} - 2g_{\lambda \alpha, \beta} \right) \dot{x}^\alpha \dot{x}^\beta.$$  

(2.6)

Now contract the known expression for the connection coefficients [see Eq. (8.19) in Hartle] with $\dot{x}^\alpha \dot{x}^\beta$,

$$\Gamma^\sigma_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} g^{\lambda \sigma} \left( g_{\alpha \beta, \lambda} + g_{\lambda \alpha, \beta} + g_{\lambda \beta, \alpha} \right) \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} g^{\lambda \sigma} \left( -g_{\alpha \beta, \lambda} + 2g_{\lambda \alpha, \beta} \right) \dot{x}^\alpha \dot{x}^\beta,$$

(2.7)

where in the last term we have switched dummy indices $\alpha \leftrightarrow \beta$ and used $g_{\alpha \beta} = g_{\beta \alpha}$. We then recognize Eq. (2.6) as the geodesic equation

$$\ddot{x}^\sigma = -\Gamma^\sigma_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta.$$  

(2.8)
III. HARTLE CHAPTER 6, PROBLEM 14 (8 POINTS)

We will calculate the proper time along worldlines near the Earth in the static weak field (Newtonian) limit. In this limit, the line element is [see Eq. (6.20) in Hartle, using \( c = 1 \)]

\[
 ds^2 = -(1 + 2\Phi(x)) dt^2 + (1 - 2\Phi(x)) (dx^2 + dy^2 + dz^2),
\]

(3.1)

where \( \Phi(x) = -GM_\oplus / (x^2 + y^2 + z^2) \) is usual Newtonian potential for the the Earth’s gravitational field. The proper time between events \( A \) and \( B \) is then given by

\[
 \tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-ds^2} = \int_A^B dt \left[ (1 + 2\Phi) - (1 - 2\Phi)v^2 \right]^{1/2} = \int_0^P dt \ (1 + 2\Phi - v^2 + 2\Phi v^2)^{1/2},
\]

(3.2)

where \( v^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 \), and where the events \( A \) and \( B \) will always be such that \( t(B) - t(A) = P \). In the limit where \( v^2 \ll 1 \) and \( |\Phi| \ll 1 \), we have

\[
 \tau_{AB} = P + \int_0^P dt \left( \Phi - \frac{1}{2} v^2 \right) + O(\Phi^2) + O(v^4) + O(\Phi v^2).
\]

(3.3)

A.

For a circular worldline at constant radius \( R \) with period \( P \), both \( \Phi = -GM_\oplus / R \), and \( v^2 = (2\pi R/P)^2 \), are constant. Using Eq. (3.3) the proper time from \( A \) to \( B \) along this worldline is given by

\[
 \tau^c_{AB} = P - \left( \frac{GM_\oplus}{R} + \frac{2\pi^2 R^2}{P^2} \right) \int_0^P dt = P \left( 1 - \frac{2\pi^2 R^2}{P^2} - \frac{GM_\oplus}{R} \right).
\]

(3.4)

Kepler’s law is valid in the Newtonian limit. For circular Earth orbits this law takes the form \( P^2 = 4\pi^2 R^3 / (GM_\oplus) \), see Example 3.2 on page 40 in Hartle. Substituting this into Eq. (3.4) gives

\[
 \tau^c_{AB} = P \left( 1 - \frac{3 GM_\oplus}{2 R} \right).
\]

(3.5)

In the Newtonian limit, this worldline is a geodesic and must therefore be a path of extremal proper time.

B.

For a stationary observer at constant position \( x^2 + y^2 + z^2 = R^2 \), once again both \( \Phi = -GM_\oplus / R \), and \( v^2 = 0 \), are constant. The proper time from \( A \) to \( B \) along this worldline is given by Eq. (3.3)

\[
 \tau^s_{AB} = P - \frac{GM_\oplus}{R} \int_0^P dt = P \left( 1 - \frac{GM_\oplus}{R} \right).
\]

(3.6)

This worldline is not a geodesic. It might be a bit of a surprise then that \( \tau^c_{AB} < \tau^s_{AB} \). However the circular orbit, which is a geodesic, need not be a global maximum of the proper time integral. It need only be an extrema.

C.

By definition, the proper time vanishes along any photon worldline, or null curve. The proper time from \( A \) to \( B \) along this worldline is then

\[
 \tau^\gamma_{AB} = 0.
\]

(3.7)

Note that Eq. (3.3) cannot be used in this case since the assumption \( v^2 \ll 1 \) is invalid.

It is likely that the word “photon” was a typo in the text, and should have been replaced with “particle”. That is, Hartle might have meant for us to consider a particle that goes radially outward with an initial velocity that is a little bit less than the escape velocity, chosen so that the particle goes out and returns back to the staring point in a time \( P \).
D.

The worldline of a particle on any eccentric periodic orbit (which includes the events A and B) with period \( P \), and other worldlines of particles which move only radially are also extrema of the proper time integral. Although there is a theorem of uniqueness for geodesics between any two nearby events, in this case the events A and B are sufficiently separated that the extrema of the proper time integral are not unique. That is, the extrema are local rather than global. If you enjoyed this problem, you might also want to try problem 12 in chapter 6 of Hartle.

IV. Hartle Chapter 7, Problem 5 (8 Points)

Consider the spacetime with line element

\[
 ds^2 = -x dv^2 + 2dvdx. \tag{4.1}
\]

A.

We can find the null geodesics by demanding that \( ds^2 = 0 \). This condition is satisfied by lines of constant \( v \) for which \( dv = 0 \), and also by solutions of the differential equation \( dv/dx = 2/x \), or rather \( dx/dv = x/2 \), which have the form

\[
 x(v) = x_0 e^{v/2} \tag{4.2}
\]

where the initial position \( x_0 \) is an integration constant.

B.

Some representative null curves of the form (4.2) are shown in Fig. 1.

C.

A timelike geodesic must have \( ds^2 < 0 \) or

\[
 \frac{dx}{dv} < \frac{x}{2} \tag{4.3}
\]

In other words, starting at some event \( P \), for a given small future directed (positive) change in \( v \), the small change in \( x \) along any timelike geodesic originating at \( P \) must always be less than the associated small change in \( x \) along a null geodesic originating at \( P \). We can draw such curves in Fig. (1) as follows: Start at some point on one of the curved null geodesics. Move a little bit along that curve in the positive \( v \) direction, making sure to stay to the left of the original null curve. In this way, we can draw curves which cross from positive \( x \) to negative \( x \), however we can’t draw any which cross from negative \( x \) to positive \( x \).

V. Hartle Chapter 8, Problem 9 (10 Points)

We want to find the timelike geodesics for the Rindler\(^1\) spacetime with line element \( ds^2 = -x^2 dt^2 + dx^2 \). We follow the method described in problem II. That is, we will find the functions \( x^\alpha(\tau) = [t(\tau), x(\tau)] \), which extremizes the action

\[
 \int d\tau \ g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \int d\tau \ L = \int d\tau \ -x^2 \dot{t}^2 + \dot{x}^2 \tag{5.1}
\]

\(^1\) See problem set 5.
using the Euler-Lagrange equation (2.2). Note that we are again using an overdot to denote a derivative with respect to proper time $\tau$. To use the Euler-Lagrange equation, we will need

$$L,\lambda = (0, -2x\dot{t}^2), \quad \frac{\partial L}{\partial \dot{x}^\lambda} = (-2x^2\dot{t}, 2\dot{x}).$$ (5.2)

Since $L,0 = 0$, we know $\partial L/\partial \dot{t}$ is constant. The equation of motion for $\dot{t}$ is then

$$\dot{t} = \frac{\mathcal{E}}{x^2},$$ (5.3)

where $\mathcal{E}$ is constant. Note that we could have also found this result by noting that $\vec{\partial}_\dot{t}$ is a Killing vector, which means $\vec{\partial}_\dot{t} \cdot \vec{x} = -\mathcal{E}$, is constant.

The Euler-Lagrange equation also gives $\ddot{x} = -x\dot{t}^2$, or rather $\ddot{x} = -\mathcal{E}^2 x^{-3}$, if we use Eq. (5.3). However, there is an easy way to get a first order equation. Recall that $(d\ddot{x}/d\tau)^2 = g_{\alpha\beta}\dot{x}^\alpha \ddot{x}^\beta = -1$. This requirement, called 4-velocity normalization, gives $-x^2\ddot{t}^2 + \dot{x}^2 = -1$, which can be simplified using Eq. (5.3)

$$x^2 = \mathcal{E}^2 x^{-2} - 1.$$ (5.4)

We are only looking for the shape $x(t)$ of the geodesics. We can use $\dot{x} = idx/dt$ to combine Eqs. (5.4) and (5.3) into a single differential equation for $x(t)$

$$\frac{dx}{dt} = \pm\sqrt{x^2 - x^4 \mathcal{E}^{-2}}.$$ (5.5)

If we define $x = \pm z/\mathcal{E}$ we have $dz/dt = \sqrt{z^2 - z^4}$. The solutions are $z(t) = \text{sech}(t + C)$, or

$$x(t) = \pm \mathcal{E} \text{sech}(t + C),$$ (5.6)

where $C$ is a constant related to the initial position $x(0) = \pm \mathcal{E} \text{sech}(C)$. 

FIG. 1: Null curves for the spacetime with line element given by Eq. (4.1). The shape of the light cone at any point is given by intersecting null curves at that point.
You may be surprised that our solutions are determined by only two integration constants: $E$ and $C$. After all, the geodesic equations give a second order equation for each dimension, and there are two dimensions. However 4-velocity normalization, which is not guaranteed by the geodesic equation, always eliminates one constant. We eliminated another constant by asking only for the “shape” $x(t)$ of the geodesics, rather than the full parametrized solution $x^\alpha(\tau)$. In Newtonian mechanics, we sometimes also eliminate a constant by seeking only the shape of an orbit, however there is no analog to 4-velocity normalization.

Lastly note that, as was suggested in problem II, it would be a simple matter to find the connection coefficients from the equations of motion. Comparing $\ddot{x} = -x\dot{t}^2$ to Eq. (2.8) gives

$$\Gamma^x_{tt} = x, \quad \Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^x_{xx} = 0. \quad (5.7)$$

Similarly, differentiating $E$ yields $\ddot{t} = -2(1/x)\dot{t}\dot{x}$, which when compared to Eq. (2.8) gives

$$\Gamma^t_{tx} = \Gamma^t_{xt} = \frac{1}{x}, \quad \Gamma^t_{tt} = \Gamma^t_{xx} = 0. \quad (5.8)$$